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Inequalities for the Polar Derivative of A Polynomial

M.S. Pukhta

Division of Agri. Statistics, Sher-e-Kashmir University of Agricultural Sciences and Technology of Kashmir, 191121, India
*Corresponding Author: mspukhta_67@yahoo.co.in

Abstract If $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, has all its zeros on $|z| = k$, $k \le 1$, then it was recently proved by Dewan and Ahuja [3] that for every real or complex number α with $|\alpha| \le k$.

$$\max_{|z|=1} |D_\alpha P(z)| \le \frac{n(|\alpha| + s_\mu)s_\mu}{k^{n-\mu+1}(1+s_\mu)} \max_{|z|=1} |P(z)|,$$

where $s_\mu = \dfrac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$.
In this paper, we improve the above result and obtain new inequality for the polar derivative of a polynomial.

Keywords Polynomials, zeros, Maximum modulus, Polar derivative.

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1 Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ denotes its derivative. It was shown by Turan [8] that if $P(z)$ has all its zeros in $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

As an extension of (1.1), Malik [9] proved that if $P(z)$ has all its zeros in $|z| \le k$, $k \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.2}$$

As a refinement of (1.2) it was shown by Govil [7] that if $P(z)$ has all its zeros in $|z| \le k$, $k = 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |P(z)| \right\}. \tag{1.3}$$

Aziz and Shah [1] generalized (1.3) in a different direction and proved that if $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |P(z)| \right\}. \tag{1.4}$$

Let α be any complex number. If $P(z)$ is a polynomial of degree n, then the polar derivative of $P(z)$ with respect to the point α, denoted by $D_\alpha P(z)$, is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

The polynomial $D_\alpha P(z)$ is of degree atmost $n-1$ and it generalize the ordinary derivative in this sense that

$$\lim_{\alpha \to \infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} = P'(z).$$

Dewan, Singh and Lal [4] extended inequality (1.4) to the polar derivative of a polynomial and proved that if all the zeros of $P(z) = a_n z^n = \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, we have $1 \le \mu \le n$, lie in $|z| = k$, $k \le 1$, then for every real or complex number a with $|\alpha| \ge k$

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n(|\alpha| - k^\mu)}{1+k^\mu} \max_{|z|=1} |P(z)| + \frac{n(|\alpha|+1)}{k^{n-\mu}(1+k^\mu)} \min_{|z|=k} |P(z)|. \tag{1.5}$$

If we divide both sides of (1.5) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (1.4).

On the other hand, Dewan and Ahuja [3] recently proved for the class of polynomials having zeros on the boundary of a circle that

Theorem 1. *If $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, has all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \le \frac{n(|\alpha| + s_\mu)}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)| \tag{1.6}$$

where

$$s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$$

In this paper we shall prove the following result which is an improvement of Dewan and Ahuja [3]. The bound obtained by our Theorem gives a better bound than the bound of Theorem A.

Theorem 2. *If* $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \leq \mu \leq n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leq 1$, *then for every real or complex number* α *with* $|\alpha| = k$, *we have*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n(|\alpha| + s_\mu)s_\mu}{k^{n-\mu+1}(1 + s_\mu)} \max_{|z|=1} |P(z)|, \qquad (1.7)$$

where

$$s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|} \qquad (1.8)$$

To prove that the bound obtained in the above theorem is better than the bound obtained in Theorem A, we have show that

$$\frac{s_\mu}{k^{n-\mu+1}(1 + s_\mu)} \leq \frac{1}{k^{n-2\mu+1} + k^{n-\mu+1}}$$

which is equivalent to

$$s_\mu \left(1 + \frac{1}{k^\mu}\right) \leq 1 + s_\mu$$

which implies

$$s_\mu \leq k^\mu$$

or

$$\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|} \leq k^\mu$$

As $k \leq 1$, this implies

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leq k^\mu$$

which is always true (see Lemma 4).

Remark. If we divide both sides of (1.7) by $|\alpha|$ and let $|\alpha| \to \infty$, we get

$$\max_{|z|=1} |P'(z)| \leq \frac{ns_\mu}{k^{n-\mu+1}(1 + s_\mu)} \max_{|z|=1} |P(z)|, \qquad (1.9)$$

where s_μ is defined by (1.8).

2 Lemmas

For the proof of theorem, we need the following lemmas. The Lemma 1 is due to Aziz and Shah [1].

Lemma 1. *If* $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \leq \mu \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then on* $|z| = 1$

$$|q'(z)| \leq k^\mu |P'(z)| \qquad (2.1)$$

where here and throughout this paper $q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Lemma 2. *If* $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then for every real or complex number* α *with* $|\alpha| \geq k^\mu$, *we have*

$$|D_\alpha P(z)| \geq n\left(\frac{|\alpha| - k^\mu}{1 + k^\mu}\right)|P(z)|, \quad \text{for } |z| = 1. \qquad (2.2)$$

Proof of Lemma 2. Since $q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, we have

$$q'(z) = nz^{n-1}\overline{P'\left(\frac{1}{\bar{z}}\right)} - z^{n-2}\overline{P'\left(\frac{1}{\bar{z}}\right)}.$$

Therefore, for $z = e^{i\theta}$; $0 \leq \theta < 2\pi$ we have

$$q'(e^{i\theta}) = ne^{i(n-1)\theta}\overline{P(e^{i\theta})} - e^{i(n-2)\theta}\overline{P'(e^{i\theta})},$$

which gives for $|z| = 1$,

$$\begin{aligned} |q'(z)| &= |nP(z) - zP(z) - zP'(z)| \\ &\geq n|P(z)| - |P'(z)|, \end{aligned}$$

or

$$|P'(z)| + |q'(z)| \geq n|P(z)|, \quad \text{for } |z| = 1. \qquad (2.3)$$

On combining (2.1) and (2.3), we obtain

$$\begin{aligned} (1 + k^\mu)P'(z)| &\geq |P'(z)| + |q'(z)| \\ &\geq n|P(z) \end{aligned}$$

or

$$|P'(z)| \geq \left(\frac{n}{1 + k^\mu}\right)|P(z)|, \quad \text{for } |z| = 1. \qquad (2.4)$$

Now for every real or complex number α, with $|\alpha| \geq k^\mu$, the polar derivative of $P(z)$ with respect to α is

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

This implies for $|z| = 1$,

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &\geq |\alpha||P'(z)| - |q'(z)|. \end{aligned} \qquad (2.5)$$

Inequality (2.5) when combined with (2.1) of Lemma 1, gives (2.6)

$$|D_\alpha P(z)| \geq (|\alpha| - |k^\mu|)|P'(z)|, \quad \text{for } |z| = 1. \qquad (2.6)$$

The above inequality (2.6) in conjunction with (2.4) gives

$$|D_\alpha P(z)| \geq n\left(\frac{|\alpha| - k^\mu}{1 + k^\mu}\right)|P(z)|, \quad \text{for } |z| = 1.$$

which proves the Lemma. □

Lemma 3. *If* $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \leq \mu < n$, *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \leq 1$, *then for* $|z| = 1$,

$$k^{n-\mu+1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |q'(z)|. \qquad (2.7)$$

The above lemma is due to Dewan and Hans [5].

Lemma 4. *If* $P(z) = a_n z^n + \sum_{v=u}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$ *then for* $|z| = 1$,

$$|q'(z)| \le s_\mu |P'(z)| \tag{2.8}$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \le k^\mu, \tag{2.9}$$

where s_μ *is defined by* (1.8)*).*

The above lemma is due to Aziz and Rather [2].

Lemma 5. *If* $P(z)$ *is a polynomial of degree* n*, then for* $|z| = 1$.

$$|P'(z)| + |q'(z)| \le n \max_{|z|=1} |P(z)|. \tag{2.10}$$

The above lemma is a special case of a result due to Govil and Rahman [6].

3 Proof of Theorem

Proof of Theorem. Let z_0 be a point on $|z| = 1$ such that $|q'(z_0)| = \max_{|z|=1} |q'(z)|$, then by Lemma 5, we get

$$|P'(z_0)| + \max_{|z|=1} |q'(z)| \le n \max_{|z|=1} |P(z)|. \tag{3.1}$$

Combining the above inequality (3.1) with inequality (2.8) of Lemma 4, we have

$$\frac{1}{s_\mu}|q'(z_0)| + \max_{|z|=1} |q'(z)| \le n \max_{|z|=1} |P(z)|,$$

which is equivalent to

$$\left(\frac{1}{s_\mu} + 1\right) \max_{|z|=1} |q'(z)| \le n \max_{|z|=1} |P(z)|. \tag{3.2}$$

The above inequality (3.2) when combined with Lemma 3, gives

$$k^{n-\mu+1}\left(\frac{1}{s_\mu} + 1\right) \max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z).$$

which implies

$$\max_{|z|=1} |P'(z)| \le \frac{n s_\mu}{k^{n-\mu+1}(1 + s_\mu)} \max_{|z|=1} |P(z).\tag{3.3}$$

Now if $q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then as in the proof of Lemma 2, it easily follows that

$$|q'(z)| = nP(z) + (\alpha - z)P'(z).$$

Also for every real or complex number α with $|\alpha| \ge k$, we have

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

This implies with the help of Lemma 4 that for $|z| = 1$,

$$\begin{aligned} |D_\alpha P(z)| &\le |\alpha||P'(z)| + |nP(z) - zP'(z)| \\ &= |\alpha||P'(z)| + |q'(z)| \\ &\le (|\alpha| + s_\mu)|P'(z)|. \end{aligned} \tag{3.4}$$

Inequality (3.4) in conjunction with inequality (3.3) gives

$$\max_{|z|=1} |D_\alpha P(z)| \le \frac{n(|\alpha| + s_\mu)s_\mu}{k^{n-\mu+1}(1 + s_\mu)} \max_{|z|=1} |P(z)|,$$

This completes the proof of main theorem. □

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