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A Space-Time Finite Element Approximation of a Two-Step Chemical Kinetics Model

Abdellatif Agouzal[1], **Karam Allali**[2,*]

[1]Department of Mathematics, Institut Camille Jordan, University Lyon I, 43 Bd 11 November, 69100 Villeurbanne, France

[2]Department of Mathematics, Faculty of Sciences and Technologies, University Hassan II, Po.Box 146, Mohammedia, Morocco

*Corresponding Author: allali@fstm.ac.ma

Abstract In this paper we suggest a fully discretized problem of a model describing two-step chemical kinetics. The model considered is a system of equations coupling Navier-Stokes equations with three non-linear reaction-diffusion equations. A space-time finite elements approximations are presented. The stability of the fully discretized problem is studied. Optimal error estimates are given.

Keywords A priori error estimates, Boussinesq approximation, Chemical kinetics, Mixed finite elements

1 Introduction

Chemical kinetics can be defined as the study of chemical or biological systems that its compositions change with time. For example, in biological systems, the cellular behaviour involves a very large number of chemical reactions, usually, catalyzed by proteins. Those reactions, for simplicity, can be described as a simple chemical reaction between two or more states of the system. Several works have been devoted to chemical kinetics, see for instance [5, 6, 11, 17].

The numerical analysis of one-step chemical kinetics is studied in [1]. The authors prove the existence, uniqueness of the solutions and also give some a priori error estimates. In this paper, we are interested in a two-step chemical kinetics. In the first step, the specie A gives a product B which undergoes a second reaction and gives the specie C as result of the second step. For example, in biology reactions, A could be the unfolded state of a protein, B the compact denatured state or "molten globule" state and C the final native state of the protein [4, 9].

It is worth noticing that when two time curves A and C or B are known, the solution is easy, but in many cases the simultaneous determination of the substrate, intermediate and the final product is difficult or impossible. So, it is very significant to develop a numerical research method for this kind of complex reaction.

We consider both steps be exothermic first-order reactions with the corresponding reaction rates, given by the Arrhenius law [18]: $k_i = k_{i_0} exp(E_i/RT)$, here E_i is the activation energy, k_{i_0} the pre-exponential factor, R the universal gas constant, T the temperature and the index i denotes the step number ($i = 1, 2$). It was established that when the activation energy of the first reaction is much greater than the activation energy of the second reaction (the second reaction is faster that the first) the second specie B will be immediately consumed, as quickly as it is formed. In the other case, when the second reaction is slower than the first one, all the two reactants and the two products will appear (it is the case of Nitrogen dioxide which can give the final product Nitrogen via the intermediate specie Nitrogen monoxide) [16, 19]. Same phenomenon is observed in some biochemistry problems when using Michaelis-Menten kinetics [15], in the model, the unknowns are the concentrations of enzymes.

Perhaps the simplest model of such process can be described by the following system of differential equations [13]:

$$\frac{d\,[A]}{d\,t} = -k_1[A],$$
$$\frac{d\,[B]}{d\,t} = k_1[A] - k_2[B].$$

Here $[A]$ and $[B]$ are respectively the concentrations of species A and B. It is well known that the concentration of the final product is given by: $1 - [A] - [B]$. In this paper, we will take into account the influence of the hydrodynamic and energy on those kind of reactions. To this end, we will consider a more generalized model. The semi-descritized problem of such model has been studied in [2]. The authors prove the existence of the solution and give the a priori error estimates on the different unknowns of the model. This work is devoted to the fully discretized problem.

The paper is organized as follows. In Section 2, we give the model describing the two-step chemical kinetics, in Section 3 and 4, we give our variational problem and study its stability. In the Section 5, we give the a priori error estimates on the different unknowns of the problem.

2 Model presentation

For our two-step chemical kinetics, we will consider the incompressibility assumption on all the reactants and the priducts; therefore, we will consider a coupling between the hydrodynamic and the reaction-diffusion equations, under the classical Boussinesq approximation [14].

The model for such process, is given by:

$$(P) \qquad \begin{cases} \partial_t T - \lambda \Delta T + u.\nabla T - C_1 g_1(T) - C_2 g_2(T) = 0, \\ \partial_t C_1 - d_1 \Delta C_1 + u.\nabla C_1 + C_1 g_1(T) = 0, \\ \partial_t C_2 - d_2 \Delta C_2 + u.\nabla C_2 + C_2 g_2(T) - C_1 g_1(T) = 0, \\ \partial_t u - \nu \Delta u + (u\nabla)\, u + \nabla p = f(T), \\ div\ u = 0, \end{cases}$$

where the unknown factors are speed u, the pressure p, the temperature T, the concentrations C_1, C_2 of species A and B respectively; the coefficients ν, λ, d_1 and d_2 are assumed to be a positive constants (physically, d_i indicates the diffusion of the concentration C_i (i=1,2), λ the thermal diffusion and ν the viscosity of the fluid). The datas are a regular function f of $\mathbb{R}$ to $\mathbb{R}^d$ (typically, the function f is a gravity force proportional to the variations of density, therefore dependents on the temperature) and an other regular function g_i (i=1,2) of $\mathbb{R}$ to $\mathbb{R}^d$ (typically, the function g_i (i=1,2) is the source term of the reaction depending on the temperature and also on energy).

The gradient, divergence and laplacian operators can be defined as following:

$$\nabla\ v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_d}\right) \qquad div\ \vec{\mathbf{v}} = \sum_{i=1}^{d} \frac{\partial \mathbf{v}_i}{\partial x_i}, \qquad \Delta\ v = \sum_{i=1}^{d} \frac{\partial^2 v}{\partial^2 x_i}.$$

The boundary conditions are of *Dirichlet* type for speed and *Dirichlet-Neumann* type for the temperature same as for the concentration. They are written:

$$T|_{\Gamma_1} = C_i|_{\Gamma_1}, \ \frac{\partial T}{\partial n}|_{\Gamma_2} = \frac{\partial C_i}{\partial n}|_{\Gamma_2} = 0, \text{ and } u|_{\partial\Omega} = 0 \ \ i = 1, 2,$$

and the initial conditions are given by :

$$u|_{t=0} = u_0, \ T|_{t=0} = T_0, \text{ and } C_i|_{t=0} = C_{i_0} \ \ i = 1, 2,$$

where Γ_1 and Γ_2 are disjoined opens parts of $\partial\Omega$ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$.

It will be useful to note that for the following assumption:

$$(2.1) \qquad f(T) = \beta(T - T_0), \ \lambda = 0, \ \alpha(t) = g\gamma \text{ and } g_i(T) = k_i exp(-\frac{E_i}{RT}),$$

the system of equations (P) give the chemical model which is known under the name of *thermo-diffusif* model [3]. Where β is the coefficient of thermal expansion, T_0 the average temperature and g is the gravity constant. In our model, we announce that the function g_i is multiplied by the concentration C_i, therefore the reactions are considered of the first order; this functions are obtained by the *Arrhenius* law [18]. Here E_i indicates the activation energy, R the universal gas constant, g indicates the constant of gravity, γ is the ascending unit vector, T_0 the initial temperature of the product, the index i denotes the step number of the reaction. The existence and uniqueness of the continued problem solution is studied in [12].

In the next section, we establish the existence and the uniqueness of fully discritized problem and study its stability.

3 Presentation of the fully discretized problem

Before giving our discretized problem. First, we will give some assumption which is in good agreement with the physical background and present our functional framework.
Assume that :

$$\begin{cases} \text{The reals} \quad \nu, \lambda \ d_1, \ \ d_2 \ \text{ and } \lambda \ \text{ are strictly positives,} \\ \alpha \in W^{1,\infty}(\mathbb{R}), \\ g_i \in W^{1,\infty}(\mathbb{R}), \ \ C_{g_i} = \parallel g_i' \parallel_{L^\infty(\Omega)}, \ \text{ and } \ g_i \geq 0, \ \ \parallel g_i \parallel_{L^\infty(\Omega)} = 1, \ \ i = 1, 2 \\ f \in W^{1,\infty}(\mathbb{R}), \ \ f(0) = 0, \\ \forall (T_1, T_2) \in (H^1_{0,\Gamma_1}(\Omega))^2, \ \ \parallel f(T_1) - f(T_2) \parallel_{L^2(\Omega)} \leq \parallel \nabla (T_1 - T_2) \parallel_{L^2(\Omega)} . \end{cases}$$

We specify now the functional framework in which is carried out our analysis of the problem. The speeds space V defined by:

$$V = \{u \in (H_0^1(\Omega))^d; \ divu = 0 \ \ in \ \Omega\}.$$

The temperatures and the concentrations space is $H^1_{0,\Gamma_1}(\Omega) = \left\{v \in H^1(\Omega)/v|_{\Gamma_1} = 0\right\}$.
We introduce now the constant of $Friedrichs - Poincaré$ related to the domain geometry:

$$\rho = \sup_{u \in H_0^1(\Omega)} \frac{\parallel u \parallel_{L^2(\Omega)}}{\parallel \nabla u \parallel_{L^2(\Omega)}}.$$

Throughout the paper, we often use the following notation:
For each ζ, $\eta > 0$: $\zeta \lesssim \eta \ \Leftrightarrow \ \exists C^\star > 0 \ : \ \zeta \ \leq \ C^\star \eta$; *without further specification, we intend the constant* $C^\star$ *independent of the mesh-size and the solutions.*

Now, we give our functional framework by introducing some spaces, next, we give our fully discretized problem. First, we introduce the spaces that we need for our studies :
For any value of the real parameter $h > 0$, we consider three spaces X_h, M_h and W_h such as

$$X_h \subset (H_0^1(\Omega))^d \ , \ M_h \subset L_0^2(\Omega) \ \text{ and } W_h \subset H^1_{0,\Gamma_1}(\Omega),$$

we set

$$V_h := \{v_h, \ \ \forall q_h \in M_h, \int_\Omega q_h div \ v_h dx = 0\},$$

and we assume that they satisfy the following conditions:

1. For any $0 < \sigma \leq 1$, there exist a linear continuous operator P_h from $H^\sigma(\Omega) \cap L_0^2(\Omega)$ onto M_h such as

$$\forall q \in H^\sigma(\Omega) \cap L_0^2(\Omega), \|q - P_h q\|_{0,\Omega} \lesssim h^\sigma |q|_{\sigma,\Omega},$$

2. For all $0 < \sigma \leq 1$, there exist a linear continuous operator $\mathcal{I}_h$ from $(H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d$ onto X_h such as

$$\forall u \in (H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d, \ \ \|u - \mathcal{I}_h u\|_{1,\Omega} \lesssim h^\sigma |u|_{1+\sigma,\Omega}.$$

3. there exist a constant β independent of h, such as

$$\forall q_h \in M_h, \exists v_h \in X_h, \ \ \text{such as } (divv_h, q_h)_{0,\Omega} \geq \beta \|q_h\|_{0,\Omega} \|v_h\|_{1,\Omega}.$$

4. For all $0 < \sigma \leq 1$, there exist a linear continuous operator i_h from $H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ onto W_h such as

$$\forall T \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega), \ \ \|T - i_h T\|_{1,\Omega} \lesssim h^\sigma |T|_{1+\sigma,\Omega}.$$

We give now some examples of such couple of spaces verifying the last conditions [8]. We assume that the open Ω is polyhedric and we assume a regular triangulations family τ_h of Ω, where for all h the triangulation τ_h is d-simplexes set of diameters bounded above by h. For all K of τ_h, we define by $P_k(K)$ the polynomial space of total degree k on K, where k is a strictly positive real.

Let the space W_h defined as following:

$$W_h = \left\{T_h \in C^0(\Omega) \cap H_0^1(\Omega); \forall K \in \tau_h; T_{h|K} \in P_1(K)\right\}.$$

For the spaces X_h and M_h we have the following examples:

Example 1 In dimension $d = 2$, we set

$$X_h = \{v_h \in (C_0(\Omega))^2 \cap (H_0^1(\Omega))^2, \forall K \in \tau_h; v_{h|K} \in (P_2(K))^2\},$$

$$M_h = \{q_h \in L_0^2(\Omega), \forall K \in \tau_h; q_{h|K} \in P_0(K)\},$$

Example 2 In dimension $d = 2$, we set

$$X_h = \{v_h \in (C_0(\Omega))^2 \cap (H_0^1(\Omega))^2, \forall K \in \tau_h; v_{h|K} \in (P_2(K))^2\},$$

$$M_h = \{q_h \in L_0^2(\Omega) \cup C_0(\Omega), \forall K \in \tau_h; q_{h|K} \in P_1(K)\},$$

Example 3 For all T of τ_h of vertices a_i, $1 \leq i \leq d+1$, we note by λ_i the barycentric coordinate associated to the vertices a_i and by n_i the normal vector on the face not containing a_i. We set P_T the space engendered by the polynoms of $(P_1(T))^d$ and by the functions

$$p_i = \{ \prod_{j=1, j\neq i}^{d+1} \lambda_i n_i, 1 \leq i \leq d+1\}.$$

We set then

$$X_h = \{v_h \in (C_0(\Omega))^d \cap (H_0^1(\Omega))^d, \forall K \in \tau_h; v_{h|K} \in P_K\},$$

$$M_h = \{q_h \in L_0^2(\Omega) \cup C_0(\Omega), \forall K \in \tau_h; q_{h|K} \in P_1(K)\},$$

In the three examples above, the assumptions $(1) - (3)$ are satisfied and the constant β is independent of h [7, 10].

Let the following spaces:

$$X = \left(H^1_{0,\Gamma_1}(\Omega)\right)^d, \ \ W = H^1_{0,\Gamma_1}(\Omega), \ \ M = L_0^2(\Omega), \ \ V = \{v \in X, divv = 0\}.$$

and we introduce the forms defined by:
for all $(u, v, w) \in X^3$, $(T, \phi, C, \psi) \in W^4$, $p \in M$,

$$a_1(u, v, w) = \frac{1}{2}\left(\int_\Omega (u\nabla)vw \, dx - \int_\Omega (u\nabla)wv \, dx\right),$$

$$a_2(u, T, \phi) = \frac{1}{2}\left(\int_\Omega (u\nabla)T\phi \, dx - \int_\Omega (u\nabla)\phi T \, dx\right),$$

$$d(T, \psi) = \int_\Omega \nabla T \nabla \psi \, dx, \ \ b(p, v) = \int_\Omega pdiv \ v \, dx,$$

and:

$$k_i(C, T, \psi) = \int_\Omega C g_i(T)\psi dx.$$

In order to give the fully discretized problem, we will use in time the finite difference method with implicit scheme.

Let $k = \Delta t$. We consider the following fully discretized schema problem:
For $C^n_{1_h}, C^n_{2_h}, T^n_h, u^n_h$ known, find $C^{n+1}_{i_h} \in W_h (i = 1, 2), T^{n+1}_h \in W_h, u^{n+1}_h \in X_h$ and $p^{n+1}_h \in M_h$ such that:

$$(P_h^n)\begin{cases} \forall \phi_h \in W_h, \ \ (C^{n+1}_{1_h}, \phi_h)_{0,\Omega} + kd_1 d(C^{n+1}_{1_h}, \phi_h) + ka_1(u^n_h, C^{n+1}_{1_h}, \phi_h) \\ \qquad\qquad +kK(C^{n+1}_{1_h}, T^n_h, \phi_h) = (C^n_{1_h}, \phi_h)_{0,\Omega}, \\ \forall \phi_h \in W_h, \ \ (C^{n+1}_{2_h}, \phi_h)_{0,\Omega} + kd_2 d(C^{n+1}_{2_h}, \phi_h) + ka_1(u^n_h, C^{n+1}_{2_h}, \phi_h) \\ \qquad\qquad +kK_2(C^{n+1}_{2_h}, T^n_h, \phi_{2_h}) - kK_1(C^{n+1}_{1_h}, T^n_h, \phi_{2_h}) = (C^n_{2_h}, \phi_{2_h})_{0,\Omega}, \\ \forall \phi_h \in W_h, \ (T^{n+1}_h, \phi_h)_{0,\Omega} + k\lambda d(T^{n+1}_h, \phi_h) + ka_1(u^n_h, T^{n+1}_h, \phi_h) \\ \qquad\qquad -kK_1(C^{n+1}_{1_h}, T^n_h, \phi_h) - kK_2(C^{n+1}_{2_h}, T^n_h, \phi_h) = (T^n_h, \phi_h)_{0,\Omega}, \\ \forall v_h \in X_h, \ \ (u^{n+1}_h, v_h)_{0,\Omega} + k\mu d(u^{n+1}_h, v_h) + ka_1(u^n_h, u^{n+1}_h, v_h) - kb(p^{n+1}_h, v_h) = \\ \qquad\qquad k\alpha(t^{n+1})(f(T^{n+1}_h), v_h)_{0,\Omega} + (u^n_h, v_h)_{0,\Omega} \\ \forall q_h \in M_h, \ \ b(q_h, u^{n+1}_h) = 0. \end{cases}$$

As usual, we consider that the couple of spaces (X_h, M_h) is compatible. By classical arguments [7, 10], we have the following lemma:

Lemma 3.1. *For all $n \in \mathbf{N}^*$, the problem admits a unique solution*

$$(u_h^n, p_h^n, T_h^n, C_{1_h}^n, C_{2_h}^n) \in X_h \times M_h \times W_h^3.$$

In the sequel of this paper, for each n positive integer, we denoted by $C_{1_h}^n$, $C_{2_h}^n$, T_h^n, u_h^n and p_h^n the discret solution of the problem (P_h^n).

4 Study of stability

The main result of this section is to prove stability of our fully disretized schema. To prove that we need some lemmas. First we have:

Lemma 4.1. *For all $n \in \mathbf{N}^*$, we have:*

$$\|C_{1_h}^n\|_{0,\Omega}^2 + \sum_{i=0}^{n-1} \|C_{1_h}^{i+1} - C_{1_h}^i\|_{0,\Omega}^2 + 2kd_1 \sum_{i=1}^{n} \|C_{1_h}^i\|_{1,\Omega}^2 \leq \|C_{1_h}^0\|_{0,\Omega}^2.$$

Proof. First of all, let us notice that:

$$a_1(u_{1_h}^n, C_{1_h}^{n+1}, C_{1_h}^{n+1}) = 0 \quad \text{ and } K(C_{1_h}^{n+1}, T_h^n, C_{1_h}^{n+1}) \geq 0.$$

Consequently, by choosing $C_{1_h}^{n+1}$ as test function in the first equation of the problem (P_h^n) and by using the identity $a(a-b) = \dfrac{1}{2}(a^2 - b^2 + (a-b)^2)$, we have:

$$\|C_{1_h}^{n+1}\|_{0,\Omega}^2 + \|C_{1_h}^{n+1} - C_{1_h}^n\|_{0,\Omega}^2 + 2kd_1\|C_{1_h}^{n+1}\|_{1,\Omega}^2 \leq \|C_{1_h}^n\|_{0,\Omega}^2.$$

By summing over the time, we obtain:

$$\|C_{1_h}^{n+1}\|_{0,\Omega}^2 + \sum_{i=0}^{n} \|C_{1_h}^{i+1} - C_{1_h}^i\|_{0,\Omega}^2 + 2kd_1 \sum_{i=1}^{n+1} \|C_{1_h}^i\|_{1,\Omega}^2 \leq \|C_{1_h}^0\|_{0,\Omega}^2.$$

□

We have also the following:

Lemma 4.2. *For all $n \in \mathbf{N}^*$, we have:*

$$\|C_{2_h}^n\|_{0,\Omega}^2 + \sum_{i=0}^{n-1} \|C_{2_h}^{i+1} - C_{2_h}^i\|_{0,\Omega}^2 + d_2 k \sum_{i=1}^{n} \|C_{2_h}^i\|_{1,\Omega}^2 \leq \|C_{2_h}^0\|_{0,\Omega}^2 + \frac{\rho^4}{2d_1 d_2}\|C_{1_h}^0\|_{0,\Omega}^2.$$

Proof. By choosing $C_{2_h}^{n+1}$ as test function in the second equation of the problem (P_h^n), while noticing that : $a_1(u_h^n, C_{2_h}^{n+1}, C_{2_h}^{n+1}) = 0$, $K_2(C_{2_h}^{n+1}, T_h^n, C_{2_h}^{n+1}) \leq 0$ and by using the identity $a(a-b) = \dfrac{1}{2}(a^2 - b^2 + (a-b)^2)$, we have:

$$\|C_{2_h}^{n+1}\|_{0,\Omega}^2 + \|C_{2_h}^{n+1} - C_{2_h}^n\|_{0,\Omega}^2 + 2kd_2\|C_{2_h}^{n+1}\|_{1,\Omega}^2 \leq 2kK_1(C_{1_h}^{n+1}, T_h^n, C_{2_h}^{n+1}) + \|C_{2_h}^n\|_{0,\Omega}^2.$$

However:

$$K_1(C_{1_h}^{n+1}, T_h^n, C_{2_h}^{n+1}) \leq \frac{\rho^4}{2d_2}\|C_{1_h}^{n+1}\|_{1,\Omega}^2 + \frac{d_2}{2}\|C_{2_h}^{n+1}\|_{1,\Omega}^2.$$

Therefore:

$$\|C_{2_h}^{n+1}\|_{0,\Omega}^2 + \|C_{2_h}^{n+1} - C_{2_h}^n\|_{0,\Omega}^2 + kd_2\|C_{2_h}^{n+1}\|_{1,\Omega}^2 \leq \|C_{2_h}^n\|_{0,\Omega}^2 + k\frac{\rho^4}{d_2}\|C_{1_h}^{n+1}\|_{1,\Omega}^2.$$

By summing over the steps time and by using Lemma 4.1, we deduce the following estimate:

$$\|C_{2_h}^{n+1}\|_{0,\Omega}^2 + \sum_{i=0}^{n} \|C_{2_h}^{i+1} - C_{2_h}^i\|_{0,\Omega}^2 + kd_2 \sum_{i=1}^{n+1} \|C_{2_h}^i\|_{1,\Omega}^2 \leq \|C_{2_h}^0\|_{0,\Omega}^2 + \frac{\rho^4}{2d_1 d_2}\|C_{1_h}^0\|_{0,\Omega}^2.$$

□

We have also the following:

Lemma 4.3. *For all $n \in \mathbf{N}^*$, we have:*

$$\|T_h^n\|_{0,\Omega}^2 + \sum_{i=0}^{n-1} \|T_h^{i+1} - T_h^i\|_{0,\Omega}^2 + \lambda k \sum_{i=1}^{n} \|T_h^i\|_{1,\Omega}^2 \leq \|T_h^0\|_{0,\Omega}^2 + \frac{\rho^4}{\lambda d_1}(1 + \frac{\rho^4}{d_2^2})\|C_{1_h}^0\|_{0,\Omega}^2 + \frac{2\rho^4}{\lambda d_2}\|C_{2_h}^0\|_{0,\Omega}^2.$$

Proof. By choosing T_h^{n+1} as test function in the second equation of the problem (P_h^n), while noticing that : $a_1(u_h^n, T_h^{n+1}, T_h^{n+1}) = 0$ and by using the identity $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$, we have:

$$\|T_h^{n+1}\|_{0,\Omega}^2 + \|T_h^{n+1} - T_h^n\|_{0,\Omega}^2 + 2k\lambda\|T_h^{n+1}\|_{1,\Omega}^2 \leq 2kK_1(C_{1_h}^{n+1}, T_h^n, T_h^{n+1}) + 2kK_2(C_{2_h}^{n+1}, T_h^n, T_h^{n+1}) + \|T_h^n\|_{0,\Omega}^2.$$

However:

$$K(C_{1_h}^{n+1}, T_h^n, T_h^{n+1}) \leq \frac{\rho^4}{\lambda}\|C_{1_h}^{n+1}\|_{1,\Omega}^2 + \frac{\lambda}{4}\|T_h^{n+1}\|_{1,\Omega}^2.$$

Therefore:

$$\|T_h^{n+1}\|_{0,\Omega}^2 + \|T_h^{n+1} - T_h^n\|_{0,\Omega}^2 + k\lambda\|T_h^{n+1}\|_{1,\Omega}^2 \leq \|T_h^n\|_{0,\Omega}^2 + k\frac{2\rho^4}{\lambda}(\|C_{1_h}^{n+1}\|_{1,\Omega}^2 + \|C_{2_h}^{n+1}\|_{1,\Omega}^2).$$

By summing over the steps time and by using Lemma 4.1 and Lemma 4.2, we deduce the following estimate:

$$\begin{aligned}\|T_h^{n+1}\|_{0,\Omega}^2 &+ \sum_{i=0}^n \|T_h^{i+1} - T_h^i\|_{0,\Omega}^2 + k\lambda \sum_{i=1}^{n+1} \|T_h^i\|_{1,\Omega}^2 \leq \|T_h^0\|_{0,\Omega}^2 \\ &+ \frac{2k\rho^4}{\lambda}\left(\frac{1}{2kd_1}\|C_{1_h}^0\| + \frac{\rho^4}{2kd_1d_1^2}\|C_{1_h}^0\| + \frac{1}{kd_2}\|C_{2_h}^0\|_{0,\Omega}^2\right).\end{aligned}$$

□

Also, we have the following:

Lemma 4.4. *For all $n \in \boldsymbol{N}^*$, we have:*

$$\left\{\begin{aligned}\|u_h^n\|_{0,\Omega}^2 + \sum_{i=0}^{n-1}\|u_h^{i+1} - u_h^i\|_{0,\Omega}^2 + k\nu\sum_{i=1}^n \|u_h^i\|_{1,\Omega} \leq \ & \frac{\alpha^2\rho^2}{\lambda\nu}\left(\|T_h^0\|_{0,\Omega}^2 + \frac{\rho^4}{\lambda d_1}(1 + \frac{\rho^4}{d_2^2})\|C_{1_h}^0\|_{0,\Omega}^2\right. \\ & \left.+ \frac{2\rho^4}{\lambda d_2}\|C_{2_h}^0\|_{0,\Omega}^2\right) + \|u_h^0\|_{0,\Omega}^2.\end{aligned}\right.$$

Proof. By choosing $u_h^{n+1} \in V_h$ as test function in the third equation of the problem (P_h^n), while noticing that $a_1(u_h^n, u_h^{n+1}, u_h^{n+1}) = 0$ and by re-using the identity $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$, we have:

$$\|u_h^{n+1}\|_{0,\Omega}^2 + \|u_h^{n+1} - u_h^n\|_{0,\Omega}^2 + 2k\nu\|u_h^{n+1}\|_{1,\Omega}^2 = 2k\alpha(t^{n+1})(f(T_h^{n+1}), u_h^{n+1}) + \|u_h^n\|_{0,\Omega}^2.$$

However:

$$2k\alpha(t^{n+1})(f(T_h^{n+1}, u_h^{n+1}) \leq k\nu\|u_h^{n+1}\|_{1,\Omega}^2 + \frac{\alpha^2\rho^2}{\nu}k\|T_h^{n+1}\|_{1,\Omega}^2.$$

Therefore by using Lemma 4.1, Lemma 4.2 and Lemma 4.3, we deduce that:

$$\begin{aligned}\|u_h^n\|_{0,\Omega}^2 + \sum_{i=0}^{n-1}\|u_h^{i+1} - u_h^i\|_{0,\Omega}^2 + k\nu\sum_{i=1}^n |u_h^i|_{1,\Omega} \leq \ & \frac{\alpha^2\rho^2}{\lambda\nu}\left(\|T_h^0\|_{0,\Omega}^2 + \frac{\rho^4}{\lambda d_1}(1 + \frac{\rho^4}{d_2^2})\|C_{1_h}^0\|_{0,\Omega}^2\right. \\ & \left.+ \frac{2\rho^4}{\lambda d_2}\|C_{2_h}^0\|_{0,\Omega}^2\right) + \|u_h^0\|_{0,\Omega}^2.\end{aligned}$$

□

Now, using the Lemmas 4.1, 4.2, 4.3 and 4.4, we easily deduce stability result:

Theorem 4.1. *For all n positive integer the fully discitized problem (P_h^n) is stable.*

In the next section, we obtain some error estimates at the same time on speed, the pressure, the temperature and the concentrations.

5 Error stimates

We consider the operators R_h and r_h respectively definite from $\left(H_0^1(\Omega)\right)^d$ and $H_{0,\Gamma_1}^1(\Omega)$ onto X_h and W_h by:

$$\forall u \in \left(H_0^1(\Omega)\right)^d, \ \ \forall v_h \in X_h, \ \ \int_\Omega \nabla(R_h u - u).\nabla v_h = 0,$$

and

$$\forall s \in H_{0,\Gamma_1}^1(\Omega), \ \ \forall \phi_h \in W_h, \ \ \int_\Omega \nabla(r_h s - s).\nabla \phi_h = 0.$$

We remind that:

$$\forall\, u \in \left(H_0^1(\Omega)\right)^d, \quad \| \nabla R_h u \|_{1,\Omega} \leq \| \nabla u \|_{1,\Omega} .$$

Moreover, if $u \in \left(H^{1+\sigma}(\Omega)\right)^d$, with $1 < \sigma \leq 2$, we have [8]:

$$\|u - R_h u\|_{1,\Omega} \lesssim h^\sigma \|u\|_{1+\sigma,\Omega}.$$

We have also [8],

$$\forall\, s \in \left(H_{0,\Gamma_1}^1(\Omega)\right)^d, \quad \| \nabla r_h s \|_{1,\Omega} \leq \| \nabla s \|_{1,\Omega} .$$

Moreover, if $s \in \left(H^{1+\sigma}(\Omega)\right)^d$, with $1 < \sigma \leq 2$, we have, [8]:

$$\|r_h s - s\|_{1,\Omega} \lesssim h^\sigma \|s\|_{1+\sigma,\Omega}.$$

In the following, we set:

$$\zeta^n = R_h u^n - u_h^n \ , \ \eta^n = r_h T^n - T_h^n \ \text{ and } \epsilon_i^n = r_{i_h} C^n - C_{i_h}^n, \quad (i = 1, 2)$$

and

$$\overline{\partial}_t \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

Finally, we set:

$$A = \|u\|_{L^\infty(0,t,(H^1(\Omega))^2)} \ , \ M = \sup_{0 \leq k \leq N} \|u_h^k\|_{1,\Omega} \ , B = \|T\|_{L^\infty(0,t,H^1(\Omega))},$$

$$M_c = \max_{i=1,2}(\|C_i\|_{L^\infty(0,t,H^1(\Omega))}), C_g = \|g\|_{L^\infty(\mathbb{R})}.$$

In this section, our main theorem is the following:

Theorem 5.1. *Assume that:*

$$NA < \nu + NM, \quad NB < \frac{\lambda}{2}, \quad NM_C < \frac{\lambda}{2}$$

and

$$16 M_C^2 C_g^2 < \min\left(\lambda, \frac{\lambda}{2\rho^2}\right), \quad k \leq \frac{1}{4}, \quad w = max(\frac{120}{7}k, \frac{304}{7}k M_C^2 C_g^2) < 1.$$

If the solution of the problem (P) admits the following regularity:

$$\begin{cases} u \in W^{2,\infty}(0,t,(L^2(\Omega))^d) \cap W^{1,\infty}(0,t,(H^1(\Omega))^d) \cap L^\infty(0,t,(H^{1+\sigma}(\Omega))^d), \\ p \in L^\infty(0,t,H^\sigma(\Omega)), \\ T, C_1, C_2 \in W^{2,\infty}(0,t,(L^2(\Omega))) \cap W^{1,\infty}(0,t,H^1(\Omega)) \cap L^\infty(0,t,H^{1+\sigma}(\Omega)), \end{cases}$$

then we have the following error estimates:

$$\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2 n^2)$$

for the velocity,

$$\|p^n - p_h^n\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2 n^2 + k^3 n^3)$$

for the pressure,

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \tfrac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2 n^2 + k^3 n^3)$$

for the temperature and

$$\|C_i^n - C_{i_h}^n\|_{0,\Omega}^2 + \tfrac{16}{7(1-w)} k(\lambda - 3\rho^2)\|\nabla(C_i^n - C_{i_h}^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2 n^2 + k^3 n^3),$$

for the concentrations (i=1,2).

In the sequel of this paper, we assume:

$$\begin{cases} u \in W^{2,\infty}(0,t,(L^2(\Omega))^d) \cap W^{1,\infty}(0,t,(H^1(\Omega))^d) \cap L^\infty(0,t,(H^{1+\sigma}(\Omega))^d), \\ \\ p \in L^\infty(0,t,H^\sigma(\Omega)), \\ \\ T, C_1, C_2 \in W^{2,\infty}(0,t,(L^2(\Omega))) \cap W^{1,\infty}(0,t,H^1(\Omega)(\Omega)) \cap L^\infty(0,t,H^{1+\sigma}(\Omega)), \end{cases}$$

To prove the a priori estimate on the pressure, we need some lemmas, first we have:

Lemma 5.1. *For all $v_h \in V_h$ and $n \in \boldsymbol{N}^*$, we have:*

$$|a_1(u_h^{n-1}, u_h^n, v_h) - a_1(u^n, u^n, v_h)| \le \big(NM\|u^n - u_h^n\|_{1,\Omega} + NA\|u^{n-1} - u_h^{n-1}\|_{1,\Omega}$$

$$+NAk\|u\|_{W^{1,\infty}(0,t,(H^1(\Omega))^2))}\big)\,|v_h|_{1,\Omega}.$$

Proof. By the triangular inequality, we have:

$$|a_1(u_h^{n-1}, u_h^n, v_h) - a_1(u^n, u^n, v_h)| \le |a_1(u_h^{n-1}, u_h^n, v_h) - a_1(u_h^{n-1}, u^n, v_h)|$$

$$+|a_1(u_h^{n-1}, u^n, v_h) - a_1(u^{n-1}, u^n, v_h| + |a_1(u^{n-1}, u^n, v_h) - a_1(u^n, u^n, v_h)|.$$

The proof of this lemma is obtained by using the three following inequalities:

$$|a_1(u_h^{n-1}, u_h^n, v_h) - a_1(u_h^{n-1}, u^n, v_h)| \le NM\|u_h^n - u^n\|_{1,\Omega}\|v_h\|_{1,\Omega},$$

$$|a_1(u_h^{n-1}, u^n, v_h) - a_1(u^{n-1}, u^n, v_h| \le NA\|u_h^{n-1} - u^{n-1}\|_{1,\Omega}\|v_h\|_{1,\Omega},$$

and

$$|a_1(u^{n-1}, u^n, v_h) - a_1(u^n, u^n, v_h)| \le NAk\|u\|_{W^{1,\infty}(0,t,(H^1(\Omega))^2))}\|v_h\|_{1,\Omega}.$$

□

Again, we have the following:

Lemma 5.2. *For all real θ_0 strictly positive independently of h and k, we have:*

$$\begin{cases} |a_1(u_h^{n-1}, u_h^n, \zeta^n) - a_1(u^n, u^n, \zeta^n)| \le & (\theta_0 NM + NA)\|u^n - u_h^n\|_{1,\Omega}^2 \\ & +NA\|u^{n-1} - u_h^{n-1}\|_{1,\Omega}^2 + O(h^{2\sigma} + k^2). \end{cases}$$

Proof. By the triangular inequality, we have:

$$|a_1(u_h^{n-1}, u_h^n, \zeta^n) - a_1(u^n, u^n, \zeta^n)| \le |a_1(u_h^{n-1}, u_h^n, \zeta^n) - a_1(u_h^{n-1}, u^n, \zeta^n)|$$

$$+|a_1(u_h^{n-1} - u^{n-1}, u^n, R_h u^n - u^n + u^n - u_h^n)| + |a_1(u^{n-1} - u^n, u^n, R_h u^n - u^n + u^n - u_h^n)|.$$

We have also:

$$\|u^n - u^{n-1}\|_{1,\Omega} \lesssim k\|u\|_{W^{1,\infty}(0,t,(H^1(\Omega))^d)},$$

$$\|u^n - R_h u^n\|_{1,\Omega} \lesssim h^\sigma \|u\|_{L^\infty(0,t,(H^{1+\sigma}(\Omega))^d)}.$$

However

$$|a_1(u_h^{n-1}, u_h^n - u^n, \zeta^n) \le NM\|u_h^n - u^n\|_{1,\Omega}\|\zeta^n\|_{1,\Omega},$$

$$|a_1(u_h^{n-1} - u^{n-1}, u^n, R_h u^n - u^n + u^n - u_h^n)| \le NA\|u_h^{n-1} - u^{n-1}\|_{1,\Omega}\|R_h u^n - u^n + u^n - u_h^n\|_{1,\Omega},$$

and

$$|a_1(u^{n-1} - u^n, u^n, R_h u^n - u^n + u^n - u_h^n)| \le NA\|u^{n-1} - u^n\|_{1,\Omega}\|R_h u^n - u^n + u^n - u_h^n\|_{1,\Omega}.$$

Therefore by using the previous lemma, the last three inequalities and Young inequality, we obtain:

$$\begin{aligned} |a_1(u_h^{n-1}, u_h^n, \zeta^n) - a_1(u^n, u^n, \zeta^n)| \ \le\ & (\theta_0 NM + NA)\|u^n - u_h^n\|_{1,\Omega}^2 \\ & +NA\|u^{n-1} - u_h^{n-1}\|_{1,\Omega}^2 + O(h^{2\sigma} + k^2). \end{aligned}$$

□

We have again the following technical estimate:

Lemma 5.3. *For n positive integer, we have:*

$$\|\partial_t u^n - \overline{\partial}_t u_h^n\|_{-1,\Omega} \lesssim \quad (h^\sigma + k) + \rho(\nu + NM)\|u^n - u_h^n\|_{1,\Omega}$$

$$+\rho NA\|u^{n-1} - u_h^{n-1}\|_{1,\Omega} + \rho^2\alpha\|T^n - T_h^n\|_{1,\Omega}.$$

Proof. First of all, we have:

$$\begin{aligned}\|\partial_t u^n - \overline{\partial}_t u_h^n\|_{-1,\Omega} \le & \|\partial_t u^n - \rho_h \partial_t u^n\|_{-1,\Omega} + \|\rho_h \partial_t u^n - \overline{\partial}_t u_h^n\|_{-1,\Omega} \\ \le & \|\partial_t u^n - \rho_h \partial_t u^n\|_{-1,\Omega} + \sup_{v\in V} \frac{(\rho_h \partial_t u^n - \overline{\partial}_t u_h^n, v)_{0,\Omega}}{\|v\|_{1,\Omega}}.\end{aligned}$$

However, for all $v_h \in V_h$, we have:

$$\begin{aligned}(\rho_h \partial_t u^n - \overline{\partial}_t u_h^n, v_h)_{0,\Omega} =& (\partial_t u^n, v_h)_{0,\Omega} - (\overline{\partial}_t u_h^n, v_h)_{0,\Omega} \\ =& -\nu d(u^n - u_h^n, v_h) + a_1(u_h^{n-1}, u_h^n, v_h) - a_1(u^n, u^n, v_h) \\ &+ b(p^n - p_h^n, v_h) + \alpha(t^n)(f(T^n) - f(T_h^n), v_h).\end{aligned}$$

On the one hand, we have:

$$b(p^n - p_h^n, v_h) = b(p^n - P_h p^n, v_h),$$

Therefore:

$$b(p^n - p_h^n, v_h) \le \|p^n - P_h p^n\|_{0,\Omega} \|v_h\|_{1,\Omega},$$

then

$$b(p^n - p_h^n, v_h) \lesssim h^\sigma \|p\|_{L^\infty(0,t,H^\sigma(\Omega))} \|v_h\|_{1,\Omega}.$$

On the other hand, by using the following estimates:

$$\nu |d(u^n - u_h^n, v_h)| \le \nu \|u^n - u_h^n\|_{1,\Omega} \|v_h\|_{1,\Omega},$$

and

$$|\alpha(t^n)(f(T^n) - f(T_h^n), v_h)| \le \rho\alpha \|T^n - T_h^n\|_{1,\Omega} \|v_h\|_{1,\Omega}.$$

By using the previous lemmas we have the following, which prove our lemma:

$$\begin{aligned}\sup_{v_h\in V_h} \frac{(\rho_h \partial_t u^n - \overline{\partial}_t u_h^n, v_h)_{0,\Omega}}{\|v_h\|_{1,\Omega}} \lesssim & \quad (h^\sigma + k) + (\nu + NM)\|u^n - u_h^n\|_{1,\Omega} \\ & + NA\|u^{n-1} - u_h^{n-1}\|_{1,\Omega} + \rho\alpha \|T^n - T_h^n\|_{1,\Omega}.\end{aligned}$$

□

Finally, we are able to give the error estimate on the pressure:

Lemma 5.4. *We have the following estimate:*

$$\begin{aligned}\|p^n - p_h^n\|_{0,\Omega} \lesssim & \quad (h^\sigma + k) + \frac{1}{\beta}(1+\rho)(\nu + NM)\|\nabla(u^n - u_h^n)\|_{0,\Omega} \\ & + \frac{1+\rho}{\beta} NA \|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega} + \frac{\alpha\rho}{\beta}(1+\rho)\|\nabla(T^n - T_h^n)\|_{0,\Omega}.\end{aligned}$$

Proof. First of all, we have:

$$\begin{aligned}\|p^n - p_h^n\|_{0,\Omega} \le & \|p^n - P_h p^n\|_{0,\Omega} + \|P_h p^n - p_h^n\|_{0,\Omega} \\ \lesssim & h^\sigma \|p\|_{L^\infty(H^\sigma)} + \beta^{-1} \sup_{v_h\in X_h} \frac{b(P_h p^n - p_h^n, v_h)}{\|\nabla v_h\|_{0,\Omega}} \\ \lesssim & h^\sigma \|p\|_{L^\infty(H^\sigma)} + \beta^{-1} \sup_{v_h\in X_h} \frac{b(p^n - p_h^n, v_h)}{\|\nabla v_h\|_{0,\Omega}}.\end{aligned}$$

We look for finding estimate on $b(p^n - p_h^n, v_h)$
However, we have:

$$b(p^n - p_h^n, v_h) = b(p^n, v_h) - b(p_h^n, v_h),$$

$$b(p^n, v_h) = \nu d(u^n, v_h) + a_1(u^n, u^n, v_h) - \alpha(t^n)(f(T_h^n), v_h) + (\partial_t u^n, v_h)$$

and

$$b(p_h^n, v_h) = \nu d(u_h^n, v_h) + a_1(u_h^{n-1}, u_h^n, v_h) - \alpha(t^n)(f(T_h^n), v_h) + (\overline{\partial}_t u_h^n, v_h).$$

From where:

$$\begin{aligned}b(p^n - p_h^n, v_h) =& (\partial_t u^n - \overline{\partial}_t u_h^n, v_h) + \nu d(u^n - u_h^n, v_h) \\ &+ (a_1(u^n, u^n, v_h) - a_1(u_h^{n-1}, u_h^n, v_h)) \\ &- \alpha(t^n)(f(T^n) - f(T_h^n), v_h).\end{aligned}$$

We obtain then:

$$b(p^n - p_h^n, v_h) \lesssim \|\partial_t u^n - \overline{\partial}_t u_h^n\|_{-1,\Omega}\|\nabla v_h\|_{0,\Omega} + (\nu + NM)\|\nabla(u^n - u_h^n)\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega}$$

$$+ k\|\nabla v_h\|_{0,\Omega} + NA\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega}$$

$$+ \alpha\rho\|\nabla(T^n - T_h^n)\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega},$$

therefore:

$$b(p^n - p_h^n, v_h) \lesssim ((h^\sigma + k) + (1+\rho)(\nu + NM)\|\nabla(u^n - u_h^n)_{0,\Omega}\|$$

$$+(1+\rho)(NA)\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega} + (1+\rho)\alpha\rho\|\nabla(T^n - T_h^n)\|_{0,\Omega}\big)\, \|\nabla v_h\|_{0,\Omega}.$$

From where, we deduce:

$$\|p^n - p_h^n\|_{0,\Omega} \lesssim (h^\sigma + k) + \frac{1}{\beta}(1+\rho)(\nu + NM)\|\nabla(u^n - u_h^n)\|_{0,\Omega}$$
$$+ \frac{1+\rho}{\beta}NA\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega} + \frac{\alpha\rho}{\beta}(1+\rho)\|\nabla(T^n - T_h^n)\|_{0,\Omega}.$$

□

Lemma 5.5. *We have the following estimate:*

$$\|\overline{\partial}_t R_h u^n - \partial_t u^n\|_{0,\Omega} \lesssim k\|u\|_{W^{2,\infty}(0,t,(L^2(\Omega))^d)} + h^\sigma\|u\|_{W^{1,\infty}(0,t,(H_0^1(\Omega))^d)}.$$

Proof. by triangular inequality we have:

$$\|\overline{\partial}_t R_h u^n - \partial_t u^n\|_{0,\Omega} \le \|\overline{\partial}_t R_h u^n - \overline{\partial}_t u^n\|_{0,\Omega} + \|\overline{\partial}_t u^n - \partial_t u^n\|_{0,\Omega}.$$

By using Taylor-Young formula with integral remainder, we obtain:

$$\|\overline{\partial}_t R_h u^n - \overline{\partial}_t u^n\|_{0,\Omega} + \|\overline{\partial}_t u^n - \partial_t u^n\|_{0,\Omega} = \|\frac{1}{k}\int_{t^n-k}^{t^n}(R_h\partial_t u - \partial_t u)ds\|_{0,\Omega}$$
$$+\|\int_{t^n-k}^{t^n}(s - t^{n-1})\frac{\partial^2}{\partial t^2}u ds\|_{0,\Omega}.$$

Therefore:

$$\|\overline{\partial}_t R_h u^n - \partial_t u^n\|_{0,\Omega} \lesssim h^\sigma\|u\|_{W^{1,\infty}(0,t,H_0^1(\Omega))^d)} + k\|u\|_{W^{2,\infty}(0,t,L^2(\Omega))^d)}.$$

□

Lemma 5.6. *Assume that $NA < \nu + NM$, and $k \le \frac{1}{4}$. Then:*

$$\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2) + k\sum_{i=1}^{n}\|\nabla(T^i - T_i^n)\|_{0,\Omega}^2.$$

Proof. For the proof of this lemma, we recall the following lemma, known as the *Gronwal* lemma:

Lemma 5.7. *Let a_n, b_n and c_n three positive sequences, c_n not decreasing sequence, Assume that*

$$a_n + b_n \le c_n + \overline{\nu}\sum_{i=0}^{n-1}a_i, \quad \overline{\nu} > 0$$

and

$$a_0 + b_0 \le c_0.$$

We have the following result:

$$a_n \le c_n \exp(\overline{\nu}n) \quad .$$

For $\xi^n = R_h u^n - u_h^n \in X_h$, we have:

$$(\overline{\partial}_t\xi^n, \xi^n) + \nu d(u^n - u_h^n, u^n - u_h^n) = \nu d(u^n - u_h^n, u^n - u_h^n) + (\overline{\partial}_t R_h u^n, \xi^n) - (\overline{\partial}_t u_h^n, \xi^n).$$

However, if we choose ξ^n as test function in the third equation of (P_h^n), we obtain the following equation:

$$(\overline{\partial}_t u_h^n, \xi^n) + \nu d(u_h^n, \xi^n) + a_1(u_h^{n-1}, u_h^n, \xi^n) - b(p_h^n, \xi^n) = \alpha(t^n)(f(T^n), \xi^n).$$

By replacing the value of $(\overline{\partial}_t u_h^n, \xi^n)$, we obtain:

$$(\overline{\partial}_t \xi^n, \xi^n) + \nu d(u^n - u_h^n, u^n - u_h^n) = \nu d(u^n - u_h^n, u^n - u_h^n) + (\overline{\partial}_t R_h u^n, \xi^n) + \nu d(u_h^n, \xi^n)$$

$$+ a_1(u_h^{n-1}, u_h^n, \xi^n) - b(p_h^n, \xi^n) - \alpha(t^n)(f(T_h^n), \xi^n),$$

therefore:

$$(\overline{\partial}_t \xi^n, \xi^n) + \nu d(u^n - u_h^n, u^n - u_h^n) = \nu d(u^n - u_h^n, u^n - u_h^n) + (\overline{\partial}_t R_h u^n - \partial_t u^n, \xi^n)$$

$$+\nu d(u_h^n, \xi^n) + a_1(u_h^{n-1}, u_h^n, \xi^n) - b(p_h^n, \xi^n) - \alpha(t^n)(f(T_h^n), \xi^n)$$

$$-\nu d(u^n, \xi^n) - a_1(u^n, u^n, \xi^n) + b(p^n, \xi^n) + \alpha(t^n)(f(T^n), \xi^n),$$

because:

$$(\partial_t u^n, \xi^n) = -\nu d(u^n, \xi^n) - a_1(u^n, u^n, \xi^n) + b(p^n, \xi^n) + \alpha(t^n)(f(T^n), \xi^n).$$

Therefore:

$$(\overline{\partial}_t \xi^n, \xi^n) + \nu d(u^n - u_h^n, u^n - u_h^n) = (\overline{\partial}_t R_h u^n - \partial_t u^n, \xi^n) + \alpha(t^n)(f(T^n) - f(T_h^n), \xi^n)$$

$$+(a_1(u_h^{n-1}, u_h^n, \xi^n) - a_1(u^n, u^n, \xi^n)) + (b(p^n, \xi^n) - b(p_h^n, \xi^n))$$

$$+\nu \left(d(u^n - u_h^n, u^n - u_h^n) + d(u_h^n, \xi^n) - d(u^n, \xi^n)\right).$$

we have:

$$\begin{aligned} b(p^n, \xi^n) - b(p_h^n, \xi^n)) = & \quad b(p^n, R_h u^n - u_h^n) - b(p_h^n, R_h u^n - u_h^n) \\ = & \quad b(p^n - p_h^n, R_h u^n - u^n + u^n - u_h^n) \\ = & \quad b(p^n - p_h^n, R_h u^n - u^n) + b(p^n - P_h p^n, u^n - u_h^n) \\ & \quad +b(P_h p^n - p_h^n, u^n - u_h^n). \end{aligned}$$

Since $P_h p^n - p_h^n \in M_h$, we have the following equality:

$$b(P_h p^n - p_h^n, u^n - u_h^n) = 0.$$

We have also:

$$d(u^n - u_h^n, u^n - u_h^n) + d(u_h^n, \xi^n) - d(u^n, \xi^n) = d(u^n - u_h^n, u^n - u_h^n - (R_h u^n - u_h^n)),$$

then

$$d(u^n - u_h^n, u^n - u_h^n) + d(u_h^n, \xi^n) - d(u^n, \xi^n) = d(u^n - u_h^n, u^n - R_h u^n).$$

Finally, we have the following identity:

$$\begin{aligned} (\overline{\partial}_t \xi^n, \xi^n) + \nu \|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 = & \quad \frac{1}{2k}\left(\|\xi^n\|_{0,\Omega}^2 - \|\xi^{n-1}\|_{0,\Omega}^2 + \|\xi^n - \xi^{n-1}\|_{0,\Omega}^2\right) \\ & \quad +\nu \|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \end{aligned}$$

By using the previous lemmas, we have for all θ_1, $\theta_1 > 0$ independents of h and k :

$$\begin{aligned} (\overline{\partial}_t \xi^n, \xi^n) + \nu \|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim & \left(\|\nabla(u^n - R_h u^n)\|_{0,\Omega}^2 + \|\overline{\partial}_t R_h u^n - \partial_t u^n\|_{0,\Omega}^2 + \|p^n - P_h p^n\|_{1,\Omega}^2\right) \\ & + \|\xi^n\|_{0,\Omega}^2 + \theta_1 \|p^n - p_h^n\|_{1,\Omega}^2 + \frac{\alpha^2}{2}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ & + |a_1(u_h^{n-1}, u_h^n, \xi^n) - a_1(u^n, u^n, \xi^n)| + \theta_2 \|\nabla(u^n - u_h^n)\|_{0,\Omega}^2. \end{aligned}$$

By using the previous lemmas, we obtain:

$$\begin{aligned}\frac{1}{2k}(\|\xi^n\|_{0,\Omega}^2 - \|\xi^{n-1}\|_{0,\Omega}^2 + \|\xi^n - \xi^{n-1}\|_{0,\Omega}^2) + \nu\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \\ \lesssim (h^{2\sigma} + k^2) + \|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \|\xi^n\|_{0,\Omega}^2 \\ + (NA + \theta_0 NM + \theta_2 + \frac{4\theta_1}{\beta^2}(1+\rho)^2(\nu + NM)^2)\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \\ + \left(NA + \frac{4\theta_1}{\beta^2}(1+\rho)^2 N^2 A^2\right)\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2\end{aligned}$$

$$\begin{aligned}\frac{1}{2k}(\|\xi^n\|_{0,\Omega}^2 - \|\xi^{n-1}\|_{0,\Omega}^2 + \|\xi^n - \xi^{n-1}\|_{0,\Omega}^2) + \nu\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \\ \lesssim (h^{2\sigma} + k^2) + \|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \|\xi^n\|_{0,\Omega}^2 \\ + (NA + \theta_0 NM + \theta_2 + \frac{4\theta_1}{\beta^2}(1+\rho)^2(\nu + NM)^2)\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \\ + (NA + \frac{4\theta_1}{\beta^2}(1+\rho)^2 N^2 A^2)\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2\end{aligned}$$

Using the hypothesis, $NA < \nu + MN$, then for

$$\theta_0 = \theta_2 = \frac{\nu}{8(1+NM)},$$

and

$$\theta_1 = \nu\beta^2\left(32(1+\rho^2)(\nu + NM + NA)(\nu + NM - NA)\right)^{-1},$$

we have:

$$NA + \frac{4\theta_1}{\beta^2}(1+\rho)^2 N^2 A^2 \leq \frac{\nu}{4}$$

and

$$NA + \theta_0 NM + \theta_2 + \frac{4\theta_1}{\beta^2}(1+\rho)^2(\nu + NM)^2 \leq \frac{\nu}{2}.$$

Therefore, we have:

$$\begin{aligned}\|\xi^n\|_{0,\Omega}^2 + \nu k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim \|\xi^{n-1}\|_{0,\Omega}^2 + 2k\|\xi^n\|_{0,\Omega}^2 + k\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ + \frac{\nu k}{2}\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2 + k(h^{2\sigma} + k^2).\end{aligned}$$

By summing over the time, we have:

$$\begin{aligned}\|\xi^n\|_{0,\Omega}^2 + \nu k\sum_{i=1}^{n}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 \quad &\lesssim \quad 2k\sum_{i=1}^{n}\|\xi^n\|_{0,\Omega}^2 + k\sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|_{0,\Omega}^2 \\ &+ \frac{\nu k}{2}\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 + kn(h^{2\sigma} + k^2).\end{aligned}$$

If $2k \leq \frac{1}{2}$, we obtain:

$$\begin{aligned}\|\xi^n\|_{0,\Omega}^2 + 2\nu k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \quad &\lesssim \quad kn(h^{2\sigma} + k^2) \\ &+ 4k\sum_{i=1}^{n-1}\|\xi^i\|_{0,\Omega}^2 + k\sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|_{0,\Omega}^2\end{aligned}$$

Finally, by using *Gronwal* lemma with:

$$a_n = \|\xi^n\|_{0,\Omega}^2, \quad b_n = 2\nu k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2,$$

$$c_n = kn(h^{2\sigma} + k^2) + k\sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|_{0,\Omega}^2,$$

we obtain:

$$\|\xi^n\|_{0,\Omega}^2 + 2\nu k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim \left(kn(h^{2\sigma} + k^2) + k\sum_{i=1}^{n} \|\nabla(T^i - T_h^i)\|_{0,\Omega}^2 \right) \exp(4kn),$$

that we can rewrite it in the form:

$$\begin{aligned}\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 &\leq \|u^n - R_h u^n\|_{0,\Omega}^2 + \|\xi^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \\ &\lesssim kn(h^{2\sigma} + k^2) + k\sum_{i=1}^{n} \|\nabla(T^i - T_h^i)\|_{0,\Omega}^2.\end{aligned}$$

. □

Now for temperature estimation we need the following lemmas.

Lemma 5.8. *For all constants θ_3, θ_4 independents of k and h, we have:*

$$\begin{aligned}|a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u^n, T^n, \eta^n)| \lesssim &(h^{2\sigma} + k^2) + (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ &+ (\theta_4 NB + NB/2)\|u^{n-1} - u_h^{n-1}\|_{0,\Omega}^2.\end{aligned}$$

Proof. We set $\eta^n = r_h T^n - T_h^n$. By the triangular inequality, we have:

$$|a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u^n, T^n, \eta^n)| \leq |a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u_h^{n-1}, T^n, \eta^n)|$$

$$+ |a_1(u_h^{n-1}, T^n, \eta^n) - a_1(u^{n-1}, T^n, \eta^n)| + |a_1(u^{n-1}, T^n, \eta^n) - a_1(u^n, T^n, \eta^n)|$$

$$\leq NM\|\nabla(T^n - T_h^n)\|_{0,\Omega}\|\nabla(T_h^n - r_h T^n)\|_{0,\Omega} + NB\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}\|\nabla(T^n - r_h T^n)\|_{0,\Omega}$$

$$+ NB\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}\|\nabla(T^n - T_h^n)\|_{0,\Omega} + NB\|\nabla(u^n - u^{n-1})\|_{0,\Omega}\|\nabla(T^n - r_h T^n)\|_{0,\Omega}$$

$$+ NB\|\nabla(u^n - u^{n-1})\|_{0,\Omega}\|\nabla(T^n - T_h^n)\|_{0,\Omega},$$

then

$$|a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u^n, T^n, \eta^n)| \leq (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \|\nabla(T^n - r_h T^n)\|_{0,\Omega}^2$$

$$(\theta_4 NB + NB/2)\|\nabla(u_h^{n-1} - u^{n-1})\|_{0,\Omega}^2 + \|\nabla(u^n - u^{n-1})\|_{0,\Omega}^2$$

$$\lesssim (h^{2\sigma} + k^2) + (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + (\theta_4 NB + NB/2)\|u^{n-1} - u_h^{n-1}\|_{0,\Omega}^2.$$

□

Lemma 5.9. *We have the following a priori estimate:*

$$\|\overline{\partial}_t r_h T^n - \partial_t T^n\|_0 \lesssim (h^\sigma + k).$$

Proof. By applying the development of Taylor with remainder integral, we obtain:

$$\begin{aligned}\|\overline{\partial}_t r_h T^n - \partial_t T^n\|_{0,\Omega} &\leq \|\overline{\partial}_t r_h T^n - \overline{\partial}_t T^n\|_{0,\Omega} + \|\overline{\partial}_t T^n - \partial_t T^n\|_{0,\Omega} \\ &= \|\frac{1}{k}\int_{t_{n-1}}^{t_n} (r_h \partial_t T(s) - \partial_t T(s))\, ds\|_{0,\Omega} \\ &\quad + \|\int_{t_{n-1}}^{t_n} \left(s - t^{n-1}\right) \partial_{t^2}^2 T(s) ds\|_{0,\Omega} \\ &\lesssim \left(h^\sigma \|T\|_{W^{1,\infty}(H^\sigma)} + k\|T\|_{W^{2,\infty}(L^2)}\right) \\ &\lesssim (h^\sigma + k).\end{aligned}$$

□

Lemma 5.10. *For all constants $\theta_3, \theta_4, \theta_5, \theta_6, \theta_7$, independents of h and k, we have:*

$$\frac{1}{2k}(\|\eta^n\|_{0,\Omega}^2 - \|\eta^{n-1}\|_{0,\Omega}^2 + \|\eta^n - \eta^{n-1}\|_{0,\Omega}^2) + \lambda\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)$$

$$+ (\theta_5 + \theta_6 + \theta_7 + \theta_8)\|\eta^n\|_{0,\Omega}^2 + (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2$$

$$+ \rho^2 \frac{M_C^2 C_g^2}{\theta_8}\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}^2 + \frac{\rho^2}{4\theta_6}\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2$$

$$+ \frac{\rho^2}{4\theta_7}\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 + (\theta_4 NB + \frac{NB}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2.$$

Proof. We set $\eta^n = r_h T^n - T_h^n$. First of all, we have the two following equalities:

$$(\overline{\partial}_t \eta^n, \eta^n) + \lambda d(T^n - T_h^n, T^n - T_h^n) = (\overline{\partial}_t r_h T^n - \overline{\partial}_t T_h^n, \eta^n) + \lambda d(T^n - T_h^n, T^n - T_h^n),$$

and

$$(\overline{\partial}_t T_h^n, \eta^n) = -\lambda d(T_h^n, \eta^n) - a_1(u_h^{n-1}, T_h^n, \eta^n) + K_1(C_{1_h}^n, T_h^{n-1}, \eta^n) + K_2(C_{2_h}^n, T_h^{n-1}, \eta^n).$$

Therefore, we have:

$$(\overline{\partial}_t \eta^n, \eta^n) + \lambda d(T^n - T_h^n, T^n - T_h^n) = (\overline{\partial}_t r_h T^n, \eta^n) + \lambda d(T_h^n, \eta^n) + \lambda d(T^n - T_h^n, T^n - T_h^n)$$
$$+ a_1(u_h^{n-1}, T_h^n, \eta^n) - K_1(C_{1_h}^n, T_h^{n-1}, \eta^n) - K(C_{2_h}^n, T_h^{n-1}, \eta^n)$$

then:

$$(\overline{\partial}_t \eta^n, \eta^n) + \lambda d(T^n - T_h^n, T^n - T_h^n) = (\overline{\partial}_t r_h T^n - \partial_t T^n, \eta^n) + \lambda d(T_h^n - T^n, \eta^n)$$

$$+ \lambda d(T^n - T_h^n, T^n - T_h^n) + a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u^n, T^n, \eta^n)$$

$$+ K_1(C_1^n, T^n, \eta^n) - K(C_{1_h}^n, T_h^{n-1}, \eta^n) + K_2(C_2^n, T^n, \eta^n) - K_2(C_{2_h}^n, T_h^{n-1}, \eta^n).$$

However:

$$\begin{cases} d(T^n - T_h^n, T^n - T_h^n) + d(T_h^n - T^n, \eta^n) = d(T^n - T_h^n, T^n - r_h T^n) \\ \qquad = d(T^n - r_h T^n, T^n - r_h T^n) + d(r_h T^n - T_h^n, T^n - r_h T^n) \end{cases}$$

by using the identity:

$$a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2),$$

we have:

$$\frac{1}{2k}(\|\eta^n\|_{0,\Omega}^2 - \|\eta^{n-1}\|_{0,\Omega}^2 + \|\eta^n - \eta^{n-1}\|_{0,\Omega}^2) + \lambda\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2$$

$$= (\overline{\partial}_t r_h T^n - \partial_t u^n, \eta^n) + \lambda\|\nabla(T^n - r_h T^n)\|_{0,\Omega}^2$$

$$+ a_1(u_h^{n-1}, T_h^n, \eta^n) - a_1(u^n, T^n, \eta^n)$$

$$+ K_1(C_1^n, T^n, \eta^n) - K_1(C_{1_h}^n, T_h^{n-1}, \eta^n)$$

$$+ K_2(C_2^n, T^n, \eta^n) - K_2(C_{2_h}^n, T_h^{n-1}, \eta^n).$$

We have also:

$$\begin{aligned} K_1(C_1^n, T^n, \eta^n) - K_1(C_{1_h}^n, T_h^{n-1}, \eta^n) &= K(C_1^n, T^n, \eta^n) - K_1(C_1^n, T^{n-1}, \eta^n) \\ &\quad + K_1(C_1^n, T^{n-1}, \eta^n) - K_1(C_{1_h}^n, T^{n-1}, \eta^n) \\ &\quad + K_1(C_{1_h}^n, T^{n-1}, \eta^n) - K_1(C_{1_h}^n, T_h^{n-1}, \eta^n) \\ &\lesssim k\|T\|_{W^{1,\infty}}\|\eta^n\|_{0,\Omega} + \rho\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}\|\eta^n\|_{0,\Omega} \\ &\quad + \rho C_g M_C\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}\|\eta^n\|_{0,\Omega}, \end{aligned}$$

we have same inequality for the second concentration. We have also:

$$\begin{aligned} &\frac{1}{2k}(\|\eta^n\|_{0,\Omega}^2 - \|\eta^{n-1}\|_{0,\Omega}^2 + \|\eta^n - \eta^{n-1}\|_{0,\Omega}^2) + \lambda\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ &\lesssim (h^\sigma + k)\|\eta^n\|_{0,\Omega} + \lambda\|\nabla(T^n - r_h T^n)\|_{0,\Omega}^2 + \\ &(h^{2\sigma} + k^2) + (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ &+ (\theta_4 NB + \frac{NB}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2 \\ &+ 2k\|\eta^n\|_{0,\Omega} + \rho\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}\|\eta^n\|_{0,\Omega} + \rho\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}\|\eta^n\|_{0,\Omega} \\ &+ 2\rho M_C C_g\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}\|\eta^n\|_{0,\Omega}. \end{aligned}$$

Therefore:

$$\begin{aligned} &\frac{1}{2k}(\|\eta^n\|_{0,\Omega}^2 - \|\eta^{n-1}\|_{0,\Omega}^2 + \|\eta^n - \eta^{n-1}\|_{0,\Omega}^2) + \lambda\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ &\lesssim (h^{2\sigma} + k^2) + (\theta_5 + \theta_6 + \theta_7 + \theta_8)\|\eta^n\|_{0,\Omega}^2 + (\theta_3 NM + NB)\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \\ &+ \rho^2\frac{M_C^2 C_g^2}{\theta_8}\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}^2 + \frac{\rho^2}{4\theta_6}\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{\rho^2}{4\theta_7}\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \\ &+ (\theta_4 NB + \frac{NB}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2. \end{aligned}$$

□

Lemma 5.11. *We assume that $NB < \frac{\lambda}{2}$, $6\rho^2 M_C^2 C_g^2 \leq \lambda$ and ($k \leq \frac{1}{4}$). Then we have:*

$$\begin{aligned} \|T^n - T_h^n\|_{0,\Omega}^2 + \lambda k\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 &\lesssim kn(h^{2\sigma} + k^2) + 2k(\theta_4 NB + \frac{NB}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 \\ &+ 8\rho^2 k\sum_{i=1}^{n}\|\nabla(C_1^i - C_{1_h}^i)\|_{0,\Omega}^2 + 8\rho^2 k\sum_{i=1}^{n}\|\nabla(C_2^i - C_{2_h}^i)\|_{0,\Omega}^2. \end{aligned}$$

Proof. By choosing $\theta_5 = \theta_6 = \theta_7 = \theta_8 = \frac{1}{16}$ and $\theta_3 = \frac{\lambda - 2NB}{2NM}$, we have:

$$\begin{aligned} \frac{1}{2k}(\|\eta^n\|_{0,\Omega}^2 - \|\eta^{n-1}\|_{0,\Omega}^2 + \|\eta^n - \eta^{n-1}\|_{0,\Omega}^2) + \frac{\lambda}{2}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim \frac{1}{4}\|\eta^n\|_{0,\Omega}^2 \\ + (h^{2\sigma} + k^2) + 16\rho^2 M_C^2 C_g^2\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}^2 + 4\rho^2\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ + 4\rho^2\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 + (\theta_4 NB + \frac{NB}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2, \end{aligned}$$

by summing over the time, we have:

$$\frac{1}{2k}\left(\|\eta^n\|_{0,\Omega}^2\right)+\frac{\lambda}{2}\sum_{i=1}^{n}\|\nabla(T^i-T_h^i)\|_{0,\Omega}^2 \lesssim \frac{1}{4}\sum_{i=1}^{n}\|\eta^i\|_{0,\Omega}^2+n(h^{2\sigma}+k^2)$$
$$+16\rho^2M_C^2C_g^2\sum_{i=1}^{n-1}\|\nabla(T^i-T_h^i)\|_{0,\Omega}^2+4\rho^2\sum_{i=1}^{n}\|\nabla(C_1^i-C_{1_h}^i)\|_{0,\Omega}^2$$
$$+4\rho^2\sum_{i=1}^{n}\|\nabla(C_2^i-C_{2_h}^i)\|_{0,\Omega}^2+(\theta_4NB+\frac{NB}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|_{0,\Omega}^2.$$

However:

$$16\rho^2M_C^2C_g^2\le\frac{\lambda}{2}\ \text{ and }\ k\le\frac{1}{4}.$$

We deduce that:

$$\frac{7}{8}\left(\|\eta^n\|_{0,\Omega}^2\right)+\lambda k\|\nabla(T^n-T_h^n)\|_{0,\Omega}^2 \lesssim \frac{k}{2}\sum_{i=1}^{n-1}\|\eta^i\|_{0,\Omega}^2+kn(h^{2\sigma}+k^2)+8\rho^2k\sum_{i=1}^{n}\|\nabla(C_1^i-C_{1_h}^i)\|_{0,\Omega}^2$$
$$+8\rho^2k\sum_{i=1}^{n}\|\nabla(C_2^i-C_{2_h}^i)\|_{0,\Omega}^2+2k(\theta_4NB+\frac{NB}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|_{0,\Omega}^2.$$

By using the discret *Gronwal* lemma, we obtain:

$$\|\eta^n\|_{0,\Omega}^2+k\lambda\|\nabla(T^n-T_h^n)\|_{0,\Omega}^2 \lesssim \Bigg(kn(h^{2\sigma}+k^2)+2k(\theta_4NB+\frac{NB}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|_{0,\Omega}^2$$
$$+8\rho^2k\sum_{i=1}^{n}\|\nabla(C_1^i-C_{1_h}^i)\|_{0,\Omega}^2+8\rho^2k\sum_{i=1}^{n}\|\nabla(C_2^i-C_{2_h}^i)\|_{0,\Omega}^2\Bigg)$$
$$\times\exp(kn).$$

The result comes from the triangular inequality:

$$\begin{aligned}\|T^n-T_h^n\|_{0,\Omega}^2+k\lambda\|\nabla(T^n-T_h^n)\|_{0,\Omega}^2 &\le \|T^n-r_hT^n\|_{0,\Omega}^2+\|\eta^n\|_{0,\Omega}^2+k\lambda\|\nabla(T^n-T_h^n)\|_{0,\Omega}^2\\ &\lesssim h^\sigma\|T\|_{L^\infty(0,t,H^\sigma)}+\|\eta^n\|_{0,\Omega}^2+k\lambda\|\nabla(T^n-T_h^n)\|_{0,\Omega}^2.\end{aligned}$$

□

Lemma 5.12. *For all constants $\theta_8,\theta_9,\theta_{11}$ and θ_{12} independents of h and k, we have:*

$$\left\{\begin{aligned}|a_1(u_h^{n-1},C_{1_h}^n,\epsilon_1^n)-a_1(u^n,C_1^n,\epsilon_1^n)| &\lesssim (h^{2\sigma}+k^2)+(\theta_9NM+NM_C)\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}^2\\ &\quad+(\theta_{10}NM_C+NM_C/2)\|\nabla(u^{n-1}-u_h^{n-1})\|_{0,\Omega}^2.\end{aligned}\right.$$

$$\left\{\begin{aligned}|a_1(u_h^{n-1},C_{2_h}^n,\epsilon_2^n)-a_1(u^n,C_2^n,\epsilon_2^n)| &\lesssim (h^{2\sigma}+k^2)+(\theta_{11}NM+NM_C)\|\nabla(C_2^n-C_{2_h}^n)\|_{0,\Omega}^2\\ &\quad+(\theta_{12}NM_C+NM_C/2)\|\nabla(u^{n-1}-u_h^{n-1})\|_{0,\Omega}^2.\end{aligned}\right.$$

Proof. We set $\epsilon_1^n=r_hC_1^n-C_{1_h}^n$. By using the previous lemmas and the Young inequalities, we have:

$$|a_1(u_h^{n-1},C_{1_h}^n,\epsilon_1^n)-a_1(u^n,C_1^n,\epsilon_1^n)|\le|a_1(u_h^{n-1},C_{1_h}^n,\epsilon_1^n)-a_1(u_h^{n-1},C_1^n,\epsilon_1^n)|$$
$$+|a_1(u_h^{n-1},C_1^n,\epsilon_1^n)-a_1(u^{n-1},C_1^n,\epsilon_1^n)|+|a_1(u^{n-1},C_1^n,\epsilon_1^n)-a_1(u^n,C_1^n,\epsilon_1^n)|$$
$$\le NM\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}\|\nabla(C_{1_h}^n-r_hC_1^n)\|_{0,\Omega}+NM_C|\nabla(u^{n-1}-u_h^{n-1})\|_{0,\Omega}\|\nabla(C_1^n-r_hC_1^n)\|_{0,\Omega}$$
$$+NM_C\|\nabla(u^{n-1}-u_h^{n-1})\|_{0,\Omega}\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}+NM_C\|\nabla(u^n-u^{n-1})\|_{0,\Omega}\|\nabla(C_1^n-r_hC_1^n)\|_{0,\Omega}$$
$$+NM_C\|\nabla(u^n-u^{n-1})\|_{0,\Omega}\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}$$
$$\le(\theta_8NM+NM_C)\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}^2+C_1\|\nabla(C_1^n-r_hC_1^n)\|_{0,\Omega}^2$$
$$(\theta_9NM_C+NM_C/2)\|\nabla(u_h^{n-1}-u^{n-1})\|_{0,\Omega}^2+C_1\|\nabla(u^n-u^{n-1})\|_{0,\Omega}^2$$
$$\lesssim(h^{2\sigma}+k^2)+(\theta_8NM+NM_C)\|\nabla(C_1^n-C_{1_h}^n)\|_{0,\Omega}^2+(\theta_9NM_C+NM_C/2)\|u^{n-1}-u_h^{n-1}\|_{0,\Omega}^2.$$

The proof of the second inequality of the lemma is similar to the proof of the first one. □

We have the following lemma:

Lemma 5.13. *We have the following estimate:*

$$\|\partial_t r_h C_i^n - \partial_t C_i^n\|_{0,\Omega} \lesssim (h^\sigma + k). \quad (i = 1, 2)$$

Proof. By applying the Taylor development with remainder integral, we obtain:

$$\begin{aligned}
\|\overline{\partial}_t r_h C_i^n - \partial_t C_i^n\|_{0,\Omega} &\leq \|\overline{\partial}_t r_h C_i^n - \overline{\partial}_t C_i^n\|_{0,\Omega} + \|\overline{\partial}_t C_i^n - \partial_t C_i^n\|_{0,\Omega} \\
&= \|\frac{1}{k}\int_{t_{n-1}}^{t_n} (r_h \partial_t C_i(s) - \partial_t C_i(s))\, ds\|_{0,\Omega} \\
&\quad + \|\int_{t_{n-1}}^{t_n} \left(s - t^{n-1}\right) \partial_{t^2}^2 C_i(s) ds\|_{0,\Omega} \\
&\lesssim \left(h^\sigma \|C_i\|_{W^{1,\infty}(H^\sigma)} + k\|C_i\|_{W^{2,\infty}(L^2)}\right) \\
&\lesssim (h^\sigma + k).
\end{aligned}$$

□

we have the same lemma for the second concentration.

Lemma 5.14. *For all constants $\theta_9, \theta_{10}, \theta_{13}, \theta_{14}, \theta_{15}$, independents of h and k, we have:*

$$\begin{aligned}
&\frac{1}{2k}\left(\|\epsilon^n\|_{0,\Omega}^2 - \|\epsilon^{n-1}\|_{0,\Omega}^2 + \|\epsilon^n - \epsilon^{n-1}\|_{0,\Omega}^2\right) + \lambda\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2) \\
&\quad + (\theta_{13} + \theta_{14} + \theta_{15})\|\epsilon^n\|_{0,\Omega}^2 + (\theta_9 NM + NM_C)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \rho^2\frac{M_C^2 C_g^2}{4\theta_7}\|\nabla(T^n - T_h^{n-1})\|_{0,\Omega}^2 \\
&\quad + \frac{\rho^2}{4\theta_6}\|\nabla(C_1^n - C_{1_h}^{n-1})\|_{0,\Omega}^2 (\theta_{10} NM_C + \frac{NM_C}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2.
\end{aligned}$$

Proof. We set $\epsilon^n = r_h C_1^n - C_{1_h}^n$. First of all, we have:

$$(\overline{\partial}_t \epsilon^n, \epsilon^n) + \lambda d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n) = (\overline{\partial}_t r_h C_1^n - \overline{\partial}_t C_{1_h}^n, \epsilon^n) + \lambda d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n),$$

however

$$(\overline{\partial}_t C_{1_h}^n, \epsilon^n) = -\lambda d(C_{1_h}^n, \epsilon^n) - a_1(u_h^{n-1}, C_{1_h}^n, \epsilon^n) - K(C_{1_h}^n, T_h^{n-1}, \epsilon^n).$$

Therefore:

$$\begin{aligned}
(\overline{\partial}_t \epsilon^n, \epsilon^n) + \lambda d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n) &= (\overline{\partial}_t r_h C_1^n, \epsilon^n) + \lambda d(C_{1_h}^n, \epsilon^n) + \lambda d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n) \\
&\quad + a_1(u_h^{n-1}, C_{1_h}^n, \epsilon^n) + K(C_{1_h}^n, T_h^{n-1}, \epsilon^n) \\
&= (\overline{\partial}_t r_h C_1^n - \partial_t C_1^n, \epsilon^n) + \lambda d(C_{1_h}^n - C_1^n, \epsilon^n) + \lambda d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n) \\
&\quad + a_1(u_h^{n-1}, C_{1_h}^n, \epsilon^n) - a_1(u^n, C_1^n, \epsilon^n) - K(C_1^n, T^n, \epsilon^n) + K(C_{1_h}^n, T_h^{n-1}, \epsilon^n).
\end{aligned}$$

However:

$$\begin{aligned}
d(C_1^n - C_{1_h}^n, C_1^n - C_{1_h}^n) + d(C_{1_h}^n - C_1^n, \epsilon^n) &= d(C_1^n - C_{1_h}^n, C_1^n - r_h C_1^n) \\
&= d(C_1^n - r_h C_1^n, C_1^n - r_h C_1^n) + d(r_h C_1^n - C_{1_h}^n, C_1^n - r_h C_1^n)
\end{aligned}$$

and

$$a(a - b) = \frac{1}{2}(a^2 - b^2 + (a - b)^2).$$

Therefore, we have:

$$\begin{aligned}
\tfrac{1}{2k}\left(\|\epsilon^n\|^2 - \|\epsilon^{n-1}\|^2 + \|\epsilon^n - \epsilon^{n-1}\|^2\right) + \lambda\|\nabla(C_1^n - C_{1_h}^n)\|^2 &= (\overline{\partial}_t r_h C_1^n - \partial_t u^n, \epsilon^n) \\
+ \lambda\|\nabla(C_1^n - r_h C_1^n)\|^2 + a_1(u_h^{n-1}, C_{1_h}^n, \epsilon^n) &- a_1(u^n, C_1^n, \epsilon^n) \\
-K(C_1^n, T^n, \epsilon^n) &+ K(C_{1_h}^n, T_h^{n-1}, \epsilon^n).
\end{aligned}$$

We have as well:

$$\begin{aligned} K(C_1^n, T^n, \epsilon^n) - K(C_{1_h}^n, T_h^{n-1}, \epsilon^n) &= K(C_1^n, T^n, \epsilon^n) - K(C_1^n, T^{n-1}, \epsilon^n) \\ &\quad + K(C_1^n, T^{n-1}, \epsilon^n) - K(C_{1_h}^n, T^{n-1}, \epsilon^n) \\ &\quad + K(C_{1_h}^n, T^{n-1}, \epsilon^n) - K(C_{1_h}^n, T_h^{n-1}, \epsilon^n) \\ &\lesssim k\|T\|_{W^{1,\infty}}\|\epsilon^n\|_{0,\Omega} + \rho\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}\|\epsilon^n\|_{0,\Omega} \\ &\quad + \rho C_g M_C\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}\|\epsilon^n\|_{0,\Omega}, \end{aligned}$$

From where:

$$\begin{aligned} &\tfrac{1}{2k}\left(\|\epsilon^n\|_{0,\Omega}^2 - \|\epsilon^{n-1}\|_{0,\Omega}^2 + \|\epsilon^n - \epsilon^{n-1}\|_{0,\Omega}^2\right) + \lambda\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\lesssim (h^\sigma + k)\|\epsilon^n\|_{0,\Omega} + \lambda\|\nabla(C_1^n - r_h C_1^n)\|_{0,\Omega}^2 \\ &\quad + (h^{2\sigma} + k^2) + (\theta_8 NM + NM_C)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\quad + (\theta_9 NM_C + \tfrac{NM_C}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2 \\ &\quad + k\|\epsilon^n\|_{0,\Omega} + \rho\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}\|\epsilon^n\|_{0,\Omega} \\ &\quad + \rho M_C C_g\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}\|\epsilon^n\|_{0,\Omega}. \end{aligned}$$

From where:

$$\begin{aligned} &\tfrac{1}{2k}\left(\|\epsilon^n\|_{0,\Omega}^2 - \|\epsilon^{n-1}\|_{0,\Omega}^2 + \|\epsilon^n - \epsilon^{n-1}\|_{0,\Omega}^2\right) + \lambda\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\lesssim (h^{2\sigma} + k^2) + (\theta_{10} + \theta_{11} + \theta_{12})\|\epsilon^n\|_{0,\Omega}^2 \\ &\quad + (\theta_8 NM + NM_C)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\quad + \rho^2\tfrac{M_C^2 C_g^2}{4\theta_{12}}\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}^2 + \tfrac{\rho^2}{4\theta_{11}}\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\quad (\theta_9 NM_C + \tfrac{NM_C}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2. \end{aligned}$$

□

Lemma 5.15. *For all constants $\theta_{11}, \theta_{12}, \theta_{16}, \theta_{17}, \theta_{18}, \theta_{19}$, independents of h and k, we have:*

$$\left\{ \begin{aligned} \tfrac{1}{2k} & \left(\|\epsilon^n\|_{0,\Omega}^2 - \|\epsilon^{n-1}\|_{0,\Omega}^2 + \|\epsilon^n - \epsilon^{n-1}\|_{0,\Omega}^2\right) + \lambda\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \\ \lesssim & (h^{2\sigma} + k^2) + (\theta_{16} + \theta_{17} + \theta_{18} + \theta_{19})\|\epsilon^n\|_{0,\Omega}^2 + (\theta_{11} NM + NM_C)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \\ & + \rho^2\tfrac{M_C^2 C_g^2}{\theta_{16}}\|\nabla(T^n - T_h^{n-1})\|_{0,\Omega}^2 + \tfrac{\rho^2}{4\theta_{14}}\|\nabla(C_2^n - C_{2_h}^{n-1})\|_{0,\Omega}^2 \\ & + \tfrac{\rho^2}{4\theta_{15}}\|\nabla(C_1^n - C_{1_h}^{n-1})\|_{0,\Omega}^2 + (\theta_{12} NM_C + \tfrac{NM_C}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2. \end{aligned} \right.$$

Proof. The proof of this lemma is similar to the proof of the previous lemma. □

Lemma 5.16. *We assume that $NM_C < \frac{\lambda}{2}$, $6M_C^2 C_g^2 \leq \lambda$ and $(k \leq \frac{1}{4})$. We have:*

$$\left\{ \begin{aligned} \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 & + \tfrac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \lesssim kn(h^{2\sigma} + k^2) \\ & + \frac{16k}{7}(\theta_9 NM_C + \frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 \\ & + \frac{48}{7}\rho^2 k M_C^2 C_g^2 \sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|_{0,\Omega}^2. \end{aligned} \right.$$

Proof. For $\theta_{10} = \theta_{11} = \theta_{12} = \frac{1}{12}$, $\theta_8 = \frac{\lambda - 2NM_C}{2NM}$ (because $NM_C < \frac{\lambda}{2}$), we have:

$$\begin{aligned} &\tfrac{1}{2k}\left(\|\epsilon^n\|_{0,\Omega}^2 - \|\epsilon^{n-1}\|_{0,\Omega}^2 + \|\epsilon^n - \epsilon^{n-1}\|_{0,\Omega}^2\right) + \tfrac{\lambda}{2}\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 \\ &\lesssim \tfrac{1}{4}\|\epsilon^n\|_{0,\Omega}^2 + (h^{2\sigma} + k^2) + 3\rho^2 M_C^2 C_g^2\|\nabla(T^{n-1} - T_h^{n-1})\|_{0,\Omega}^2 \\ &\quad + 3\rho^2\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + (\theta_4 NB + \tfrac{NB}{2})\|\nabla(u^{n-1} - u_h^{n-1})\|_{0,\Omega}^2. \end{aligned}$$

From where

$$\frac{1}{2k}\left(\|\epsilon^n\|^2_{0,\Omega}\right)+\frac{\lambda}{2}\sum_{i=1}^n\|\nabla(C_1^i-C_{1_h}^i)\|^2_{0,\Omega}\lesssim\frac{1}{4}\sum_{i=1}^n\|\epsilon^i\|^2_{0,\Omega}+n(h^{2\sigma}+k^2)$$
$$+3\rho^2M_C^2C_g^2\sum_{i=1}^{n-1}\|\nabla(T^i-T_h^i)\|^2_{0,\Omega}+3\rho^2\sum_{i=1}^n\|\nabla(C_1^i-C_{1_h}^i)\|^2_{0,\Omega}$$
$$+(\theta_9NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}.$$

While multiplying by $2k$ and by using the assumption $k\leq\frac{1}{4}$, we obtain:

$$\frac{7}{8}\left(\|\epsilon^n\|^2_{0,\Omega}\right)+2k(\lambda-3\rho^2)\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}\lesssim\frac{k}{2}\sum_{i=1}^{n-1}\|\epsilon^i\|^2_{0,\Omega}$$
$$+kn(h^{2\sigma}+k^2)+6\rho^2kM_C^2C_g^2\sum_{i=1}^n\|\nabla(T^i-T_h^i)\|^2_{0,\Omega}$$
$$+2k(\theta_9NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}.$$

While multiplying by $\frac{8}{7}$, we obtain:

$$\left(\|\epsilon^n\|^2_{0,\Omega}\right)+\frac{16}{7}k(\lambda-3\rho^2)\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}\lesssim\frac{4k}{7}\sum_{i=1}^{n-1}\|\epsilon^i\|^2_{0,\Omega}$$
$$+kn(h^{2\sigma}+k^2)+\frac{48}{7}\rho^2kM_C^2C_g^2\sum_{i=1}^n\|\nabla(T^i-T_h^i)\|^2_{0,\Omega}$$
$$+\frac{16k}{7}(\theta_9NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}.$$

By using the discret lemma of *Gronwal*, we obtain:

$$\|\epsilon^n\|^2_{0,\Omega}+\frac{16}{7}k(\lambda-3\rho^2)\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}\lesssim\Big(kn(h^{2\sigma}+k^2)$$
$$+\frac{48}{7}\rho^2kM_C^2C_g^2\sum_{i=1}^n\|\nabla(T^i-T_h^i)\|^2_{0,\Omega}$$
$$+\frac{16k}{7}(\theta_9NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}\Big)\exp(kn).$$

The result comes from the triangular inequality:

$$\|C_1^n-C_{1_h}^n\|^2_{0,\Omega}+\frac{16}{7}k(\lambda-3\rho^2)\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}\leq\|C_1^n-r_hC_1^n\|^2_{0,\Omega}+\|\epsilon^n\|^2_{0,\Omega}+\frac{16}{7}k(\lambda-3\rho^2)\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}$$
$$\lesssim h^\sigma\|C_1\|_{L^\infty(H^\sigma)}+\|\epsilon^n\|^2_{0,\Omega}+k\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}.$$

□

Lemma 5.17. *We assume that* $NM_C<\frac{\lambda}{2}$, $6M_C^2C_g^2\leq\lambda$ *and* $(k\leq\frac{1}{4})$. *We have:*

$$\|C_2^n-C_{2_h}^n\|^2_{0,\Omega}+\frac{16}{7}k(\lambda-3\rho^2)\|\nabla(C_2^n-C_{2_h}^n)\|^2_{0,\Omega}\lesssim kn(h^{2\sigma}+k^2)$$
$$+\frac{16k}{7}(\theta_{12}NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}$$
$$+\frac{256}{7}kM_C^2C_g^2\sum_{i=1}^n\|T^i-T_h^i\|^2_{0,\Omega}+\frac{64}{7}k\sum_{i=1}^n\|C_1^i-C_{1_h}^i\|^2_{0,\Omega}.$$

Proof. For $\theta_{16}=\theta_{17}=\theta_{18}=\theta_{19}=\frac{1}{16}$, $\theta_{11}=\frac{\lambda-2NM_C}{2NM}$(because $NM_C<\frac{\lambda}{2}$), we have:

$$\frac{1}{2k}(\|\epsilon^n\|^2_{0,\Omega}-\|\epsilon^{n-1}\|^2_{0,\Omega}+\|\epsilon^n-\epsilon^{n-1}\|^2_{0,\Omega})+\frac{\lambda}{2}\|\nabla(C_2^n-C_{2_h}^n)\|^2_{0,\Omega}\lesssim\frac{1}{4}\|\epsilon^n\|^2_{0,\Omega}$$
$$+(h^{2\sigma}+k^2)+16\rho^2M_C^2C_g^2\|\nabla(T^{n-1}-T_h^{n-1})\|^2_{0,\Omega}+4\rho^2\|\nabla(C_1^n-C_{1_h}^n)\|^2_{0,\Omega}$$
$$+4\rho^2\|\nabla(C_2^n-C_{2_h}^n)\|^2_{0,\Omega}+(\theta_{12}NB+\frac{NB}{2})\|\nabla(u^{n-1}-u_h^{n-1})\|^2_{0,\Omega}.$$

From where

$$\frac{1}{2k}\left(\|\epsilon^n\|^2_{0,\Omega}\right)+\frac{\lambda}{2}\sum_{i=1}^n\|\nabla(C_2^i-C_{2_h}^i)\|^2_{0,\Omega}\lesssim\frac{1}{4}\sum_{i=1}^n\|\epsilon^i\|^2_{0,\Omega}+n(h^{2\sigma}+k^2)$$
$$+16\rho^2M_C^2C_g^2\sum_{i=1}^{n-1}\|\nabla(T^i-T_h^i)\|^2_{0,\Omega}+4\rho^2\sum_{i=1}^n\|\nabla(C_1^i-C_{1_h}^i)\|^2_{0,\Omega}$$
$$+4\rho^2\sum_{i=1}^n\|\nabla(C_2^i-C_{2_h}^i)\|^2_{0,\Omega}+(\theta_{12}NM_C+\frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i-u_h^i)\|^2_{0,\Omega}.$$

While multiplying by $2k$ and by using the assumption $k \leq \frac{1}{4}$, we obtain:

$$\frac{7}{8}\left(\|\epsilon^n\|^2_{0,\Omega}\right) + 2k(\lambda - 4\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega} \lesssim \frac{k}{2}\sum_{i=1}^{n-1}\|\epsilon^i\|^2_{0,\Omega} + kn(h^{2\sigma} + k^2)$$
$$+ 32\rho^2 k M_C^2 C_g^2 \sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|^2_{0,\Omega} + 8\rho^2 k \sum_{i=1}^{n}\|\nabla(C_1^i - C_{1_h}^i)\|^2_{0,\Omega}$$
$$+ 2k(\theta_{12}NM_C + \frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}.$$

While multiplying by $\frac{8}{7}$, we obtain:

$$\left(\|\epsilon^n\|^2_{0,\Omega}\right) + \frac{16}{7}k(\lambda - 4\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega} \lesssim \frac{4k}{7}\sum_{i=1}^{n-1}\|\epsilon^i\|^2_{0,\Omega} + kn(h^{2\sigma} + k^2)$$
$$+ \frac{256}{7}\rho^2 k M_C^2 C_g^2 \sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|^2_{0,\Omega} + \frac{64}{7}\rho^2 k \sum_{i=1}^{n}\|\nabla(C_1^i - C_{1_h}^i)\|^2_{0,\Omega}$$
$$+ \frac{16k}{7}(\theta_{12}NM_C + \frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}.$$

By using the discret lemma of *Gronwal*, we obtain:

$$\|\epsilon^n\|^2_{0,\Omega} + \tfrac{16}{7}k(\lambda - 4\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega} \lesssim \Big(kn(h^{2\sigma} + k^2)$$
$$+\tfrac{256}{7}\rho^2 k M_C^2 C_g^2 \textstyle\sum_{i=1}^{n}\|\nabla(T^i - T_h^i)\|^2_{0,\Omega} + \tfrac{64}{7}\rho^2 k \sum_{i=1}^{n}\|\nabla(C_1^i - C_{1_h}^i)\|^2_{0,\Omega}$$
$$+\tfrac{16k}{7}(\theta_{12}NM_C + \tfrac{NM_C}{2})\textstyle\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}\Big)\exp(kn).$$

The result comes from the triangular inequality:

$$\|C_2^n - C_{2_h}^n\|^2_{0,\Omega} + \tfrac{16}{7}k(\lambda - 4\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega} \leq \|C_2^n - r_h C_2^n\|^2_{0,\Omega} + \|\epsilon^n\|^2_{0,\Omega}$$
$$+\tfrac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega}$$
$$\lesssim h^\sigma\|C_2\|_{L^\infty(H^\sigma)} + \|\epsilon^n\|^2_{0,\Omega} + k\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega}.$$

□

Lemma 5.18. *For the velocity, we have the following estimate:*

$$\|u^n - u_h^n\|^2_{0,\Omega} + k\|\nabla(u^n - u_h^n)\|^2_{0,\Omega} \lesssim \left(kn(h^{2\sigma} + k^2) + k^2n^2(h^{2\sigma} + k^2)\right)$$
$$\exp\left(k^2n^2\left(2(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12}NM_C + NM_C)\right)\right).$$

Proof. According to the previous lemmas, we have:

$$\|T^n - T_h^n\|^2_{0,\Omega} + \lambda k\|\nabla(T^n - T_h^n)\|^2_{0,\Omega} \lesssim kn(h^{2\sigma} + k^2) + 2k(\theta_4 NB + \tfrac{NB}{2})\textstyle\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}$$
$$+8k\textstyle\sum_{i=1}^{n}\|C_1^i - C_{1_h}^i\|^2_{0,\Omega} + 8k\sum_{i=1}^{n}\|C_2^i - C_{2_h}^i\|^2_{0,\Omega},$$

$$\|C_1^n - C_{1_h}^n\|^2_{0,\Omega} \quad +\tfrac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_1^n - C_{1_h}^n)\|^2_{0,\Omega} \lesssim kn(h^{2\sigma} + k^2)$$
$$+\frac{16k}{7}(\theta_9 NM_C + \frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}$$
$$+\frac{48}{7}kM_C^2C_g^2\sum_{i=1}^{n}\|T^i - T_h^i\|^2_{0,\Omega}.$$

and

$$\|C_2^n - C_{2_h}^n\|^2_{0,\Omega} \quad +\tfrac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|^2_{0,\Omega} \lesssim kn(h^{2\sigma} + k^2)$$
$$+\frac{16k}{7}(\theta_{12} NM_C + \frac{NM_C}{2})\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|^2_{0,\Omega}$$
$$+\frac{256}{7}kM_C^2C_g^2\sum_{i=1}^{n}\|T^i - T_h^i\|^2_{0,\Omega} + \frac{64}{7}k\sum_{i=1}^{n}\|C_1^i - C_{1_h}^i\|^2_{0,\Omega}.$$

By summing, we obtain:

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2 + \lambda k \|\nabla(T^n - T_h^n)\|_{0,\Omega}^2$$
$$+ \frac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \lesssim kn(h^{2\sigma} + k^2)$$
$$+ \left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)$$
$$\times \sum_{i=1}^{n-1} \|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 + \frac{120}{7}k \sum_{i=1}^{n} \|C_1^i - C_{1_h}^i\|_{0,\Omega}^2$$
$$+ 8k \sum_{i=1}^{n} \|C_2^i - C_{2_h}^i\|_{0,\Omega}^2 + \frac{304}{7}kM_C^2 C_g^2 \sum_{i=1}^{n} \|T^i - T_h^i\|_{0,\Omega}^2.$$

If we take $w = max(\frac{120}{7}k, \frac{304}{7}kM_C^2 C_g^2)$, we obtain

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2 + \frac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)$$
$$\times \|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \lesssim \frac{1}{1-w}kn(h^{2\sigma} + k^2)$$
$$+ \frac{1}{1-w}\left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right) \sum_{i=1}^{n-1} \|\nabla(u^i - u_h^i)\|_{0,\Omega}^2$$
$$+ \frac{w}{1-w}\left(\sum_{i=1}^{n}(\|C_1^i - C_{1_h}^i\|_{0,\Omega}^2 + \|C_2^i - C_{2_h}^i\|_{0,\Omega}^2 + \|T^i - T_h^i\|_{0,\Omega}^2)\right).$$

By using the *Gronwal* lemma with

$$a_n = \|T^n - T_h^n\|_{0,\Omega}^2 + \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2,$$

$$b_n = \frac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)$$
$$\times \|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2$$

and

$$c_n = \frac{1}{1-w}kn(h^{2\sigma} + k^2) + \frac{1}{1-w}\left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)$$
$$\times \sum_{i=1}^{n-1} \|\nabla(u^i - u_h^i)\|_{0,\Omega}^2,$$

we obtain:

$$\|T^n - T_h^n\|_{0,\Omega}^2 \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2 + \frac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)$$
$$\times \|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \lesssim \left(\frac{1}{1-w}kn(h^{2\sigma} + k^2)\right.$$
$$\left. + \frac{1}{1-w}\left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right) \sum_{i=1}^{n-1} \|\nabla(u^i - u_h^i)\|_{0,\Omega}^2\right)$$
$$\times \exp(\frac{wn}{1-w}).$$

We have:

$$\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim kn(h^{2\sigma} + k^2) + k\left(kn\tfrac{1}{1-w}(h^{2\sigma} + k^2)\right.$$
$$\left. + \tfrac{1}{1-w}\left(2k(\theta_4 NB + \tfrac{NB}{2}) + \tfrac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)\right) \textstyle\sum_{j=2}^{n} \sum_{i=1}^{j-1} \|\nabla(u^i - u_h^i)\|_{0,\Omega}^2.$$

Finally, we have:

$$\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim \left(kn(h^{2\sigma} + k^2) + k^2n^2(h^{2\sigma} + k^2)\right)$$
$$\exp\left(k^2n^2\left(2(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)\right).$$

□

Lemma 5.19. *We have the following at the same time for the temperature and for the concentrations :*

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \tfrac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3)$$

and

$$\|C_i^n - C_{i_h}^n\|_{0,\Omega}^2 + \tfrac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_i^n - C_{i_h}^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3). \quad (i = 1, 2)$$

Proof. We have

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2 + \frac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)$$

$$\times\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \lesssim \frac{1}{1-w}kn(h^{2\sigma} + k^2)$$

$$+ \frac{1}{1-w}\left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2.$$

According to the previous lemma, we can remark that:

$$k\sum_{i=1}^{n-1}\|\nabla(u^i - u_h^i)\|_{0,\Omega}^2 \lesssim kn\left(kn(h^{2\sigma} + k^2) + k^2n^2(h^{2\sigma} + k^2)\right)$$

Therefore:

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \|C_1^n - C_{1_h}^n\|_{0,\Omega}^2 + \|C_2^n - C_{2_h}^n\|_{0,\Omega}^2 + \frac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)$$

$$\times\|\nabla(C_1^n - C_{1_h}^n)\|_{0,\Omega}^2 + \frac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_2^n - C_{2_h}^n)\|_{0,\Omega}^2 \lesssim \frac{1}{1-w}n(h^{2\sigma} + k^2)$$

$$+ \frac{1}{1-w}\left(2k(\theta_4 NB + \frac{NB}{2}) + \frac{16k}{7}(\theta_9 NM_C + \theta_{12} NM_C + NM_C)\right)$$

$$n\left(kn(h^{2\sigma} + k^2) + k^2n^2(h^{2\sigma} + k^2)\right)$$

Conclusion:

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \tfrac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3)$$

We have also:

$$\|C_i^n - C_{i_h}^n\|_{0,\Omega}^2 + \tfrac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_i^n - C_{i_h}^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3) \quad (i = 1, 2)$$

□

Finally, from the previous lemmas, we are able to state our following mean result:

Theorem 5.2. *Assume that:*

$$NA < \nu + NM, \quad NB < \frac{\lambda}{2}, \quad NM_C < \frac{\lambda}{2}$$

and

$$16M_C^2C_g^2 < \min\left(\lambda, \frac{\lambda}{2\rho^2}\right), \qquad k \leq \frac{1}{4}, \qquad w = max(\frac{120}{7}k, \frac{304}{7}kM_C^2C_g^2) < 1.$$

If the solution of the problem (P) admits the following regularity:

$$\begin{cases} u \in W^{2,\infty}(0,t,(L^2(\Omega))^d) \cap W^{1,\infty}(0,t,(H^1(\Omega))^d) \cap L^\infty(0,t,(H^{1+\sigma}(\Omega))^d), \\ p \in L^\infty(0,t,H^\sigma(\Omega)), \\ T, C_1, C_2 \in W^{2,\infty}(0,t,(L^2(\Omega))) \cap W^{1,\infty}(0,t,H^1(\Omega)) \cap L^\infty(0,t,H^{1+\sigma}(\Omega)), \end{cases}$$

then we will have the following error estimates:

$$\|u^n - u_h^n\|_{0,\Omega}^2 + k\|\nabla(u^n - u_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2)$$

for the velocity,

$$\|p^n - p_h^n\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3)$$

for the pressure,

$$\|T^n - T_h^n\|_{0,\Omega}^2 + \tfrac{\lambda k}{1-w}\|\nabla(T^n - T_h^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3)$$

for the temperature and

$$\|C_i^n - C_{i_h}^n\|_{0,\Omega}^2 + \tfrac{16}{7(1-w)}k(\lambda - 3\rho^2)\|\nabla(C_i^n - C_{i_h}^n)\|_{0,\Omega}^2 \lesssim (h^{2\sigma} + k^2)(kn + k^2n^2 + k^3n^3), \quad (i = 1, 2)$$

for the concentrations.

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