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Influence of Fractal Embedding in Three-Dimensional Euclidean Space on Wave Propogation in Electro-Chromodynamics

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Abstract In this paper two zero-dimensional compact sets with equal topological and fractal dimensions but embedded in Euclidean space by different ways are under study. Diffraction of plane electromagnetic wave propagated and reflected by fractal surfaces is considered for each of these compact sets placed in vacuum. It is obtained, that the embedding of compact influences on characteristics of wave in final state. Thus, the embedding of Cantor set in Euclidean space is additional property of a fractal which can be important both for applications of fractal electrodynamics and for physics of strong interactions.

Keywords Fractals, Quantum chromodynamics, Diffraction, Cantor Discontinuum

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1 Introduction

Geometry is believed to play a crucial role in the formulation and understanding of mathematical methods of modern physics theories. This statement concerns, in particular, electrodynamics and quantum chromodynamics (QCD). Objects with high irregular geometry possessing properties impossible to describe in the framework of Euclid's geometry are wide spread in nature and are encountered in many fields of science also. Usually, these objects are fractals. It seems the search is important for connections between geometrical characteristics of these structures and physical quantities which describe its (structures) properties with respect to electromagnetic and / or strong interactions [1].

QCD as non-Abelian theory contains non-trivial topological gauge field configurations, which are localized objects (instantons, sphalerons) with average linear size $\sim 1/3$ fm in Euclidian space. Study in the field of pion femtoscopy allows to get the following numerical estimations for volume of matter created in relativistic heavy ion collisions: $V_f \sim 2.5 \times 10^3$ fm^3 at highest Relativistic heavy ion collider (RHIC, BNL) energy $\sqrt{s_{\rm NN}} = 200$ GeV [2, 3, 4] and $V_f \sim 5.0 \times 10^3$ fm^3 at Large hadron collider (LHC, CERN) energy $\sqrt{s_{\rm NN}} = 2.76$ TeV [4]. In spite of these estimations correspond to the freeze-out stage one can expect significant amount of instantons / sphalerons can be inside such volume and these topologically non-trivial objects can form clusters and structures with highly irregular geometry. One can suggest at qualitative level at least that such structures can be characterized by fractional topology charge. This hypothesis does not contradict with general case of solutions on solitons in quantum field theory [5]. It was found in [5] that the fermion number on kinks in one dimension or on magnetic monopoles in three dimensions is, in general, a transcendental function of the coupling constant of the gauge theories. On the one hand, as was indicated in [5], the direct utility of results for fractional topology charge for high energy physics seems problematical and it is more likely that corresponding effects do arise in condensed matter systems. But on the other hand in [6] the hypothetical objects with topological charge 1/2, so called merons, were mentioned. The merons are objects into which instantons can dissociate at large sizes, when the coupling and quantum fluctuations get big enough. Unlike instantons, which interact at large distances weakly, as dipole, and cannot effectively disorder the Wilson loop, the merons interact like charge and can do so [6]. Moreover the intensive investigations on RHIC and LHC demonstrate that the final state matter created in relativistic heavy ion collisions in energy domain, at least, $\sqrt{s_{\rm NN}} \simeq 0.04 - 2.76$ TeV is quark-gluon system characterized by very low viscosity and strong coupling, i.e. (quasi)ideal quark-gluon liquid – so called strongly coupled quark-gluon plasma (sQGP). The theoretical and experimental studies indicate that this matter will remains strongly coupled system up to highest experimentally available energies. The new corresponding physics realm called "QCD of condensed matter" is developed rapidly at the present time [7]. Thus the objects with fractional topological charge can be some interesting in point of view both the problem of confinement of color charges in QCD and for study of strongly coupled systems (sQGP). Suggestion with respect to production of large amount of topologically non-trivial objects is confirmed by estimations which show creation about 400 sphaleron-type clusters in Au+Au collisions at $\sqrt{s_{\rm NN}} = 200$ GeV [8]. Hypothesis concerned on complex and highly irregular geometry of QCD vacuum in Euclidean space does not contradict both lattice calculations [9, 10] and fractal-like approximation of structures in distributions of topological charge density [11]. The picture that has emerged on basis of these investigations is quite different from earlier conceptions like the dilute instanton gas or even the instanton liquid [12].

Structures with non-trivial topology in QCD vacuum are believed to relate with some nonperturbative phenomena like large mass of η' meson, the axial $U(1)$ anomaly, violation of some fundamental symmetries ($\mathcal{P}/\mathcal{CP}$) in hot QCD matter. The important note is that the coupling of these non-trivial topological field configurations to electromagnetic field induced by the axial anomaly in the framework of Maxwell-Chern-Simons electrodynamics can lead to the experimentally observable effects in the presence of external strong magnetic field (so called "chiral magnetic effect") [13]. At present it is obtained that extremely intense (electro)magnetic field is generated in relativistic heavy ion collisions [14, 15]. The evidence for the chiral magnetic effect has been found recently both from numerical lattice QCD calculations [16] and from several experiments at RHIC [17, 18] and LHC [19] for the field of strong interaction physics. Therefore the study seems important for interaction of electromagnetic field with topologically non-trivial structures which can be characterized as fractal-like ones [1].

2 Wave propagation in various fractals

In this paper the problem of electromagnetic wave propagation in fractal environment is considered. At present it is unknown what structures namely can be formed by topologically non-trivial QCD-objects, i.e. type of possible fractals is unknown exactly. Thus one can makes only some hypothesis concerning the type of possible highly irregural structures or consider some examples based on known fractals. It is supposed that two zero-dimensional compact sets $\mathbf{A}$ and $\mathbf{C}$ with different topological properties and equal fractal dimensions, i.e.

$$\left.\begin{array}{l}\dim\mathbf{A} = 0, \dim\mathbf{C} = 0,\\ \dim_{\rm H}\mathbf{A} = \dim_{\rm H}\mathbf{C},\end{array}\right\} \qquad (1)$$

are placed in vacuum. Here dim – topological dimension, $\dim_{\rm H}$ – Hausdorff dimension.

Definition 1.

A zero-dimensional compact set $\mathbf{K}$ is named *wild* or *wild embedded* in three-dimensional Euclidean space $\mathbf{E}^3$ if there is no homeomorphism of the space $\mathbf{E}^3$ on itself at which the compact $\mathbf{K}$ would be transformed in zero-dimensional compact lied on a straight line $\mathbf{E}^1 \subset \mathbf{E}^3$. In the opposite case the compact $\mathbf{K}$ is named *tame*.

In the framework of group theory this definition can be reformulated by the following way: a zero-dimensional compact set is defined as *wild* in three-dimensional Euclidean space $\mathbf{E}^3$ if its complement in $\mathbf{E}^3$ has non trivial fundamental group, otherwise the zero-dimensional compact set is *tame* in $\mathbf{E}^3$.

2.1 Wild Compact Set

Lets compact set $\mathbf{A}$ is the zero-dimensional Antoine compact set [20]. The Antoine compact set can be represented by the follow way:

$$\mathbf{A} = \bigcap_{k=1}^{+\infty} \bigcup_{i=1}^{3^k} \mathbf{T}_{ki}, \qquad (2)$$

where $\mathbf{T}_{ki}$ is the polyhedral complete torus for which the relations are valid $\forall\, i = 1, 2, \ldots, 3^k,\ k = 0, 1, \ldots :\ \mathbf{T}_{ki} \cap \mathbf{T}_{kj} = \emptyset$ at $i \neq j$; $\mathrm{diam}\mathbf{T}_{ki} < (k+1)^{-1}$; the three complete tori of next "level" $\mathbf{T}_{k+1 i_j},\ j = 1, 2, 3$ are inside of each complete torus $\mathbf{T}_{ki}$ with that the each of complete tori $\mathbf{T}_{k+1 i_j}$ is locking with two another ones and chain of complete tori $\bigcup_{j=1}^{3} \mathbf{T}_{k+1 i_j}$ can not be drawn together into point in $\mathbf{T}_{ki}$. It should be noted that the chain of complete tori $\mathbf{T}_{11} \cup \mathbf{T}_{12} \cup \mathbf{T}_{13}$ lies in the complete torus $\mathbf{T}_0 \subset \mathbf{E}^3$, $\mathrm{diam}\mathbf{T}_0 < 1$. The Antoine compact set $\mathbf{A}$ is wild compact one [20].

2.2 Tame Compact Set

The zero-dimensional compact set $\mathbf{C}$, homeomorphous to the compact set $\mathbf{A}$, was constructed in [21] as following:

$$\mathbf{C} = \bigcap_{k=0}^{+\infty} \mathbf{C}_k = \bigcap_{k=0}^{+\infty} \bigcup_{j=1}^{3^k} \mathbf{U}_{kj}. \qquad (3)$$

Here $\mathbf{C}_0 = \mathbf{T}'_0$ is the complete torus in $\mathbf{E}^3$ and $\mathrm{diam}\mathbf{T}'_0 < 1$. Following relations are valid: $\mathbf{U}_{kj} = \mathbf{T}'_{kj},\ j \neq 3n$; $\mathbf{U}_{kj} = \mathbf{Q}'_{kj}, j = 3n$; $\mathbf{U}_{ki} \cup \mathbf{U}_{kj} = \emptyset,\ i \neq j$; and $\left(\bigcup_{j=3m+1}^{3m+3} \mathbf{U}_{kj}\right) \subset \mathrm{int}\mathbf{U}_{k-1l}$ for arbitrary l, where $\mathbf{T}'_{kj}$ – polyhedral complete torus with $\mathrm{diam}\mathbf{T}'_{k+1_j} < (k+2)^{-1}$, $\mathbf{Q}'_{kj}$ is the polyhedral three-dimensional cell with $\mathrm{diam}\mathbf{Q}'_{k+1_j} < (k+2)^{-1}$. At construction of the $\mathbf{C}_{k+1}$ two complete tori $\mathbf{T}'_{k+1i_1}$, $\mathbf{T}'_{k+1i_2}$ locked each to other and without intersection and cell $\mathbf{Q}'_{k+1i_3}$ is constructed inside of each $\mathbf{U}_{ki}$ so as to $\left(\bigcup_{j=1}^{2} \mathbf{T}'_{k+1i_j}\right) \bigcap \mathbf{Q}'_{k+1i_3} = \emptyset$. Thus $\mathbf{C}_{k+1} = \bigcup_{i=1}^{3^{k+1}} \mathbf{U}_{k+1i}$ i.e. $\mathbf{C}_{k+1}$ is the unification of the pairs of polyhedral complete tori without intersections $\mathbf{T}'_{k+1i_j},\ j = 1, 2$ and polyhedral three-dimensional cells $\mathbf{Q}'_{k+1i_3}$ with that $\mathrm{diam}\mathbf{U}_{k+1i} \to 0$ together with $(k+2)^{-1}$ at $k \to \infty$. The object $\mathbf{C}$ defined by (3) is compact set because each of its elements $\mathbf{C}_k$ is compact set. By the construction $\mathbf{C} \subset \bigcup_{i=1}^{3^k} \mathrm{int}\mathbf{U}_{ki}$ for each k. It follows that $\mathbf{C}$ is the sum of pairs non-intersected openly-closed sets: $\mathbf{C} \cap \mathrm{int}\mathbf{U}_k$, which diameters are tend to zero with increasing of k. Therefore, one can obtain $\mathrm{diam}\mathbf{C} = 0$. In [21] it was proved that compact set $\mathbf{C}$ is the tame compact set in $\mathbf{E}^3$.

2.3 Wave Propagation

It seems the circumstance is very important that both the wild Antoine compact set $\mathbf{A}$ and the tame compact $\mathbf{C}$ are fractals. Compact sets $\mathbf{A}$ and $\mathbf{C}$ are homeomorphous with the Cantor perfect set $\tilde{\mathbf{K}}$ lied on a straight line $\mathbf{E}^1 \subset \mathbf{E}^3$.

Definition 2.

Psedoisotopy $\mathcal{F}_t$ of the space $\mathbf{E}^3$ is named *homotopy* $\mathcal{F}_t, t \in [0,1]$, $\mathcal{F}_t : \mathbf{E}^3 \to \mathbf{E}^3$ such as to $\mathbf{F}_t$ is homeomorphous mapping of the space $\mathbf{E}^3$ on itself for $t \in [0,1)$ but at $t = 1$ the $\mathcal{F}_1$ is continuous mapping of the $\mathbf{E}^3$ on itself. If $\mathcal{F}_0 = \mathcal{I}$, where $\mathcal{I}$ is the identity map, then one can speaks that $\mathcal{F}_t$ is pseudoisotopy from identity map.

In [21] it was shown, that for the compact set (3) such pseudoisotopy can be constructed $\mathcal{F}_t, t \in [0,1]$, $\mathcal{F}_0 = \mathcal{I}, \mathcal{F}_t : \mathbf{E}^3 \to \mathbf{E}^3$, which transforms the tame compact set (3) in the wild compact set (2). In [22] it was shown, that the set of points of wildness for zero-dimensional compact set $\mathbf{K}$ in $\mathbf{E}^3$ is either non-counting or empty. In [23, 24] the following theorem was proved.

Theorem.

Lets $\mathbf{K}_1$ and $\mathbf{K}_2$ are compact sets in $\mathbf{E}^3$ and lets $\mathbf{f} : \mathbf{K_1} \to \mathbf{K_2}$, where $\mathbf{f}$ is homeomorphism. Moreover $\mathbf{f}$ and $\mathbf{f^{-1}}$ satisfy the Lipshitz condition. Then $\mathbf{K}_1$ and $\mathbf{K}_2$ have equal fractal dimensions.

As homeomorphism $\mathbf{f} : \mathbf{K_1} \to \mathbf{K_2}$ one can takes the mapping $\mathcal{F}_1$ constructed by the pseudoisotopy $\mathcal{F}_t, t \in [0,1]$, $\mathcal{F}_t : \mathbf{E}^3 \to \mathbf{E}^3$ so as to it (i.e. this map) satisfies the Lipshitz condition [21]. It follows that $\dim_{\mathrm{H}}\mathbf{A} = \dim_{\mathrm{H}}\mathbf{C}$, i.e. compact sets (2) and (3) are fractals with equal fractal dimension D.

Lets the complete torus $\mathbf{T}_0$ is filled by some matter with refraction index n_1 differs from the refraction index of environment (vacuum) n_0. The right Cartesian coordinates are chosen as the coordinate system with axis OZ directed horizontally, with origin of coordinates coincided with the center of symmetry of complete torus $\mathbf{T}_0$, and with the plane OXY included the mean line of the complete torus under consideration. Lets the plane harmonic wave $\Psi(x,y,z)$ falls on the $\mathbf{T}_0$ with that the wave length satisfies the relation $\lambda \ll \mathrm{diam}\mathbf{T}_0$.

At the construction step with arbitrary number k for Antoine compact set $\mathbf{A}$ one considers the chain of complete tori of corresponding level $\bigcup_{j=1}^{3} \mathbf{T}_{ki_j}$ inside some fixed, for example, i-th complete torus $\mathbf{T}_{ki}$ of previous level. Note that the chain $\bigcup_{j=1}^{3} \mathbf{T}_{ki_j}$ is not drawn together into point in $\mathbf{T}_{ki}$. The wave propagated in $\mathbf{T}_{ki}$ and passed through complete torus $\mathbf{T}_{ki_j}$ $(j = 1,2,3)$, collides with obstacle in the places of locking $\mathbf{T}_{ki_1}$ with $\mathbf{T}_{ki_2}$ and $\mathbf{T}_{ki_3}$. Thus the wave diffraction is formed on the each step with number up to some $\tilde{k}$, where $\tilde{k}$ is defined by the relation $\mathrm{diam}\mathbf{T}_{\tilde{k}\ldots1} < (\tilde{k}+1)^{-1} < \lambda$, which means the wave can not distinguishes complete tori with diameters smaller than wave length.

In the case of $\mathbf{C}$ the compact set $\mathbf{U}_{kj} = \mathbf{Q}'_{kj}$ at $j = 3n$ is the polyhedral three-dimensional cell which creates splits with complete tori $\mathbf{U}_{kj} = \mathbf{T}_{kj}$ at $j \neq 3n$. Because $\forall j : \mathbf{U}_{kj}$ are polyhedrons for the compact set $\mathbf{C}$ the construction can be made so that the width of appeared split will be smaller than up to some step number $\tilde{k}$. Thus diffraction from split will be generated in this case instead of diffraction from obstacle.

3 Conclusions

The main aim of this paper is qualitative study of influence of fractal embedding on physics characteristics of wave propagation. It seems that the relations between geometrical (embedding) properties of some fractal object and physical properties of waves interacted with this object can be important both for fundamental studies and for applications.

Therefore, based on the qualitative consideration the following conclusion can be made. In spite of (1), embedding way of the wild compact set $\mathbf{A}$ and tame compact set $\mathbf{C}$ in $\mathbf{E}^3$ can influences on physics characteristics of wave and on distribution of intensity, i.e. embedding is the one more characteristic of fractal which can be important for applications of fractal geometry in physics. There are various experiments for propagation of electromagnetic waves in fractal environments characterized by different Hausdorff dimensions (see, for example, [25]). But unfortunately there is no experimental study of influence of fractal embedding on physics characteristics of wave propagation. This paper is the first qualitative study of the problem. Therefore future investigations are necessary for derivation of numerical estimations and verification of this hypothesis at quantitative level.

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