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Triple Coincidence Point Theorems for Multi - Valued Maps in Partially Ordered Metric Spaces

K.P.R.Rao[1], G.N.V.Kishore[2,*], P.R.Sobhana Babu[3]

[1]Department of Mathematics, Acharya Nagarjuna University,Nagarjuna Nagar,
Guntur - 522 510, Andhra Pradesh, India
[2]Department of Mathematics, Baba Institute of Technology and Sciences,P.M.Palem,
Madhurawada Visakhapatnam - 530048, Andhra Pradesh, India
[3]Department of Mathematics, Ramachandra College of Engineering, Vatluru(V),
Eluru-534007, West Godavari Dist., Andhra Pradesh, India
*Corresponding Author: kishore.apr2@gmail.com

Abstract In this paper we prove a triple coincidence point theorem for multi - valued and single-valued mappings in a partially ordered metric space based on the concepts of [5]. Also we give an example which supports our main result. Our result generalizes several results relating to coupled fixed point theorems.

Keywords Triple fixed point, complete space, w - compatible, set-valued mapping, Δ - Symmetric Property

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1 Introduction

The study of fixed points for multi - valued contraction mappings using the Hausdorff metric was initiated by Nadler [9].

Let (X, d) be a metric space. We denote $CB(X)$ the family of all non - empty closed and bounded sub sets of X and $CL(X)$ the set of all non - empty closed sub sets of X. For $A, B \in CB(X)$ and $x \in X$, we denote $D(x, A) = \inf\{d(x, a) : a \in A\}$. Let H be the Hausdorff metric induced by the metric d on X, that is

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in CB(X)$.

Definition 1.1 *An element $x \in X$ is said to be a fixed point of a set - valued mapping $T : X \to CB(X)$ if and only if $x \in Tx$.*

In 1969, Nadler [9] extended the famous Banach Contraction Principle [8] from single - valued mapping to multi - valued mapping and proved the following fixed point theorem for the multi - valued contraction.

Theorem 1.2 *([9]): Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$H(Tx, Ty) \leq c\ d(x, y),$$

for all $x, y \in X$. Then, T has a fixed point.

The existence of fixed points for various multi - valued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [1, 4, 7, 9, 11] and the references therein.

The concept of coupled fixed point for multi - valued mapping was introduced by Samet and Vetro [2] and later several authors namely Hussain and Alotaibi [6] and Aydi et. al.[3] proved coupled coincidence point theorems in partially ordered metric spaces.

Berinde and Borcut [10], introduced the concept of triple fixed points and obtained a tripled fixed point theorem for single valued map. Later we introduced Triple fixed, Triple coincidence and Triple common fixed points for multi - valued maps in our earlier paper [5] as follows.

Definition 1.3 *([5]) Let X be a non empty set, $T : X \times X \times X \to 2^X$ (Collection of all non empty subsets of X). $f : X \to X$.*

(i) The point $(x, y, z) \in X \times X \times X$ is called a tripled fixed of T if

$$x \in T(x, y, z),\ \ y \in T(y, x, y) \text{ and } z \in T(z, y, x).$$

(ii) The point $(x, y, z) \in X \times X \times X$ is called a tripled coincident point of T and f if

$$fx \in T(x, y, z), fy \in T(y, x, y) \text{ and } fz \in T(z, y, x).$$

(iii) The point $(x, y, z) \in X \times X \times X$ is called a tripled common fixed point of T and f if

$$x = fx \in T(x, y, z),\ \ y = fy \in T(y, x, y) \text{ and } z = fz \in T(z, y, x).$$

Definition 1.4 *([5]) Let $T : X \times X \times X \to 2^X$ be a multi - valued map and f be a self map on X. The Hybrid pair $\{T, f\}$ is called w - compatible if $f(T(x, y, z)) \subseteq T(fx, fy, fz)$ whenever (x, y, z) is a tripled coincidence point of T and f.*

2 Results

Let (X, d) be a metric space endowed with a partial order $\preceq$ and $G : X \to X$. Define the set $\Delta \subset X^3$ by $\Delta = \{(x, y, z) \in X^3 :\ \text{Gx R Gy, Gx R Gz and Gy R Gz}\}$.

Definition 2.1 *A mapping $F : X^3 \to X$ is said to have a Δ - Symmetric property if and only if $(x, y, z) \in \Delta \Rightarrow F(x, y, z)RF(y, x, y), F(x, y, z)RF(z, y, x)$ and $F(y, x, y)RF(z, y, z)$.*

Definition 2.2 *A mapping $f : X^3 \to [0, \infty)$ is called lower semi continuous if, for any sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X and $(x, y, z) \in X^3$, one has*

$$\lim_{n\to\infty} (x_n, y_n, z_n) = (x, y, z) \Longrightarrow f(x, y, z) \le \lim_{n\to\infty} \inf\ f(x_n, y_n, z_n).$$

Theorem 2.3 *Let (X, d) be a metric space endowed with a partial order $\preceq$ and $\Delta \neq \emptyset$. Suppose that $F : X \times X \times X \to CL(X)$ has a Δ - Symmetric property, $g : X \to X$ is continuous, $g(X)$ is complete, the function $f : g(X) \times g(X) \times g(X) \to [0, +\infty)$ defined for all $x, y, z \in X$ by*

*(2.3.1) $f(gx, gy, gz) := D(gx, F(x, y, z)) + D(gy, F(y, x, y)) + D(gz, F(z, y, x))$,
is lower semi - continuous and that there exists a function $\phi : [0, +\infty) \to [a, 1)$, $0 < a < 1$, satisfying*

(2.3.2) $\lim_{r\to t^+} \sup\ \phi(r) < 1$, for each $t \in [0, +\infty)$.

Assume that for any $(x, y, z) \in \Delta$ there exist $gu \in F(x, y, z)$, $gv \in F(y, x, y)$ and $gw \in F(z, y, x)$ satisfying

*(2.3.3) $\sqrt{\phi(f(gx, gy, gz))}[d(gx, gu) + d(gy, gv) + d(gz, gw)] \le f(gx, gy, gz)$
such that*

(2.3.4) $f(gu, gv, gw) \le \phi(f(gx, gy, gz))[d(gx, gu) + d(gy, gv) + d(gz, gw)]$.

Then, F and g have a triple coincidence point. That is, there exists $(g\alpha, g\beta, g\gamma) \in X \times X \times X$ such that $g\alpha \in F(\alpha, \beta, \gamma)$, $g\beta \in F(\beta, \alpha, \beta)$ and $g\gamma \in F(\gamma, \beta, \alpha)$.

Let $(x_0, y_0, z_0) \in \Delta$ be arbitrary and fixed, by (2.3.3) and (2.3.4), we can choose $gx_1 \in F(x_0, y_0, z_0)$, $gy_1 \in F(y_0, x_0, y_0)$ and $gz_1 \in F(z_0, y_0, x_0)$ such that

$$\sqrt{\phi(f(gx_0, gy_0, gz_0))}\,[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \le f(gx_0, gy_0, gz_0) \tag{2.1}$$

and

$$f(gx_1, gy_1, gz_1) \le \phi(f(gx_0, gy_0, gz_0))\,[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)]\,. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\begin{aligned} f(gx_1, gy_1, gz_1) &\le \phi(f(gx_0, gy_0, gz_0))[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \\ &= \sqrt{\phi(f(gx_0, gy_0, gz_0))}\left\{\sqrt{\phi(f(gx_0, gy_0, gz_0))}\right. \\ &\qquad \left.[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)]\right\} \\ &\le \sqrt{\phi(f(gx_0, gy_0, gz_0))} f(gx_0, gy_0, gz_0). \end{aligned}$$

Thus

$$f(gx_1, gy_1, gz_1) \leq \sqrt{\phi(f(gx_0, gy_0, gz_0))}f(gx_0, gy_0, gz_0). \tag{2.3}$$

Now, since F has a Δ - Symmetric property and $(x_0, y_0, z_0) \in \Delta$, we have

$$F(x_0, y_0, z_0)RF(y_0, x_0, y_0), F(x_0, y_0, z_0)RF(z_0, y_0, x_0) \text{ and } F(y_0, x_0, y_0)RF(z_0, y_0, x_0)$$

Thus

$$gx_1 \ R \ gy_1, \ \ gx_1 \ R \ gz_1 \ \text{ and } \ gy_1 \ R \ gz_1 \tag{2.4}$$

By definition of Δ, $(x_1, y_1, z_1) \in \Delta$.
Again from (2.3.3) and (2.3.4), we can choose $gx_2 \in F(x_1, y_1, z_1)$, $gy_2 \in F(y_1, x_1, y_1)$ and $gz_2 \in F(z_1, y_1, x_1)$ such that

$$\sqrt{\phi(f(gx_1, gy_1, gz_1))}\left[d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)\right] \leq f(gx_1, gy_1, gz_1)$$

and

$$f(gx_2, gy_2, gz_2) \leq \phi(f(gx_1, gy_1, gz_1))\left[d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)\right].$$

Hence, we get

$$f(gx_2, gy_2, gz_2) \leq \sqrt{\phi(f(gx_1, gy_1, gz_1))}f(gx_1, gy_1, gz_1), \text{ with } (x_2, y_2, z_2) \in \Delta.$$

Continuing this process we can choose $gx_n \in X$, $gy_n \in X$ and $gz_n \in X$ such that for all $n = 0, 1, 2, \cdots$, we have $(x_n, y_n, z_n) \in \Delta$,

$$gx_{n+1} \in F(x_n, y_n, z_n), gy_{n+1} \in F(y_n, x_n, y_n) \text{ and } gz_{n+1} \in F(z_n, y_n, x_n), \tag{2.5}$$

$$\sqrt{\phi(f(gx_n, gy_n, gz_n))}\begin{bmatrix} d(gx_n, gx_{n+1}) \\ +d(gy_n, gy_{n+1}) \\ +d(gz_n, gz_{n+1}) \end{bmatrix} \leq f(gx_n, gy_n, gz_n) \tag{2.6}$$

and

$$f(gx_{n+1}, gy_{n+1}, gz_{n+1}) \leq \sqrt{\phi(f(gx_n, gy_n, gz_n))}f(gx_n, gy_n, gz_n), \tag{2.7}$$

with $(x_{n+1}, y_{n+1}, z_{n+1}) \in \Delta$.
Now, we shall show that $f(gx_n, gy_n, gz_n) \to 0$ as $n \to \infty$.
If $f(gx_m, gy_m, gz_m) = 0$, for some m, then we get

$$\begin{array}{ll} D(gx_m, F(x_m, y_m, z_m)) & = 0, \text{ implies that } gx_m \in \overline{F(x_m, y_m, z_m)} = F(x_m, y_m, z_m), \\ D(gy_m, F(y_m, x_m, y_m)) & = 0, \text{ implies that } gy_m \in \overline{F(y_m, x_m, y_m)} = F(y_m, x_m, y_m) \\ \text{and} & \\ D(gz_m, F(z_m, y_m, x_m)) & = 0, \text{ implies that } gz_m \in \overline{F(z_m, y_m, x_m)} = F(z_m, y_m, x_m). \end{array}$$

Hence in this case (gx_m, gy_m, gz_m) is a triple coincidence point of F and g and the theorem is proved.
Suppose that $f(gx_n, gy_n, gz_n) > 0$ for all n.
Using (2.7) and $\phi(t) < 1$, we conclude that $\{f(gx_n, gy_n, gz_n)\}$ is a strictly decreasing sequence of non - negative real numbers. Thus, there exists a $\delta \geq 0$, such that

$$\lim_{n\to\infty} f(gx_n, gy_n, gz_n) = \delta.$$

Now, we will show that $\delta = 0$. On contrary assume that $\delta > 0$,
Letting $n \to \infty$ in (2.7), we have that

$$\delta \leq \lim_{f(gx_n, gy_n, gz_n)\to\delta^+} \sup \sqrt{\phi(f(gx_n, gy_n, gz_n))}\ \delta < \delta,$$

a contradiction. Hence $\delta = 0$. That is

$$\lim_{n\to\infty} f(gx_n, gy_n, gz_n) = 0. \tag{2.8}$$

Now, we prove $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in (X, d).
Suppose

$$\delta \leq \lim_{f(gx_n, gy_n, gz_n)\to 0^+} \sup \sqrt{\phi(f(gx_n, gy_n, gz_n))}\ .$$

Then by assumption (2.3.2), we have $\delta < 1$.
Let k be such that $\delta < k < 1$. Then, there exists $n_0 \in N$ such that

$$\sqrt{\phi(f(gx_n, gy_n, gz_n))} < k, \text{ for each } n \geq n_0$$

Thus from (2.7), we get

$$f(gx_{n+1}, gy_{n+1}, gz_{n+1}) < k.f(gx_n, gy_n, gz_n), \text{ for each } n \geq n_0.$$

Hence by induction, for each $n \geq n_0$, we have

$$f(gx_{n+1}, gy_{n+1}, gz_{n+1}) < k^{n+1-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}). \tag{2.9}$$

Since $\phi(t) \geq a > 0$ for all $t > 0$, from (2.6), (2.9) and for each $n \geq n_0$, we have

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) < \frac{1}{\sqrt{a}}\ k^{n-n_0}\ f(gx_{n_0}, gy_{n_0}, gz_{n_0}). \tag{2.10}$$

Now we consider for $m > n$, we have

$$\begin{aligned} & d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) \\ & \quad \leq \ d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ & \qquad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\ & \qquad + d(gz_n, gz_{n+1}) + d(gz_{n+1}, gz_{n+2}) + \cdots + d(gz_{m-1}, gz_m) \\ & \quad \leq \tfrac{1}{\sqrt{a}}\ k^{n-n_0}\ f(gx_{n_0}, gy_{n_0}, gz_{n_0}) + \tfrac{1}{\sqrt{a}}\ k^{n+1-n_0}\ f(gx_{n_0}, gy_{n_0}, gz_{n_0}) \\ & \qquad + \cdots + \tfrac{1}{\sqrt{a}}\ k^{m-1-n_0}\ f(gx_{n_0}, gy_{n_0}, gz_{n_0}) \\ & \quad \leq \tfrac{1}{\sqrt{a}}\ k^{n-n_0} \tfrac{1}{1-k}\ f(gx_{n_0}, gy_{n_0}, gz_{n_0}) \\ & \quad \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in (X, d). Suppose $g(X)$ is complete, there exist $u, v, w \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = u = g\alpha,\ \lim_{n\to\infty} gy_n = v = g\beta \text{ and } \lim_{n\to\infty} gz_n = w = g\gamma,$$

for some $\alpha, \beta, \gamma \in X$.
Now, we show that $(g\alpha, g\beta, g\gamma)$ is triple coincidence point of F and g.
Since f is lower semi continuous from (2.8), we have

$$\begin{aligned} 0 & \leq f(g\alpha, g\beta, g\gamma) \\ & = D(g\alpha, F(\alpha, \beta, \gamma)) + D(g\beta, F(\beta, \alpha, \beta)) + D(g\gamma, F(\gamma, \beta, \alpha)) \\ & \leq \lim_{n\to\infty} \inf f(gx_n, gy_n, gz_n) = 0. \end{aligned}$$

Hence, we get

$$D(g\alpha, F(\alpha, \beta, \gamma)) = D(g\beta, F(\beta, \alpha, \beta)) = D(g\gamma, F(\gamma, \beta, \alpha)) = 0,$$

which implies that

$$g\alpha \in F(\alpha, \beta, \gamma),\ g\beta \in F(\beta, \alpha, \beta) \text{ and } g\gamma \in F(\gamma, \beta, \alpha).$$

Thus (α, β, γ) is triple coincidence point of F and g.

Example 2.4 *Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$, then (X, d) is complete metric space and we define $\preceq$ by*

$$x \preceq y \iff x \leq y$$

then $\preceq$ is partial order relation.
We define $g : X \to X$ by $g(x) = \frac{x}{2}$, $F : X \times X \times X \to CB(X)$ by $F(x, y, z) = [x, 1],\ \forall x, y, z \in X$.
Then

$$\begin{aligned} f(gx, gy, gz) & = D(gx, F(x, y, z)) + D(gy, F(y, x, y)) + D(gz, F(z, y, x)) \\ & = \inf\{d(gx, a) : a \in [x, 1]\} + \inf\{d(gy, b) : b \in [y, 1]\} + \inf\{d(gz, c) : c \in [z, 1]\} \\ & = d(\tfrac{x}{2}, x) + d(\tfrac{y}{2}, y) + d(\tfrac{z}{2}, z) \\ & = \left|\tfrac{x}{2} - x\right| + \left|\tfrac{y}{2} - y\right| + \left|\tfrac{z}{2} - z\right| \\ & = \tfrac{x+y+z}{2}. \end{aligned}$$

Also let $\phi : [0, \infty) \to [a, 1], 0 \leq a < 1$ by $\phi(t) = \frac{t}{t+1}$ it is clear that $\lim_{r\to t^+} \sup \phi(t) < 1$ for each $t \in [0, +\infty)$.
Without loss generality we choose $g0 = gu \preceq gx, g0 = gv \preceq gy$ and $g0 = gw \preceq gy$.
It is clear that

$$\sqrt{\phi(f(gx, gy, gz))}[d(gx, gu) + d(gy, gv) + d(gz, gw)] \leq f(gx, gy, gz)$$

such that

$$f(gu, gv, gw) \leq \phi(f(gx, gy, gz))[d(gx, gu) + d(gy, gv) + d(gz, gw)].$$

Hence all conditions of Theorem 2.3 are satisfied and (0, 0, 0) is the coincidence point of g and F.

Corollary 2.5 *Let (X, d) be a metric space endowed with a partial order $\preceq$ and $\Delta \neq \emptyset$, that is there exist $(x_0, y_0, z_0) \in \Delta$. Suppose that $F : X \times X \times X \to CL(X)$ has a Δ - property such that $f : X \times X \times X \to [0, +\infty)$ given by*

$$f(x, y, z) := D(x, F(x, y, z)) + D(y, F(y, x, y)) + D(z, F(z, y, x)),$$

is lower semi - continuous and that there exists a function $\phi : [0, +\infty) \to [a, 1)$, $0 < a < 1$, satisfying $\lim_{r \to t^+} \sup\ \phi(r) < 1$, for each $t \in [0, +\infty)$.
If for any $(x, y, z) \in \Delta$ there exist $u \in F(x, y, z)$, $v \in F(y, x, y)$ and $w \in F(z, y, x)$ satisfying

$$\sqrt{\phi(f(x, y, z))}[d(x, u) + d(y, v) + d(z, w)] \leq f(x, y, z)$$

such that

$$f(u, v, w) \leq \phi(f(x, y, z))[d(x, u) + d(y, v) + d(z, w)].$$

Then, F has a triple fixed point.

Proof. If we take $g = I$ (identity map), then remaining proof follows from Theorem 2.3.

3 Discussion and Conclusions

The main Theorem 2.3 is an extension of Theorem of [6] from coupled coincidence point to tripled coincidence point. We have given an example to illustrate our main theorem. We also obtain a corollary from our Theorem 2.3.

REFERENCES

[1] B.E. Rhoades, *A fixed point theorem for a multi - valued non - self mapping,* Comment. Math. Univ. Carolin., 37 (1996), 401 - 404.

[2] B. Samet and C. Vetro, *Coupled fixed point theorems for multi - valued nonlinear contraction mappings in partially ordered metric spaces,* Nonlinear Analysis, 74 (2011), 4260 - 4268.

[3] H. Aydi, M. Abbas and M. Postolache, *Coupled coincidence points for hybrid pair of mappings via mixed monotone property,* J. Adv. Math. Stud., 5(1), (2012), No. 1, 118 - 126.

[4] I. Altun, *A common fixed point theorem for multi - valued Círíc type mappings with new type compatibility,* An. St. Univ. Ovidius Constanta., 17(2), (2009), 19 - 26.

[5] K.P.R.Rao, G.N.V.Kishore and K. Tas, *A unique common triple fixed point theorem for hybrid pair of mappings,* Abstract and Applied Analysis, Volume 2012, Article ID 750403, 9 pages, doi:10.1155/2012/750403.

[6] N. Hussain and A. Alotaibi, *Coupled coincidences for multi - valued contractions in partially ordered metric spaces,* Fixed Point Theory and Applications, 2011, 2011 : 82.

[7] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multi - valued mappings on complete metric spaces,* J. Math. Anal. Appl., 141 (1989), 177 - 188.

[8] S.Banach, *Surles operations dans les ensembles abstraits et leur applications aux equations,* integrals, Fund. Math., 3(1922), 133 - 181.

[9] S. B. Jr. Nadler, *Multi - valued contraction mappings,* Pacific J. Math., XXX (1969), 475 - 488.

[10] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces,* Nonlinear Analysis, 74(15), (2011), 4889 - 4897.

[11] Wei - Shih Du, *Some generalizations of Mizoguchi - Takahashi's fixed point theorem,* Int. J. Contemp. Math. Sci., 3 (2008) 1283 - 1288.