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Some New Hermite-Hadamard Type Inequalities for Geometrically Convex Functions

İmdat İşcan

Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, Turkey
*Corresponding Author: imdat.iscan@giresun.edu.tr

Abstract In this paper, some new integral inequalities of Hermite-Hadamard type related to the geometrically convex functions are established and some applications to special means of positive real numbers are also given.

Keywords Geometrically convex function, Hermite - Hadamard type inequality

1. Introduction

This In this section, we firstly list several definitions and some known results.

Definition 1. Let I be an interval in $\mathbb{R}$. Then $f: I \to \mathbb{R}$ is said to be convex function if

$$f\left(tx+(1-t)y\right) \le tf(x)+(1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality [1]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

For several recent results concerning Hadamard's inequality we refer the interested reader to [1- 4, 6-8].

Definition 2 ([5]). A function $f: I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be GG-convex function (called in [8] geometrically convex function) if

$$f\left(x^t y^{1-t}\right) \le f(x)^t f(y)^{1-t}$$

for $x, y \in I$ and $t \in [0,1]$.

In this paper, we will establish some new integral inequalities of Hermite-Hadamard-like type related to the geometrically convex functions and then apply these inequalities to special means.

2. Main Results

Theorem 1. Suppose that $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is geometrically convex and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then one has the inequalities :

$$f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\sqrt{f(x)f\left(\frac{ab}{x}\right)}dx \le \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx$$
$$\le \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} \le \frac{f(a)+f(b)}{2}. \tag{2}$$

Proof. As f is geometrically convex, we have, for all $x, y \in I$

$$f\left(\sqrt{xy}\right) \le \sqrt{f(x)f(y)} \le \frac{f(x)+f(y)}{2}. \tag{3}$$

Now, let $x = a^{1-t}b^t$ and $y = a^t b^{1-t}$ with $t \in [0,1]$. Then we get by (3) that:

$$f\left(\sqrt{ab}\right) \le \sqrt{f(a^{1-t}b^t)f(a^t b^{1-t})} \le \frac{f(a^{1-t}b^t)+f(a^t b^{1-t})}{2}$$

for all $t \in [0,1]$. Integrating this inequality on $[0,1]$, we deduce the inequalities (2).

Secondly, since f is geometrically convex, we have, for all $t \in [0,1]$

$$f(a^t b^{1-t}) \le f(a)^t f(b)^{1-t} \le tf(a)+(1-t)f(b).$$

Integrating this inequality on $[0,1]$, we get

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx \le \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} \le \frac{f(a)+f(b)}{2}.$$

In order to prove next theorems, we need the following identity for differentiable functions. A consequence of the identities is that the author establishes some new inequalities connected with the inequalities (2) for the geometrically convex functions.

Lemma 1. Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I°, and $a,b\in I$,with $a<b$. If $f'\in L[a,b]$, then

$$f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx$$
$$=\frac{(\ln b-\ln a)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)dt\right.$$
$$\left.-b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}} f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)dt\right],$$

$$\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx$$
$$=\frac{(\ln b-\ln a)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^t\right)dt\right.$$
$$\left.-b\int_0^1 t\left(\frac{a}{b}\right)^{t} f'\left(b^{1-t}a^t\right)\right].$$

Proof. Integrating by part and changing variables of integration yields

$$\frac{(\ln b-\ln a)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)dt\right.$$
$$\left.-b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}} f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)dt\right]$$
$$=\frac{1}{2}\left[\int_0^1 tdf\left(a^{1-t}(ab)^{\frac{t}{2}}\right)+\int_0^1 tdf\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right]$$
$$=\frac{1}{2}\left[tf\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\Big|_0^1-\int_0^1 f\left(a^{1-t}(ab)^{\frac{t}{2}}\right)dt\right]$$
$$+\frac{1}{2}\left[tf\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\Big|_0^1-\int_0^1 f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)dt\right]$$
$$=f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx,$$

and

$$\frac{(\ln b-\ln a)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^t\right)dtdt\right.$$
$$\left.-b\int_0^1 t\left(\frac{a}{b}\right)^{t} f'\left(b^{1-t}a^t\right)dt\right]$$
$$=\frac{1}{2}\left[\int_0^1 tdf\left(a^{1-t}b^t\right)+\int_0^1 tdf\left(b^{1-t}a^t\right)\right]$$
$$=\frac{1}{2}\left[tf\left(a^{1-t}b^t\right)\Big|_0^1-\int_0^1 f\left(a^{1-t}b^t\right)dt\right]$$
$$+\frac{1}{2}\left[tf\left(b^{1-t}a^t\right)\Big|_0^1-\int_0^1 f\left(b^{1-t}a^t\right)dt\right]$$
$$=\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx.$$

This completes the proof.

Theorem 2. Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is geometrically convex on $[a,b]$ for $q\geq 1$, then

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|\leq\frac{\ln b-\ln a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{a\left|f'(a)\right|\left[g_1\left(\alpha\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_1\left(\gamma\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}\right\}, \quad (4)$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|\leq\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{a\left|f'(a)\right|\left[g_1(\alpha(q))\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_1(\gamma(q))\right]^{\frac{1}{q}}\right\}, \quad (5)$$

Where

$$g_1(\alpha)=\begin{cases}\frac{1}{2}, & \alpha=1\\ \frac{\alpha\ln\alpha-\alpha+1}{(\ln\alpha)^2}, & \alpha\neq 1\end{cases} \quad (6)$$

and

$$\alpha(u)=\left(\frac{b\left|f'(b)\right|}{a\left|f'(a)\right|}\right)^{u},$$

$$\gamma(u)=\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{u},\ u>0. \tag{7}$$

Proof. (1) Since $|f'|^q$ is geometrically convex on $[a,b]$, from lemma 1 and power mean inequality, we have

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$=\frac{(\ln b-\ln a)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right.$$

$$\left.+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]$$

$$\le\frac{a}{4}\ln\left(\frac{b}{a}\right)\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'\left(a^{\frac{2-t}{2}}b^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{4}\ln\left(\frac{b}{a}\right)\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'\left(b^{\frac{2-t}{2}}a^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le\frac{a}{4}\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{q(t/2)}\left|f'(a)\right|^{q((2-t)/2)}dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{4}\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)}\left|f'(b)\right|^{q((2-t)/2)}dt\right)^{\frac{1}{q}}$$

$$\le\frac{\ln b-\ln a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{a\left|f'(a)\right|\left[g_1\left(\alpha\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_1\left(\gamma\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}\right\}.$$

(2) Since $|f'|^q$ is geometrically convex on $[a,b]$, from lemma 1 and power mean inequality, we have we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\le\frac{(\ln b-\ln a)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t\left|f'\left(a^{1-t}b^t\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^t\left|f'\left(b^{1-t}a^t\right)\right|dt\right]$$

$$\le\frac{a}{2}\ln\left(\frac{b}{a}\right)\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left|f'\left(a^{1-t}b^t\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left|f'\left(b^{1-t}a^t\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le\frac{a}{2}\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left|f'(b)\right|^{qt}\left|f'(a)\right|^{q(1-t)}dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left|f'(a)\right|^{qt}\left|f'(b)\right|^{q(1-t)}dt\right)^{\frac{1}{q}}$$

$$\le\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{a\left|f'(a)\right|\left[g_1(\alpha(q))\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_1(\gamma(q))\right]^{\frac{1}{q}}\right\}.$$

This completes the proof.

If taking $q=1$ in Theorem 2, we can derive the following corollary.

Corollary 1. Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is geometrically convex on $[a,b]$, then [1]

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq \frac{\ln b-\ln a}{4}\left\{a\left|f'(a)\right|g_1\left(\alpha\left(\frac{1}{2}\right)\right)+b\left|f'(b)\right|g_1\left(\gamma\left(\frac{1}{2}\right)\right)\right\},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$

$$\leq \frac{\ln b-\ln a}{2}\left\{a\left|f'(a)\right|g_1(\alpha(1))+b\left|f'(b)\right|g_1(\gamma(1))\right\}.$$

Theorem 3. Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is geometrically convex on $[a,b]$ for $q>1$, then

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \tag{8}$$

$$\leq \frac{\ln b-\ln a}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{a\left|f'(a)\right|\left[g_2\left(\alpha\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_2\left(\gamma\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}\right\},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \tag{9}$$

$$\leq \frac{\ln b-\ln a}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{a\left|f'(a)\right|\left[g_2(\alpha(q))\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_2(\gamma(q))\right]^{\frac{1}{q}}\right\}$$

where inequality [1]

$$g_2(\alpha)=\begin{cases}1, & \alpha=1\\ \dfrac{\alpha-1}{\ln\alpha}, & \alpha\neq 1\end{cases}, \tag{10}$$

and $\alpha(u)$, $\gamma(u)$ are the same as in (7).

Proof. (1) Since $|f'|^q$ is geometrically convex on $[a,b]$, from lemma 1 and Hölder inequality, we have then

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right.$$

$$\left.+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right.$$

$$\left.+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]$$

$$\leq \frac{a}{4}\ln\left(\frac{b}{a}\right)\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{4}\ln\left(\frac{b}{a}\right)\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{a}{4}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{\frac{qt}{2}}\left|f'(a)\right|^{q((2-t)/2)}dt\right)^{\frac{1}{q}}$$

$$+\frac{b}{4}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{\frac{qt}{2}}\left|f'(b)\right|^{q((2-t)/2)}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\ln b-\ln a}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{a\left|f'(a)\right|\left[g_2\left(\alpha\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}+b\left|f'(b)\right|\left[g_2\left(\gamma\left(\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}\right\}.$$

3. Application to Special Means

Let us recall the following special means of two nonnegative number a,b with $b>a$:

1. The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2}.$$

2. The Geometric mean

$$G=G(a,b):=\sqrt{ab}.$$

3. The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

4. The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 1. Let $0 < a < b \le 1$ and $q \ge 1$. Then for $n > 1$

$$\left|G(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right| \le \left(\frac{(b-a)(n+1)q}{4L(a,b)}\right)^{1-\frac{1}{q}} \frac{1}{2q}$$
$$\times \left\{ a^{\frac{n+1}{2}} \left[b^{\frac{(n+1)q}{2}} - L_{\frac{(n+1)q}{2}-1}^{\frac{(n+1)q}{2}-1}(a,b)L(a,b) \right]^{\frac{1}{q}} \right.$$
$$\left. + b^{\frac{n+1}{2}} \left[L_{\frac{(n+1)q}{2}-1}^{\frac{(n+1)q}{2}-1}(a,b)L(a,b) - a^{\frac{(n+1)q}{2}} \right]^{\frac{1}{q}} \right\},$$

$$\left|A(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right| \le \left(\frac{(b-a)(n+1)q}{2L(a,b)}\right)^{1-\frac{1}{q}} \frac{1}{2q}$$
$$\times \left\{ \left[b^{(n+1)q} - L_{(n+1)q-1}^{(n+1)q-1}(a,b)L(a,b) \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[L_{(n+1)q-1}^{(n+1)q-1}(a,b)L(a,b) - a^{(n+1)q} \right]^{\frac{1}{q}} \right\}.$$

In particular, if $q = 1$, one has

$$\left|G(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right|$$
$$\le \frac{1}{2}\left\{ a^{\frac{n+1}{2}} \left[b^{\frac{n+1}{2}} - L_{\frac{n-1}{2}}^{\frac{n-1}{2}}(a,b)L(a,b) \right] \right.$$
$$\left. + b^{\frac{n+1}{2}} \left[L_{\frac{n-1}{2}}^{\frac{n-1}{2}}(a,b)L(a,b) - a^{\frac{n+1}{2}} \right] \right\},$$

$$\left|A(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right| \le \frac{b^{n+1} - a^{n+1}}{2}.$$

Proof. The assertion follows from the inequalities (4) and (5) in Theorem 2 for $f(x) = x^{n+1}$, $x > 0$, $n > 1$.

Proposition 2. Let $0 < a < b \le 1$ and $q > 1$. Then for $n > 1$

$$\left|G(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right|$$
$$\le \frac{(n+1)(b-a)}{2}\left(\frac{q-1}{(2q-1)L(a,b)}\right)^{1-\frac{1}{q}}$$
$$\times A\left(a^{\frac{n+1}{2}}, b^{\frac{n+1}{2}}\right) L_{\frac{n+1}{2}-\frac{1}{q}}^{\frac{n+1}{2}-\frac{1}{q}}(a,b)$$

$$\left|A(a^{n+1}, b^{n+1}) - L_n^n(a,b)L(a,b)\right|$$
$$\le (n+1)(b-a)\left(\frac{q-1}{(2q-1)L(a,b)}\right)^{1-\frac{1}{q}} L_{n+1-\frac{1}{q}}^{n+1-\frac{1}{q}}(a,b).$$

Proof. The assertion follows from the inequalities (8) and (9) in Theorem 3 for $f(x) = x^{n+1}$, $x > 0$, $n > 1$.

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