

# On Decay of Solutions to Systems of Integro-differential Equations with Strongly Damped

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**Abstract** We study the system of nonlinear integro-differential equations with strong damping and weak damping terms, in a bounded domain with the initial and Dirichlet boundary conditions. The existence of global solutions by using the potential well method, and the energy decay estimate by applying a lemma of Komornik [3]

**Keywords** Decay, Integro-differential Equations, Strong Damping

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## 1 Introduction

In this paper we consider the following initial-boundary value problem

$$\left\{ \begin{array}{l} u_{tt} - \Delta u + \int_0^t g_1(t-\tau) \Delta u(\tau) d\tau - \Delta u_t - \Delta u_{tt} + u_t = f_1(u, v), \quad (x, t) \in \Omega \times (0, T), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau) \Delta v(\tau) d\tau - \Delta v_t - \Delta v_{tt} + v_t = f_2(u, v), \quad (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, \quad x \in \partial\Omega, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \end{array} \right. \quad (1)$$

where  $\Omega$  is a bounded domain with smooth boundary  $\partial\Omega$  in  $R^n$ ;  $f_i(\cdot, \cdot) : R^2 \rightarrow R$  are given functions to be specified later.

A single wave equation of the problem (1) becomes as following

$$u_{tt} - \Delta u + \int_0^t g_1(t-\tau) \Delta u(\tau) d\tau - \Delta u_t - \Delta u_{tt} + u_t = |u|^{r-1} u. \quad (2)$$

The global existence and blow up of solution for (2) were established [7]. In the absence of the dispersive term  $\Delta u_{tt}$  and the weak damping term  $u_t$ , were established [8]. In the absence of the dispersive term  $\Delta u_{tt}$  and the strong damping term  $\Delta u_t$ , were established [11, 9]. Also, Liang and Gao [12] studied the global existence, decay and blow up of solution problem (2) with the absence of the dispersive term  $\Delta u_{tt}$ .

Liang and Gao [10] studied the global existence, decay and blow up of solution problem (1) with the absence of the dispersive term  $\Delta u_{tt}$  and the weak damping term  $u_t$ .

In this paper, under some restrictions on the initial data, we establish the global existence and the decay of solutions.

This paper is organized as follows. In section 2, we present some lemmas, and the local existence theorem. In section 3, the global existence and energy decay of the solution are given.

## 2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this work. Let  $\|\cdot\|$  and  $\|\cdot\|_p$  denote the usual  $L^2(\Omega)$  norm and  $L^p(\Omega)$  norm, respectively. First, we make the following assumptions:

(A1)  $g_i(t) : R^+ \rightarrow R^+$  belong to  $C^1(R^+)$  and satisfy

$$g_i(t) \geq 0, \quad g'_i(t) \leq 0, \quad \text{for } t \geq 0$$

and

$$1 - \int_0^\infty g_i(s) ds = l_i > 0.$$

Concerning the functions  $f_1(u, v)$  and  $f_2(u, v)$ , we take

$$\begin{aligned} f_1(u, v) &= \left[ k |u + v|^{2(r+1)} (u + v) + l |u|^r u |v|^{r+2} \right], \\ f_2(u, v) &= \left[ k |u + v|^{2(r+1)} (u + v) + l |u|^{r+2} |v|^r v \right], \end{aligned}$$

where  $k, l > 0$  are constants and  $r$  satisfies

$$\begin{cases} -1 < r & \text{if } n = 1, 2, \\ -1 < r \leq \frac{3-n}{n-2} & \text{if } n \geq 3. \end{cases} \quad (3)$$

According to the above equalities they can easily verify that

$$u f_1(u, v) + v f_2(u, v) = 2(r+2) F(u, v), \quad \forall (u, v) \in R^2, \quad (4)$$

where

$$F(u, v) = \frac{1}{2(r+2)} \left[ k |u + v|^{2(r+2)} + 2l |uv|^{r+2} \right]. \quad (5)$$

We have the following result.

**Lemma 1** [5]. *There exist two positive constants  $c_0$  and  $c_1$  such that*

$$c_0 \left( |u|^{2(r+2)} + |v|^{2(r+2)} \right) \leq 2(r+2) F(u, v) \leq c_1 \left( |u|^{2(r+2)} + |v|^{2(r+2)} \right) \quad (6)$$

is satisfied.

Let's define

$$\begin{aligned} J(t) &= \frac{1}{2} \left[ \left( 1 - \int_0^t g_1(\tau) d\tau \right) \|\nabla u\|^2 + \left( 1 - \int_0^t g_2(\tau) d\tau \right) \|\nabla v\|^2 \right] \\ &\quad + \frac{1}{2} [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] - \int_\Omega F(u, v) dx, \end{aligned} \quad (7)$$

$$\begin{aligned}
 I(t) &= \left(1 - \int_0^t g_1(\tau) d\tau\right) \|\nabla u\|^2 + \left(1 - \int_0^t g_2(\tau) d\tau\right) \|\nabla v\|^2 \\
 &\quad + [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] - 2(r+2) \int_{\Omega} F(u, v) dx,
 \end{aligned}
 \tag{8}$$

and also the energy function as follows

$$\begin{aligned}
 E(t) &= \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2) + \frac{1}{2} (\|\nabla u_t\|^2 + \|\nabla v_t\|^2) \\
 &\quad + \frac{1}{2} \left[ \left(1 - \int_0^t g_1(\tau) d\tau\right) \|\nabla u\|^2 + \left(1 - \int_0^t g_2(\tau) d\tau\right) \|\nabla v\|^2 \right] \\
 &\quad + \frac{1}{2} [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] - \int_{\Omega} F(u, v) dx,
 \end{aligned}
 \tag{9}$$

where

$$(g_i \circ w)(t) = \int_0^t g_i(t-\tau) \|w(t) - w(\tau)\|^2 d\tau.$$

**Lemma 2**  $E(t)$  is a nonincreasing function for  $t \geq 0$  and

$$\begin{aligned}
 E'(t) &= -(\|u_t\|^2 + \|v_t\|^2) - (\|\nabla u_t\|^2 + \|\nabla v_t\|^2) \\
 &\quad + \frac{1}{2} [(g'_1 \circ \nabla u)(t) + (g'_2 \circ \nabla v)] - \frac{1}{2} [g_1(t) \|\nabla u\|^2 + g_2(t) \|\nabla v\|^2] \\
 &\leq 0.
 \end{aligned}
 \tag{10}$$

**Proof.** The proof is almost the same that of [5], so omit it here. ■

Moreover, the following energy inequality holds:

$$E(t) + \int_s^t (\|u_{\tau}\|^2 + \|v_{\tau}\|^2 + \|\nabla u_{\tau}\|^2 + \|\nabla v_{\tau}\|^2) d\tau \leq E(s), \text{ for } 0 \leq s \leq t < T.$$

**Lemma 3** (Sobolev-Poincare inequality) [1]. Let  $p$  be a number with  $2 \leq p < \infty$  ( $n = 1, 2$ ) or  $2 \leq p \leq 2n/(n-2)$  ( $n \geq 3$ ), then there is a constant  $C_* = C_*(\Omega, p)$  such that

$$\|u\|_p \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega).$$

The following integral inequality plays an important role in our proof of the energy decay of the solutions to problem (1).

**Lemma 4** [3]. Let  $h : [0, \infty) \rightarrow [0, \infty)$  be a nonincreasing function and assume that there exists a constant  $c > 0$  such that

$$\int_t^{\infty} h(\tau) d\tau \leq ch(t), \quad \forall t \in [0, \infty).$$

Then we have

$$h(t) \leq h(0) e^{1-tc^{-1}}, \quad \forall t \geq c.$$

Next, we state the local existence theorem that can be established by combining arguments of [2, 4].

**Theorem 5** (Local existence). Suppose that (3) holds, and further  $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ ,  $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ . Then problem (1) has a unique local solution

$$u, v \in C([0, T]; H_0^1(\Omega)) \text{ and } u_t, v_t \in C([0, T]; H_0^1(\Omega)).$$

Moreover, at least one of the following statements holds true:

- i)  $T = \infty$ ,
- ii)  $\|u_t\|^2 + \|v_t\|^2 + l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \rightarrow \infty$  as  $t \rightarrow T^-$ .

**Remark 6** We denote by  $C$  various positive constants which may be different at different occurrences.

### 3 Global existence and energy decay

In this section, we consider the global existence and energy decay of solutions for problem (1).

**Lemma 7** [5]. *Suppose that (3) holds. Then there exists  $\eta > 0$  such that for any  $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ , the inequality*

$$\|u + v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \leq \eta \left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right)^{r+2} \quad (11)$$

is satisfied.

For the sake of simplicity and to prove our result, we take  $k = l = 1$  and introduce

$$B = \eta^{\frac{1}{2(r+2)}}, \quad \alpha_* = B^{-\frac{r+2}{r+1}}, \quad E_1 = \left( \frac{1}{2} - \frac{1}{2(r+2)} \right) \alpha_*^2, \quad (12)$$

where  $\eta$  is the optimal constant in (11). Next, we will state and prove a lemma which is similar to the one introduced firstly by Vitillaro in [6] to study a class of a single wave equation.

**Lemma 8** *Suppose that (3) holds. Let  $(u, v)$  be the solution of system (1). Assume further that  $E(0) < E_1$  and*

$$\left( l_1 \|\nabla u_0\|^2 + l_2 \|\nabla v_0\|^2 \right)^{\frac{1}{2}} < \alpha_*. \quad (13)$$

Then

$$\left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right)^{\frac{1}{2}} < \alpha_*, \quad (14)$$

for all  $t \in [0, T)$ .

**Proof.** First from (8), (9), (11) and the definition of  $B$ , we have

$$\begin{aligned} E(t) &\geq \frac{1}{2} \left[ l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right] - \int_{\Omega} F(u, v) dx \\ &= \frac{1}{2} \left[ l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right] \\ &\quad - \frac{1}{2(r+2)} \left( \|u + v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \right) \\ &\geq \frac{1}{2} \left[ l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right] \\ &\quad - \frac{1}{2(r+2)} \eta \left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right)^{r+2} \\ &\geq \frac{1}{2} \left[ l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right] \\ &\quad - \frac{B^{2(r+2)}}{2(r+2)} \left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right)^{r+2}. \end{aligned} \quad (15)$$

So, we get

$$E(t) \geq G \left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) \text{ for } t \geq 0, \quad (16)$$

where  $G(\alpha) = \frac{1}{2}\alpha^2 - \frac{B^{2(r+2)}}{2(r+2)}\alpha^{2(r+2)}$ . Note that  $G(\alpha)$  has the maximum at  $\alpha_* = B^{-\frac{r+2}{r+1}}$  and maximum value is

$$E_1 = G(\alpha_*) = \left( \frac{1}{2} - \frac{1}{2(r+2)} \right) \alpha_*^2. \quad (17)$$

Now we establish (14) by contradiction. Suppose (14) does not hold, then it follows from the continuity of  $(u(t), v(t))$  that there exists  $t_0 \in (0, T)$  such that

$$\left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right)^{\frac{1}{2}} = \alpha_*. \quad (18)$$

By (15), we see that

$$E(t_0) \geq G \left[ \left( l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2 + (g_1 \circ \nabla u)(t_0) + (g_2 \circ \nabla v)(t_0) \right)^{\frac{1}{2}} \right] = G(\alpha_*) = E_1. \quad (19)$$

This is impossible since  $E(t) \leq E(0) < E_1$ ,  $t \geq 0$ . Hence (14) is established. ■

**Theorem 9** (Global existence and energy decay). Assume that (3) hold. If the initial data  $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ ,  $(v_0, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ , satisfy  $E(0) < E_1$  and

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} < \alpha_*, \tag{20}$$

where the constants  $\alpha_*$  and  $E_1$  are defined in (12), then the corresponding solution to system (1) globally exists, i.e.,  $T = \infty$ .

Moreover, if

$$1 - \eta \left(\frac{2(r+2)}{r+1} E(0)\right)^{r+1} - \frac{5(1-l)(r+2)}{2l(r+1)} > 0, \tag{21}$$

then we have the following decay estimates

$$E(t) \leq E(0) e^{1-\mu C_7^{-1}t} \tag{22}$$

for every  $t \geq \frac{C_7}{\mu}$ , where  $C_7$  is positive constant.

**Proof.** First, we prove that  $T = \infty$ . Since  $E(0) < E_1$  and

$$\left(l_1 \|\nabla u_0\|^2 + l_2 \|\nabla v_0\|^2\right)^{\frac{1}{2}} < \alpha_*,$$

it follows from Lemma 8 that

$$\begin{aligned} & l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \\ & \leq l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \\ & < \alpha_*^2 = \eta^{-\frac{1}{r+1}} \end{aligned}$$

which implies that

$$\begin{aligned} I(t) &= l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) - 2(r+2) \int_{\Omega} F(u, v) dx \\ &= l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 - \left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right) \\ &\geq l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 - \eta \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2\right)^{r+2} \geq 0 \end{aligned}$$

for  $t \in [0, T)$ . Furthermore, (7) and (8), we get

$$\begin{aligned} J(t) &= \frac{r+1}{2(r+2)} \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right) + \frac{1}{2(r+2)} I(t) \\ &\geq \frac{r+1}{2(r+2)} \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right). \end{aligned}$$

From (10) and  $E(t) \leq E(0)$ , we deduce that

$$\begin{aligned} l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) &\leq \frac{2(r+2)}{r+1} J(t) \\ &\leq \frac{2(r+2)}{r+1} E(t) \\ &\leq \frac{2(r+2)}{r+1} E(0) \end{aligned} \tag{23}$$

for  $t \in [0, T)$ . So it follows from (23) and (10)

$$\begin{aligned} & \frac{r+1}{2(r+2)} \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right) + \frac{1}{2} \left(\|u_t\|^2 + \|v_t\|^2\right) \\ & \leq J(t) + \frac{1}{2} \left(\|u_t\|^2 + \|v_t\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\ & = E(t) \leq E(0) < E_1, \quad \forall t \in [0, T) \end{aligned}$$

which implies

$$l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 + \|u_t\|^2 + \|v_t\|^2 < C' E_1, \quad (24)$$

where  $C' = \max\left\{\frac{1}{2}, \frac{2(r+2)}{r+1}\right\}$ . Then, by Theorem 5, we have the global existence result.

Next, we want to derive the decay rate of energy function for problem (1). Multiplying the first equation of system (1) by  $u$  and the second equation of system (1) by  $v$ , integrating them over  $\Omega \times [t_1, t_2]$  ( $0 \leq t_1 \leq t_2$ ), using integration by parts and summing up, we have

$$\begin{aligned} & \int_{\Omega} uu_t dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|u_t\|^2 dt + \int_{\Omega} vv_t dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|v_t\|^2 dt \\ & \int_{\Omega} \nabla u \nabla u_t dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|\nabla u_t\|^2 dt + \int_{\Omega} \nabla v \nabla v_t dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|\nabla v_t\|^2 dt \\ & + \int_{t_1}^{t_2} \|\nabla u\|^2 dt + \int_{t_1}^{t_2} \|\nabla v\|^2 dt \\ & + \int_{t_1}^{t_2} \int_{\Omega} uu_t dx dt + \int_{t_1}^{t_2} \int_{\Omega} vv_t dx dt + \int_{t_1}^{t_2} \int_{\Omega} \nabla u \nabla u_t dx dt + \int_{t_1}^{t_2} \int_{\Omega} \nabla v \nabla v_t dx dt \\ & + \int_0^t \int_{t_1}^{t_2} \int_{\Omega} g_1(t-\tau) \Delta u(\tau) d\tau u(t) dx dt + \int_0^t \int_{t_1}^{t_2} \int_{\Omega} g_2(t-\tau) \Delta v(\tau) d\tau v(t) dx dt \\ = & \int_{t_1}^{t_2} \int_{\Omega} [uf_1(u, v) + vf_2(u, v)] dx dt. \end{aligned}$$

It follows from (9)

$$\begin{aligned} & 2 \int_{t_1}^{t_2} E(t) dt - 2(r+1) \int_{t_1}^{t_2} \int_{\Omega} F(u, v) dx dt \\ = & - \int_{\Omega} (uu_t + vv_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} (\|u_t\|^2 + \|v_t\|^2) dt \\ & - \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} (\|\nabla u_t\|^2 + \|\nabla v_t\|^2) dt \\ & - \int_{t_1}^{t_2} \int_{\Omega} (uu_t + vv_t) dx dt + \int_{t_1}^{t_2} \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx dt \\ & + \int_{t_1}^{t_2} (g_1 \circ \nabla u)(t) dt + \int_{t_1}^{t_2} (g_2 \circ \nabla v)(t) dt \\ & - \int_{t_1}^{t_2} \int_0^t g_1(\tau) d\tau \|\nabla u(t)\| dt - \int_{t_1}^{t_2} \int_0^t g_2(\tau) d\tau \|\nabla v(t)\| dt \\ & - \int_0^t \int_{t_1}^{t_2} \int_{\Omega} g_1(t-\tau) \Delta u(\tau) d\tau u(t) dx dt \\ & - \int_0^t \int_{t_1}^{t_2} \int_{\Omega} g_2(t-\tau) \Delta v(\tau) d\tau v(t) dx dt. \end{aligned} \quad (25)$$

For the eleventh term on the right hand side of (25), we have

$$\begin{aligned} & - \int_{\Omega} \int_0^t g_1(t-\tau) \Delta u(\tau) u(t) d\tau dx \\ = & \int_{\Omega} \int_0^t g_1(t-\tau) \nabla u(\tau) \nabla u(t) d\tau dx \\ = & \frac{1}{2} \left[ \int_0^t g_1(t-\tau) (\|\nabla u(t)\|^2 + \|\nabla u(\tau)\|^2) d\tau - (g_1 \circ \nabla u)(t) \right], \end{aligned} \quad (26)$$

and similarly, we have

$$\begin{aligned} & - \int_{\Omega} \int_0^t g_2(t-\tau) \Delta v(\tau) v(t) d\tau dx \\ = & \int_{\Omega} \int_0^t g_2(t-\tau) \nabla v(\tau) \nabla v(t) d\tau dx \\ = & \frac{1}{2} \left[ \int_0^t g_2(t-\tau) (\|\nabla v(t)\|^2 + \|\nabla v(\tau)\|^2) d\tau - (g_2 \circ \nabla v)(t) \right]. \end{aligned} \quad (27)$$

Substituting (26), (27) into (25), we have

$$\begin{aligned}
 & 2 \int_{t_1}^{t_2} E(t) dt - 2(r+1) \int_{t_1}^{t_2} \int_{\Omega} F(u, v) dx dt \\
 = & - \int_{\Omega} (uu_t + vv_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} \left( \|u_t\|^2 + \|v_t\|^2 \right) dt \\
 & - \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) dt \\
 & - \int_{t_1}^{t_2} \int_{\Omega} (uu_t + vv_t) dx dt + \int_{t_1}^{t_2} \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx dt \\
 & + \frac{1}{2} \int_0^t g_1(t-\tau) \|\nabla u(\tau)\|^2 d\tau dt + \frac{1}{2} \int_0^t g_2(t-\tau) \|\nabla v(\tau)\|^2 d\tau dt \\
 & + \frac{1}{2} \int_{t_1}^{t_2} [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] dt \\
 & - \frac{1}{2} \int_{t_1}^{t_2} \left[ \int_0^t g_1(\tau) d\tau \|\nabla u(t)\|^2 + \int_0^t g_2(\tau) d\tau \|\nabla v(t)\|^2 \right] dt \\
 \leq & - \int_{\Omega} (uu_t + vv_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} \left( \|u_t\|^2 + \|v_t\|^2 \right) dt \\
 & - \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) dt \\
 & - \int_{t_1}^{t_2} \int_{\Omega} (uu_t + vv_t) dx dt + \int_{t_1}^{t_2} \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx dt \\
 & + \frac{1}{2} \int_{t_1}^{t_2} \left[ \int_0^t g_1(t-\tau) \|\nabla u(\tau)\|^2 d\tau + \int_0^t g_2(t-\tau) \|\nabla v(\tau)\|^2 d\tau \right] dt \\
 & + \frac{1}{2} \int_{t_1}^{t_2} [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] dt \\
 = & A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8. \tag{28}
 \end{aligned}$$

In what follows we will estimate  $A_1, A_2, \dots, A_8$  in (28). Firstly, by Hölder, Young and Sobolev Poincare inequalities, we have

$$\begin{aligned}
 \int_{\Omega} (|uu_t| + |vv_t|) dx & \leq \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|v_t(t)\|^2 \\
 & \leq \frac{C}{2} \|\nabla u(t)\|^2 + \frac{1}{2} \|u_t(t)\|^2 + \frac{C}{2} \|\nabla v(t)\|^2 + \frac{1}{2} \|v_t(t)\|^2.
 \end{aligned}$$

Then, by (24), we have

$$A_1 \leq \int_{\Omega} (|uu_t| + |vv_t|) dx \Big|_{t_1}^{t_2} \leq 2C_1 E(t_1). \tag{29}$$

For  $A_2$  in (28), applying  $\|u_t\|^2 + \|v_t\|^2 \leq -E'(t)$  from (10), we have

$$A_2 = 2 \int_{t_1}^{t_2} \left( \|u_t\|^2 + \|v_t\|^2 \right) dt \leq 2C_2 E(t_1). \tag{30}$$

Similarly, we have

$$A_3 \leq \int_{\Omega} (|\nabla u \nabla u_t| + |\nabla v \nabla v_t|) dx \Big|_{t_1}^{t_2} \leq 2C_3 E(t_1), \tag{31}$$

and

$$A_4 = 2 \int_{t_1}^{t_2} \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) dt \leq 2C_4 E(t_1). \tag{32}$$

We also have the following estimate

$$\begin{aligned}
A_5 &= \int_{t_1}^{t_2} \int_{\Omega} (uu_t + vv_t) dxdt \\
&= \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \|u\|^2 dt + \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \|v\|^2 dt \\
&= \frac{1}{2} \left( \|u(t_2)\|^2 - \|u(t_1)\|^2 \right) + \frac{1}{2} \left( \|v(t_2)\|^2 - \|v(t_1)\|^2 \right) \\
&\leq \frac{2(r+2)}{r+1} E(t_1) = C_5 E(t_1).
\end{aligned} \tag{33}$$

Similarly

$$A_6 = \int_{t_1}^{t_2} \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dxdt \leq C_6 E(t_1). \tag{34}$$

Using Young inequality for convolution  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ , we have

$$\begin{aligned}
A_7 &= \frac{1}{2} \int_{t_1}^{t_2} \left[ \int_0^t g_1(t-\tau) \|\nabla u(\tau)\|^2 d\tau + \int_0^t g_2(t-\tau) \|\nabla v(\tau)\|^2 d\tau \right] dt \\
&\leq \frac{1}{2} \int_{t_1}^{t_2} \left[ \|g_1 * \|\nabla u\|^2\|_1 + \|g_2 * \|\nabla v\|^2\|_1 \right] \\
&= \frac{1}{2} \int_{t_1}^{t_2} g_1(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\|^2 dt + \frac{1}{2} \int_{t_1}^{t_2} g_2(t) dt \int_{t_1}^{t_2} \|\nabla v(t)\|^2 dt \\
&\leq \frac{1}{2} (1-l_1) \int_{t_1}^{t_2} \|\nabla u(t)\|^2 dt + \frac{1}{2} (1-l_2) \int_{t_1}^{t_2} \|\nabla v(t)\|^2 dt \\
&\leq \frac{1-l}{2} \int_{t_1}^{t_2} \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) dt \\
&\leq \frac{(1-l)(r+2)}{l(r+1)} \int_{t_1}^{t_2} E(t) dt.
\end{aligned} \tag{35}$$

From (23), we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \left[ \int_0^t g_1(t-\tau) \|\nabla u(t)\|^2 d\tau + \int_0^t g_2(t-\tau) \|\nabla v(t)\|^2 d\tau \right] dt \\
&\leq (1-l_1) \int_{t_1}^{t_2} \|\nabla u(t)\|^2 dt + (1-l_2) \int_{t_1}^{t_2} \|\nabla v(t)\|^2 dt \\
&\leq (1-l) \int_{t_1}^{t_2} \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) dt \\
&\leq \frac{2(1-l)(r+2)}{l(r+1)} \int_{t_1}^{t_2} E(t) dt.
\end{aligned} \tag{36}$$

Combining (35) and (36), we have

$$\begin{aligned}
A_8 &= \frac{1}{2} \int_{t_1}^{t_2} [(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] dt \\
&\leq \frac{1}{2} \int_{t_1}^{t_2} \int_0^t g_1(t-\tau) \left( \|\nabla u(t)\|^2 + \|\nabla u(\tau)\|^2 \right) d\tau dt \\
&\quad + \frac{1}{2} \int_{t_1}^{t_2} \int_0^t g_2(t-\tau) \left( \|\nabla v(t)\|^2 + \|\nabla v(\tau)\|^2 \right) d\tau dt \\
&\leq \frac{4(1-l)(r+2)}{l(r+1)} \int_{t_1}^{t_2} E(t) dt.
\end{aligned} \tag{37}$$

Inserting estimates (29)-(37) into (28), we arrive at

$$2 \int_{t_1}^{t_2} E(t) dt - 2(r+1) \int_{t_1}^{t_2} \int_{\Omega} F(u, v) dxdt \leq C_7 E(t_1) + \frac{5(1-l)(r+2)}{l(r+1)} \int_{t_1}^{t_2} E(t) dt. \tag{38}$$

where  $C_7 = 2C_1 + 2C_2 + 2C_3 + 2C_4 + C_5 + C_6$ .



On the other hand, from (11) and (23), we have

$$\begin{aligned} 2(r+1) \int_{\Omega} F(u, v) dx &= \frac{r+1}{r+2} \left( \|u+v\|_{2(r+2)}^{2(r+2)} + 2 \|uv\|_{r+2}^{r+2} \right) \\ &\leq \frac{r+1}{r+2} \eta \left( l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right)^{r+2} \\ &\leq 2\eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} E(t) \end{aligned}$$

which implies

$$2 \int_{t_1}^{t_2} E(t) dt - 2(r+1) \int_{t_1}^{t_2} \int_{\Omega} F(u, v) dx dt \geq 2 \left( 1 - \eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} \right) \int_{t_1}^{t_2} E(t) dt. \tag{39}$$

Noting that  $E(0) < E_1$ , we see

$$1 - \eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} > 0.$$

Thus, combining (38) and (39), we have

$$2 \left( 1 - \eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} \right) \int_{t_1}^{t_2} E(t) dt \leq C_7 E(t_1) + \frac{5(1-l)(r+2)}{l(r+1)} \int_{t_1}^{t_2} E(t) dt,$$

that is

$$\begin{aligned} 2 \left( 1 - \eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} - \frac{5(1-l)(r+2)}{2l(r+1)} \right) \int_{t_1}^{t_2} E(t) dt &\leq C_7 E(t_1). \\ \implies \mu \int_{t_1}^{t_2} E(t) dt &\leq C_7 E(t_1), \end{aligned} \tag{40}$$

where  $\mu = 2 \left( 1 - \eta \left( \frac{2(r+2)}{r+1} E(0) \right)^{r+1} - \frac{5(1-l)(r+2)}{2l(r+1)} \right)$ .

We rewrite (40) as

$$\mu \int_t^{\infty} E(t) dt \leq C_7 E(t)$$

for every  $t \in [0, \infty)$ .

Since  $\mu > 0$  from the assumption of conditions, by Lemma 4, we have

$$E(t) \leq E(0) e^{1-\mu C_7^{-1}t}$$

for every  $t \geq \frac{C_7}{\mu}$ . The proof is completed. ■

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