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A Unique Common Fixed Point Theorem of Meir-Keeler Type in a Partial Metric Space

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Abstract In this paper, we obtain a unique common fixed point theorem for four self mappings satisfying Meir-Keeler type contractive condition in partial metric spaces, which is slightly different from the result of Aydi and Karapinar [5].

Keywords Partial Metric, Weakly Compatible Maps, Complete Space

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1 Introduction and Preliminaries

The notion of partial metric space was introduced by Matthews [10] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science.

Matthews [10, 11] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map (see also [1, 3, 4, 6, 7, 8, 9]).

In this paper, we obtain a unique common fixed point theorem for four self mappings satisfying Meir-Keeler type contractive condition in partial metric spaces, which is slightly different from the result of Aydi and Karapinar [5].

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1 *(See [10, 11]) A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space (PMS).

In a partial metric space (X, p), it is clear that $p(x, y) = 0$ implies $x = y$ and $x \neq y$ implies $p(x, y) > 0$.
If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, is a metric on X.

Example 1.2 *(See e.g. [1, 7, 11]) Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.*

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x\in X, \varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X : p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [1, 2, 7, 9, 10, 11].)

Definition 1.3

1. *A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if* $p(x,x)=\lim_{n\to\infty} p(x,x_n)$.

2. *A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if* $\lim_{n,m\to\infty} p(x_n,x_m)$ *exists and is finite.*

3. *A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x\in X$ such that* $p(x,x)=\lim_{n,m\to\infty} p(x_n,x_m)$.

4. *A mapping $F:X\to X$ is said to be continuous at $x_0\in X$ if for every $\epsilon>0$, there exists $\delta>0$ such that* $F(B_p(x_0,\delta))\subseteq B_p(Fx_0,\epsilon)$.

We need the following lemmas in PMS([1, 2, 7, 9, 10, 11]).

Lemma 1.4

1. *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*

2. *A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover*

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m) \tag{1.1}$$

Lemma 1.5 *Assume $x_n\to z$ as $n\to\infty$ in a PMS (X,p) such that $p(z,z)=0$. Then $\lim_{n\to\infty} p(x_n,y)=p(z,y)$ for every $y\in X$.*

In recent year 2012, Aydi and Karapinar [5] proved the following.

Theorem 1.6 *(Corollary 2.4., [5])* **:** *Let $A,B,S,$ and T be the self maps defined on a partial metric space (X,p) satisfying the following conditions:*

(C1) $AX\subseteq TX$ and $BX\subseteq SX$,

(C2) for all $\epsilon>0$, there exists $\delta>0$ such that for all $x,y\in X$

$$\epsilon<M(x,y)<\epsilon+\delta \Rightarrow p(Ax,By)\le\epsilon, \quad \textit{where}$$

$$M(x,y)=\max\left\{p(Sx,Ty),p(Ax,Sx),p(By,Ty),\frac{1}{2}[p(Sx,By)+p(Ax,Ty)]\right\},$$

(C3) for all $x,y\in X$ with $M(x,y)>0\Rightarrow p(Ax,By)<M(x,y)$,

(C4) $p(Ax,By)<k[p(Sx,Ty)+p(Ax,Sx)+p(By,Ty)+p(Sx,By)+p(Ax,Ty]$ for all $x,y\in X$ and $0\le k<\frac{1}{3}$.

If one of $AX,BX,SX,$ or TX is a complete subspace of X, then

(I) A and S have a coincidence point,

(II) B and T have a coincidence point.

Moreover, if A and S, as well as, B and T are weakly compatible, then A,B,S and T have a unique common fixed point.

Remark 1.7 *We observed that in the proof of above theorem the authors inherently used the condition $0\le k<\frac{1}{4}$. (See the proof of our following main theorem).*

2 Main Result

Theorem 2.1 *Let (X,p) be a partial metric space and let $S,T,f,g : X \to X$ be satisfying*
(2.1.1) *for $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \max\left\{p(fx,gy),p(fx,Sx),p(gy,Ty),\frac{1}{2}\left[p(fx,Ty)+p(gy,Sx)\right]\right\} < \epsilon+\delta$$

implies $p(Sx,Ty) < \epsilon$ for all $x,y \in X$,
(2.1.2) *$S(X) \subseteq g(X), T(X) \subseteq f(X)$,*
(2.1.3) *either $f(X)$ or $g(X)$ is a complete subspace of X and*
(2.1.4) *the pairs (f,S) and (g,T) are weakly compatible.*
(2.1.5) $p(Sx,Ty) \leq k\left[\begin{array}{c} p(fx,gy)+p(fx,Sx)+p(gy,Ty) \\ +p(fx,Ty)+p(gy,Sx) \end{array}\right]$,
$\forall x,y \in X$ where $0 \leq k < \frac{1}{4}$.
Then S,T,f and g have a unique common fixed point in X.

Let $x_0 \in X$. From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= Sx_{2n} = gx_{2n+1}, \\ y_{2n+1} &= Tx_{2n+1} = fx_{2n+2}, n = 0,1,2,\cdots \end{aligned}$$

From (2.1.1), it is clear that

$$\begin{aligned} &p(Sx,Ty) < \max\left\{p(fx,gy),p(fx,Sx),p(gy,Ty),\tfrac{1}{2}\left[p(fx,Ty)+p(gy,Sx)\right]\right\}, \\ &\forall x,y \in X, \text{ whenever R. H. S is positive.} \qquad ------------(I) \end{aligned}$$

Case(a): Suppose $y_{2m+1} = y_{2m}$ for some m.
Now assume that $y_{2m+1} \neq y_{2m+2}$.
Then from (I),

$$\begin{aligned} p\left(y_{2m+1},y_{2m+2}\right) &= p\left(Sx_{2m+2},Tx_{2m+1}\right) \\ &< \max\left\{\begin{array}{c} p(y_{2m+1},y_{2m}),p(y_{2m+1},y_{2m+2}),p(y_{2m},y_{2m+1}), \\ \frac{1}{2}\left[p(y_{2m+1},y_{2m+1})+p(y_{2m},y_{2m+2})\right] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p(y_{2m+1},y_{2m+1}),p(y_{2m+1},y_{2m+2}),p(y_{2m+1},y_{2m+1}), \\ \frac{1}{2}\left[p(y_{2m+1},y_{2m+1})+p(y_{2m+1},y_{2m+2})\right] \end{array}\right\} \\ &= p(y_{2m+1},y_{2m+2}), \text{ from } (p_2). \end{aligned}$$

It is a contradiction. Hence $y_{2m+1} = y_{2m+2}$.
Continuing in this way, we get $y_n = y_{n+k}$, for all $k \geq 1$.
Thus $\{y_n\}$ is Cauchy.
Case(b): suppose $y_n \neq y_{n+1}$ for all n.

$$\begin{aligned} p\left(y_{2n},y_{2n+1}\right) &= p\left(Sx_{2n},Tx_{2n+1}\right) \\ &< \max\left\{\begin{array}{c} p(y_{2n-1},y_{2n}),p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1}), \\ \frac{1}{2}\left[p(y_{2n-1},y_{2n+1})+p(y_{2n},y_{2n})\right] \end{array}\right\} \\ &= \max\{p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1})\}, \\ \text{since } \tfrac{1}{2}\left[p(y_{2n-1},y_{2n+1})+p(y_{2n},y_{2n})\right] &\leq \tfrac{1}{2}\left[p(y_{2n-1},y_{2n})+p(y_{2n},y_{2n+1})\right] \end{aligned}$$

Thus $p\left(y_{2n},y_{2n+1}\right) < p\left(y_{2n-1},y_{2n}\right)$.
Now,

$$\begin{aligned} p\left(y_{2n+2},y_{2n+1}\right) &= p\left(Sx_{2n+2},Tx_{2n+1}\right) \\ &< \max\left\{\begin{array}{c} p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n}),p(y_{2n},y_{2n+1}), \\ \frac{1}{2}\left[p(y_{2n+1},y_{2n+1})+p(y_{2n},y_{2n+2})\right] \end{array}\right\} \\ &= \max\{p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n+2})\}. \end{aligned}$$

Thus $p\left(y_{2n+2},y_{2n+1}\right) < p\left(y_{2n},y_{2n+1}\right)$.
Hence $\{p\left(y_n,y_{n+1}\right)\}$ is a decreasing sequence of non - negative real numbers and hence converges, say, $r \geq 0$.
Suppose $r > 0$.
Then there exists $s > 0$ such that

$$r \leq \max\left\{p(fx,gy),p(fx,Sx),p(gy,Ty),\frac{1}{2}\left[p(fx,Ty)+p(gy,Sx)\right]\right\} < r+s$$

implies $p(Sx,Ty) < r$ $\qquad --------------(II)$.
Since $\{p(y_n,y_{n+1})\} \downarrow r$, there exists a positive integer n_0 such that

$$r \leq p(y_n,y_{n+1}) < r+s, \text{ for all } n \geq n_0 \qquad ------------(III)$$

Let $2k \geq n_0$. Noting that

$$\begin{aligned}\max&\left\{\begin{array}{c} p(fx_{2k+2}, gx_{2k+1}), p(fx_{2k+2}, Sx_{2k+2}), p(gx_{2k+1}, Tx_{2k+1}), \\ \frac{1}{2}\left[p(fx_{2k+2}, Tx_{2k+1}) + p(gx_{2k+1}, Sx_{2k+2})\right] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p(y_{2k+1}, y_{2k}), p(y_{2k+1}, y_{2k+2}), p(y_{2k}, y_{2k+1}), \\ \frac{1}{2}\left[p(y_{2k+1}, y_{2k+1}) + p(y_{2k}, y_{2k+2})\right] \end{array}\right\} \\ &= p(y_{2k+1}, y_{2k}), \text{ since } \{p(y_n, y_{n+1})\} \downarrow .\end{aligned}$$

From (III) and (II), we have

$$p(Sx_{2k+2}, Tx_{2k+1}) < r$$
$$\text{i. e. } p(y_{2k+2}, y_{2k+1}) < r.$$

It is a contradiction to (III).
Hence $r = 0$. Thus $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0.$ − − − − − − − − − − − − − − −(IV)
It is clear from (p_2) that $\lim_{n\to\infty} p(y_n, y_n) = 0.$ − − − − − − − − − − − − − − − −(V)
From (IV) and (V) and from the definition of d_p, we have

$$\lim_{n\to\infty} d_p(y_n, y_{n+1}) = 0. \text{ − − − − − − − − − − − − − − −}(VI)$$

Suppose $\{y_n\}$ is not Cauchy in (X, d_p).
Then there exists an $\epsilon > 0$ such that for each positive integer n_1, there exist integers m, n with $m > n > n_1$ such that

$$d_p(y_n, y_m) \geq 4\epsilon.$$

By definition of d_p, we have $d_p(x, y) \leq 2p(x, y) \;\; \forall x, y \in X \Rightarrow p(y_n, y_m) \geq 2\epsilon$.
Choose a number δ with $0 < \delta < \epsilon$ for which (2.1.1) is satisfied.
From (IV), there exists an integer n_2 depending on δ such that

$$p(y_i, y_{i+1}) < \frac{\delta}{6} \text{ for all } i \geq n_2. \text{ − − − − − − − − −}(VII)$$

With this choice of n_2, pick m, n with $m > n > n_2$ such that

$$p(y_m, y_n) \geq 2\epsilon > \epsilon + \delta. \text{ − − − − − − − − −}(VIII)$$

From (VIII), it is clear that $m - n > 6$. Without loss of generality assume that n is even.
Other wise $p(y_m, y_n) \leq \sum_{i=0}^{5} p(y_{n+i}, y_{n+i+1}) < \delta < \epsilon + \delta$, which is a contradiction to (VIII).
Suppose $p(y_n, y_{m-1}) < \epsilon + \frac{\delta}{3}$.
Then, we have

$$\begin{aligned} p(y_m, y_n) &\leq p(y_m, y_{m-1}) + p(y_{m-1}, y_n) \\ &< \frac{\delta}{6} + \epsilon + \frac{\delta}{3} < \epsilon + \delta, \end{aligned}$$

a contradiction to (VIII). Hence $p(y_n, y_{m-1}) \geq \epsilon + \frac{\delta}{3}$.
Similarly we can prove that $p(y_n, y_{m-2}) \geq \epsilon + \frac{\delta}{3}$.
Suppose $p(y_n, y_{m-2}) < \epsilon + \frac{\delta}{3}$.

$$\begin{aligned} p(y_m, y_n) &\leq p(y_m, y_{m-1}) + p(y_{m-1}, y_{m-2}) + p(y_{m-2}, y_n) \\ &< \frac{\delta}{6} + \frac{\delta}{6} + \epsilon + \frac{\delta}{3} \\ &< \epsilon + \delta, \end{aligned}$$

a contradiction to (VIII). Thus $p(y_n, y_{m-2}) \geq \epsilon + \frac{\delta}{3}$.
Thus there exists a small odd integer $j > n$ such that $p(y_n, y_j) \geq \epsilon + \frac{\delta}{3}$.
Now,

$$\begin{aligned} p(y_n, y_j) &\leq p(y_n, y_{j-2}) + p(y_{j-2}, y_{j-1}) + p(y_{j-1}, y_j) \\ &< \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} \\ &= \epsilon + \frac{2}{3}\delta. \end{aligned}$$

Thus there exists an odd integer $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} \leq p(y_n, y_j) < \epsilon + \frac{2}{3}\delta. \text{ − − − − − − − − − − −}(IX)$$

Since

$$\begin{aligned}
\epsilon &< \epsilon+\frac{\delta}{3} \\
&\leq p(y_n, y_j) \\
&\leq \max\left\{\begin{array}{c} p(y_j, y_n), p(y_j, y_{j+1}), p(y_n, y_{n+1}), \\ \frac{1}{2}\left[p(y_{j+1}, y_n)+p(y_j, y_{n+1})\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} p(fx_{2l+2}, gx_{2k+1}), p(fx_{2k+2}, Sx_{2k+2}), p(gx_{2k+1}, Tx_{2k+1}), \\ \frac{1}{2}\left[p(fx_{2l+2}, Tx_{2k+1})+p(gx_{2k+1}, Sx_{2l+2})\right] \end{array}\right\}, \\
&\qquad\qquad \text{where } n=2k \text{ and } j=2l+1 \\
&= \epsilon+\delta, \text{ from } (IX), \text{ (VII)}.
\end{aligned}$$

Fom (2.1.1), we have

$$p(y_{j+1}, y_{n+1}) = p(Sx_{2l+2}, Tx_{2k+1}) < \epsilon. ----------(X)$$

Now, we have

$$\begin{aligned}
p(y_n, y_j) &\leq p(y_n, y_{n+1})+p(y_{n+1}, y_{j+1})+p(y_{j+1}, y_j) \\
&< \frac{\delta}{6}+\epsilon+\frac{\delta}{6}=\epsilon+\frac{\delta}{3},
\end{aligned}$$

a contradiction to (IX). Hence $\{y_n\}$ is a Cauchy in (X, d_p).
Thus $\lim_{n,m\to\infty} d_p(y_n, y_m)=0$.
From (IV) and (V), we have $\lim_{n,m\to\infty} p(y_n, y_m)=0. ------(XI)$
Suppose $f(X)$ is complete.
Then $\{y_{2n+1}\}=\{fx_{2n+1}\} \subseteq f(X)$, there exists $y \in f(X)$ such that $y_{2n+1} \to y$. There exist $u \in X$ such that $y=fu$.
Since $\{y_n\}$ is Cauchy and $\{y_{2n+1}\} \to y$, it follows that $\{y_{2n}\} \to y$.
From Lemma 1.4 and (XI), we have

$$p(y, y)=\lim_{n\to\infty} p(y_{2n}, y)=\lim_{n\to\infty} p(y_{2n+1}, y)=0. ------(XII)$$

Suppose $p(Su, y)>0$.

$$\begin{aligned}
p(y, Su) &\leq p(y, Tx_{2n+1})+p(Su, Tx_{2n+1})-p(Tx_{2n+1}, Tx_{2n+1}) \\
&\leq p(y, Tx_{2n+1})+p(Su, Tx_{2n+1}) \\
&\leq p(y, Tx_{2n+1})+k\left[\begin{array}{c} p(y, y_{2n})+p(y, Su)+p(y_{2n}, y_{2n+1}) \\ +p(y, y_{2n+1})+p(y_{2n}, Su) \end{array}\right].
\end{aligned}$$

Letting $n\to\infty$, we have

$$p(y, Su) \leq 0+k[0+p(y, Su)+0+0+p(y, Su)]=k2p(y, Su).$$

It follows that $y=Su$.
Thus $y=fu=Su$.
Since $S(X) \subseteq g(X)$, we have $y=Su=gv$ for some $v \in X$.

$$\begin{aligned}
p(y, Tv) &= p(Su, Tv) \\
&\leq k[p(y, y)+p(y, y)+p(y, Tv)+p(y, Tv)+p(y, y)]=k[2p(y, Tu)].
\end{aligned}$$

It follows that $y=Tv$. Thus $y=gv=Tv$.
Since (f, S) is weakly compatible, we have $fy=Sy$, since $y=fu=Su$.

$$\begin{aligned}
p(Sy, y) &\leq p(Sy, Tx_{2n+1})+p(Tx_{2n+1}, y) \\
&\leq k\left[\begin{array}{c} p(Sy, y_{2n})+p(Sy, Sy)+p(y_{2n}, y_{2n+1}) \\ +p(Sy, y_{2n+1})+p(y_{2n}, Sy) \end{array}\right]+p(y_{2n+1}, y).
\end{aligned}$$

Letting $n\to\infty$ and from Lemma 1.5 we have

$$p(Sy, y) \leq k\left[p(Sy, y)+p(Sy, y)+0+p(Sy, y)+p(Sy, y)\right]+0.$$

It follows that $Sy=y$. Thus $fy=Sy=y$.
Since (g, T) is weakly compatible, we have $gy=Ty$.

$$p(y, Ty)=p(Sy, Tv) \leq k[p(y, Ty)+p(y, y)+p(Ty, Ty)+p(y, Ty)+p(Ty, y)] \leq 4kp(y, Ty).$$

It follows that $y = Ty$. Thus $y = gy = Ty$.
Suppose z is another common fixed point of S, T, f and g.
Then from(2.1.5), we have

$$p(y,z) = p(Sy,Tz) \leq \ k\ [p(y,z) + p(y,y) + p(z,z) + p(y,z) + p(z,y)] \leq 4k\ p(y,z).$$

Hence $y = z$.

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