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The Leibniz Rule for Fractional Derivatives Holds with Non-Differentiable Functions

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Abstract In order to convince the sceptical reader, we herein give another proof of the fact that the Leibniz rule $D^\alpha(uv) = (D^\alpha u)v + u(D^\alpha v)$ for fractional derivatives applies whenever we are dealing with non-differentiable functions, as they occur for instance, when one considers problems involving fractal space-time

Keywords Fractional Derivative, Fractional Difference, Leibniz Rule, Fractional Hadamard Theorem

1. Introduction

In the following, we shall refer to the definition of the fractional derivative $f^{(\alpha)}(x) \equiv D^\alpha f(x)$, $0 < \alpha < 1$, of a $\Re \to \Re$ function $f(x)$, given by the expression [1-2]

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \left(\Delta^\alpha f(x) / h^\alpha \right)$$

with (the fractional difference)

$$\Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\big(x + (\alpha - k)h\big),$$

and from where we can derive the alternative

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} \big(f(\xi) - f(0)\big) d\xi, \quad 0 < \alpha < 1,$$

$$= DD^{\alpha-1} f(x), \quad \alpha > 1$$

In this framework, the fractional integral is defined by the equality [1-2]

$$\int_0^x u(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} u(\xi) d\xi,$$

which yields

$$\int_0^x u^{(\alpha)}(\xi)(d\xi)^\alpha = \alpha! \big(u(x) - u(0)\big)$$

We shall use also the derivative chain rule [2]

$$D_x^\alpha f\big(u(x)\big) = D_u^\alpha \big(D_x u(x)\big)^\alpha$$

which is exactly the equality (in terms of increments)

$$\frac{d^\alpha f\big(u(x)\big)}{(dx)^\alpha} = \frac{d^\alpha f\big(u(x)\big)}{(du)^\alpha} \frac{(du)^\alpha}{(dx)^\alpha}$$

For further reading on fractional derivative, see for instance [3-6]. This being the case, it is by now well established and taken for granted that the Leibniz rule $D(uv) = (Du)v + u(Dv)$ is to be generalized in the fractional form

$$D^\alpha(uv) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} u\right)\left(D^k v\right)$$

to deal with fractional derivatives, but at first glance, it nevertheless gives rise to two remarks. First, do we have $D^\alpha(uv) = D^\alpha(vu)$? Second, what happens when both u and v are not differentiable? We shall try to answer these questions in the following.

2. Generalized Hadamard's Theorem

We denote by $C^{m\alpha}(U)$ the space of functions $f(x)$ which are m times αth differentiable on $U \subset \Re$

Generalized Hadamard's theorem *Any function $f(x) \in C^\alpha(U)$ in a neighborhood of a point x_0 can be decomposed in the form*

$$f(x) = f(x_0) + \frac{(x-x_0)^\alpha}{\alpha!} g(x), \tag{2}$$

where $g(x) \in C^{m\alpha}$ and $\alpha! := \Gamma(1+\alpha)$

Proof. Let us define the function

$$\varphi(t) := f\left(x_0 + (x - x_0)t\right),$$

which provides $\varphi(0) = f(x_0)$ and $\varphi(1) = f(x)$. We then have

$$\varphi(1) - \varphi(0) = (\alpha!)^{-1} \int_0^1 \varphi_t^{(\alpha)}(t)(dt)^\alpha$$

$$= (\alpha!)^{-1} \int_0^1 \varphi_x^{(\alpha)}(x - x_0)^\alpha (dt)^\alpha$$

$$= (\alpha!)^{-1}(x - x_0)^\alpha \int_0^1 \varphi_x^{(\alpha)}(dt)^\alpha$$

$$= (\alpha!)^{-1}(x - x_0)^\alpha g(x) \qquad (3)$$

If we once more apply (2) to $g(x)$ in (3) we eventually obtain the expansion

$$f(x) = f(x_0) + \frac{(x - x_0)^\alpha}{\alpha!} g_1(x_0) + \frac{(x - x_0)^{2\alpha}}{(\alpha!)^2} g_2(x) \qquad (4)$$

3. Application to Fractional Taylor Series of First Order

Corollary As a result of the generalized Hadamard's theorem, one has as well the first order approximation

$$f(x) = f(x_0) + \frac{(x - x_0)^\alpha}{\alpha!} f^{(\alpha)}(x_0) + o(h^{2\alpha}) .$$

where $o(h^{2\alpha})$ denotes the Landau's symbol and h is the increment $(x - x_0)$.

Proof. (1) and (4) yield, with $h := x - x_0$

$$\frac{\Delta^\alpha f(x_0)}{h^\alpha} = f^{(\alpha)}(x_0) + o_1(h^\alpha),$$

$$\frac{\alpha! \Delta f(x_0)}{h^\alpha} = g_1(x_0) + o_2(h^\alpha)$$

whereby we obtain the equality

$$\frac{\Delta^\alpha f(x_0)}{f^{(\alpha)}(x_0) + o_1(h^{2\alpha})} = \frac{\alpha! \Delta f(x_0) - o_2(h^{2\alpha})}{g_1(x_0)}$$

which provides

$$\Delta^\alpha f(x_0) = \alpha! \Delta f(x_0) - o_2(h^{2\alpha}) \qquad (5)$$

and

$$g_1(x_0) = f^{(\alpha)}(x_0) + o_1(h^{2\alpha})$$

We so once more obtain, in a very simple way, the first term of the fractional Taylor's series which we obtained previously by using fractional differences [1]

4. Application to the Leibniz Derivation Rule

We can now state the **Leibniz derivation rule for non-differentiable functions** Let $u(x)$ and $v(x)$ denote two continuous non-differentiable functions which have each one a fractional derivative of order α, $0 < \alpha < 1$. Then the fractional derivative of $u(x)v(x)$ is provided by the equality

$$D_x^\alpha \left(u(x)v(x)\right) = \left(D_x^\alpha u\right)v(x) + u(x)\left(D_x^\alpha v(x)\right)$$

Proof: We need the expression of the fractional difference $\Delta^\alpha(uv)$, to apply (1), and in order to get it, we shall refer to the formula (5) which involves $\Delta(uv)$. So, in order to obtain the latter, we shall proceed as follows. We write

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv$$

$$= (\Delta u)v + u(\Delta v) + o(h^{2\alpha}),$$

therefore (by virtue of (5))

$$\alpha! \Delta(uv) = \alpha!(\Delta u)v + u\alpha!(\Delta v) + o(h^{2\alpha}),$$

that is to say

$$\Delta^\alpha(uv) = (\Delta^\alpha u)v + u(\Delta^\alpha v) + o(h^{2\alpha})$$

on dividing both sides by h^α and making h tends to zero yields the result

Example. Assume that

$$u(x) = x^\alpha, \quad x \geq 0,$$

while $v(x)$ is the Mittag-Leffler function

$$v(x) = E_\alpha(x^\alpha), \quad x \geq 0$$

$u(x)v(x)$, $u(x)$ and $v(x)$ are not differentiable at $x = 0$, and only at $x = 0$, in such a manner that the fractional Leibniz rule should apply at this point (only), whilst, elsewhere, we are entitled to use the standard Leibniz formula. Let us check.

First, one has the series

$$u(x)v(x) = x^\alpha \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{(\alpha k)!}$$

of which the term-by;term derivative yields

$$(D^\alpha uv)_{x=0} = \alpha! \qquad (6)$$

Next, the fractional Leibniz formula provides

$$D^{\alpha}(uv) = (\alpha ! + x^{\alpha}) E_{\alpha}(x^{\alpha}) \quad (7)$$

which is equal to $\alpha!$ when $x = 0$.

5. Concluding Remarks

In words, the above result can be translated as follows: if we cannot use the standard Leibniz rule because there is no standard derivative, then we can just try again by substituting fractional derivative for derivative in the standard formula. All this rationale comes from the fact that if a function is not differentiable at $x = c$, but has a fractional derivative of order α at this point, then it is locally equivalent to the function

$$f(x) = f(c) + \frac{(x-c)^{\alpha}}{\alpha!} f^{(\alpha)}(c) + o(h^{2\alpha}), \quad (8)$$

n which case the fractional Leibniz rule applies when $x = c$ only.

The key of the result is the formula (5)

$$\Delta^{\alpha} f(x_0) = \alpha! \Delta f(x_0) - o_2(h^{2\alpha}), \quad (9)$$

which is exactly the first term of our fractional Taylor's series [2].

Some authors use (6) to define the fractional difference by means of the standard difference, but we think that this approach is very questionable on a logical standpoint, and that at first glance, it rather looks like a definition for conveniency only. Of course, one can define the fractional difference as a fraction of the standard difference, but why $\alpha!$ instead of any $\psi(\alpha)$!

There remains now to define functions which are non-differentiable everywhere instead of on some given points only as above. Of course, to this end, one can refer to the by-now classical contrivance of Kolwankar an Gangal [3] who construct their theory on the Cantor space. In other words, the problem is fractal because it is defined on a fractal space. And this point of view is fully consistent with the self-similarity of these fractal processes. Basically, the Cantor set is fractal. An alternative is to start from the remark that a function which is continuous everywhere but is nowhere differentiable exhibits random-like feature in the sense that it is impossible to duplicate the registration of this process on different intervals of definition. In other words, one could randomize the problem and consider the use of Gaussian white noise for instance.

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