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An Exactly Solvable Three Dimensional Non - linear Quantum Oscillator and $sl(2)$ Algebra

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Abstract In this paper, first we introduce the three dimensional non-linear oscillator with a position dependent mass. In that case we start by the stationary Schrödinger equation which is generated by the three dimensional Hamiltonian. The wave function depend on three spatial variables and the usual process of variable in spherical coordinates and the wave functions will be of radial and angular solutions. We can easily solve the angular part of equation but redial part of equation will be complicated. In that case, we take advantage from $sl(2)$ algebra and write the corresponding equation in terms of $P_+(r)$, $P_-(r)$ and $P_0(r)$ which are generators of generalized $sl(2)$ algebra. The information of this algebra help us to obtain the energy spectrum and wave function from radial part of equation.

Keywords Non-Linear Oscillator Potential; $sl(2)$ Algebra; First Order Operators; Energy Spectrum

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1 Introduction

Recently the one dimensional non-linear quantum oscillator has been studied by Ref.[1]. This model show us two important things in physics, the first one is non-linearity of the potential $V(x)$ and second one is position dependent mass $M(x)$. In one dimension for the quantum mechanical version of model admits exact solution for the wave functions and the energy eigenvalues [1]. This solution was achieved by using the factorization method and solution of wave function represented by Rodrigues - type of formula. Consequently, one dimensional non-linear quantum oscillator model was farther studied by Ref.[2], and have shown that the exact solution are expressed in closed - form in terms of special function, without to use the factorization method or Rodrigues formula. Here, we study on the three - dimensional generalization of the 1 D model of Refs. [1-2]. The effective potential for this 3D system is of a more complicated form, but has spherical symmetry. The application of position dependent mass will be in condensed matter and theories of gravitation especially in Brane - Dicke theory [3-5]. In this paper, we consider the nonlinear quantum oscillator [1] and [4] generalized to three dimensions. We arrive the 3D time independent Schrödinger equation which we separate into angular and radial parts. The angular solutions are the usual spherical harmonics. We then focus on the more difficult problem of solving the radial equation. In order to solve such difficulty, we take advantage from $sl(2)$ algebra. So, in this paper we are going to investigate the equation (3). By using separation variable, we solve first the angular part of equation and obtain $Y_l^m(\theta, \phi)$ which is known in differential equation. In order to achieve the energy spectrum, we have to solve equation (7) which is radial part of Schrödinger equation. Here, we note that the solution of such equation will be complicated. We use the $sl(2)$ algebra and try to write the equation (7) in terms of generators of generalized $sl(2)$ algebra as $P_+(r)$, $P_-(r)$ and $P_0(r)$. The commutation relation of such operators with comparing to usual $sl(2)$ algebra help us to achieve the energy spectrum and radial wave function.

2 Three dimensional non-linear oscillator potential

We start out by setting up the stationary Schrödinger equation that is generated by our three dimensional Hamiltonian. To this end, let us recall that the following potential,

$$V(r) = \frac{1}{2}M(x)r^2\omega^2, \tag{1}$$

where

$$M(r) = \frac{m}{1+\lambda r^2}. \tag{2}$$

For simplicity, we choose $m\omega^2 = g$ and inter the above potential in form of corresponding Schrödinger equation, which reads,

$$\left[(\lambda r^2+1)\nabla^2 + \lambda r\frac{\partial}{\partial r} + \frac{2m}{\hbar^2}(E - \frac{g}{2}\frac{r^2}{1+\lambda r^2})\right]\Psi = 0. \tag{3}$$

Now by using the separation of variables one can write Ψ as a product of radial and angular part of equation as

$$\Psi = R(r)Y(\theta,\phi). \tag{4}$$

Taking the separation constant as $L(l+1)$ for a nonnegative integer L, two functions R and Y are found to satisfy the following equations,

$$\left(1+\lambda r^2\right)R''(r) + \left(\frac{2}{r} + 3\lambda r\right)R'(r) + \left[\frac{2m}{\hbar^2}(E - \frac{g}{2}\frac{r^2}{1+\lambda r^2}) - \frac{(\lambda r^2+1)L(L+1)}{r^2}\right]R(r) = 0, \tag{5}$$

and

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + L(L+1)\right]Y = 0. \tag{6}$$

The two dimensional angular equation (6) can be solved in terms of spherical harmonics usually denoted by $Y = Y_l^m(\theta,\phi)$, where the integer m satisfies $|m| \leq l$. Now we are going to solve the equation (5) and obtain the energy spectrum. In order to solve such equation we arrange the equation (5) as $U(r) = rR(r)$,

$$\left(\lambda^2 r^6 + 2\lambda r^4 + r^2\right)U''(r) + \left(\lambda r^3 + \lambda^2 r^5\right)U'(r) + \left(\eta_1 r^2 + \eta_2 r^4 + \eta_3\right)R(r) = 0 \tag{7}$$

where

$$\eta_1 = \frac{2m\epsilon}{\hbar^2} - \frac{\lambda L(L+1)\hbar^2}{m}, \qquad \eta_2 = \frac{2m\epsilon}{\hbar^2} - \frac{g}{2} - \frac{\lambda^2 L(L+1)\hbar^2}{2m}, \qquad \eta_3 = -\frac{L(L+1)\hbar^2}{2m}, \tag{8}$$

In order to connect the redial part of equation with $sl(2)$ operators we have to choose suitable change of variables and appropriate point canonical transformations. All these help us to avoid the singularity in such equation [6-7]. Generally, one can write three dimensional non-linear oscillator potential in terms of of $f_1(r)$, $f_2(r)$, and $f_3(r)$ to connect the deformation form of $sl(2)$ algebra. So, in that case we have following equation,

$$f_1(r)\frac{d^2R}{dr^2} + f_2(r)\frac{dR}{dr} + f_3(r)R(r) = 0, \tag{9}$$

where three corresponding function will be polynomial.

$$\begin{aligned} f_1(r) &= a_0 r^6 + a_1 r^5 + a_2 r^4 + a_3 r^3 + a_4 r^2 + a_5 r + a_6 \\ f_2(r) &= a_7 r^5 + a_8 r^4 + a_9 r^3 + a_{10} r^2 + a_{11} r + a_8, \\ f_3(r) &= a_{13} r^4 + a_{14} r^3 + a_{15} r^2 + a_{16} r + a_{17} \end{aligned} \tag{10}$$

We put equation (10) in equation (9) and compare with equation (7) one can obtain $a_i (i = 1, ..., 17)$ as,

$$\begin{gathered} a_0 = \lambda^2, \quad a_1 = 0, \quad a_2 = 2\lambda, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 0, \quad a_6 = 0, \\ a_7 = \lambda^2, \quad a_8 = 0, \quad a_9 = \lambda, \quad a_{10} = 0, \quad a_{11} = 0, \quad a_{12} = 0, \\ a_{15} = \eta_1, \quad a_{14} = 0, \quad a_{16} = 0, \quad a_{13} = \eta_2, \quad a_{17} = \eta_3. \end{gathered} \tag{11}$$

The structure of usual $sl(2)$ algebra give opportunity to take following choice of operators from (9) and (10), which are given by,

$$\begin{aligned} P_+ &= \lambda^2 r^6\frac{d^2}{dr^2} + \lambda^2 r^5\frac{d}{dr} + \eta_2 r^4 \\ P_- &= 2\lambda r^4\frac{d^2}{dr^2} + \lambda r^3\frac{d}{dr} + \eta_1 r^2 \\ P_0 &= \alpha r\frac{d}{dr} - \beta, \end{aligned} \tag{12}$$

Here, instead of J^+, J^- and J^0 in usual $sl(2)$ algebra we defined the generalized generators as P_+, P_- and P_0 . Also, we can say that these generators satisfied by the corresponding commutator algebra as $[P_0, P_\pm] = \pm P_\pm$

. These commutation relation give us motivation to arrange the parameters α , β and λ as $\alpha=\frac{3}{2}$,$\beta=\frac{1}{2}$ and $3\lambda^2=\eta_2$. So, the modified operator $p_0(r)$ in equation (12) as a generalized $sl(2)$ algebra which given by,

$$P_0=\frac{3}{2}r\frac{d}{dr}-\frac{1}{2}. \tag{13}$$

The connection between operators of three dimensional non - linear quantum oscillator with $sl(2)$ algebra give us the energy spectrum as,

$$\epsilon=E-\frac{\hbar^2\lambda}{2m}, \tag{14}$$

where

$$E=\frac{\hbar^2\lambda}{2m}[1+\lambda(\frac{L(L+1)}{2m}+\frac{g}{2\lambda^2}+3)]. \tag{15}$$

Now, we use the generator of P_+, which acts on the $U(r)$ as $P_+U(r)=0$. So, by using the equation (13) one can obtain the following equation,

$$rU''(r)+\frac{1}{\lambda}U'(r)+\frac{\eta_2}{\lambda^2 r}U(r)=0. \tag{16}$$

In order to solve such equation we use the following associated Laguerre equation [8-10],

$$rL''^{(\alpha,\beta)}_{n,m}(r)+[1+\alpha-\beta r]L'^{(\alpha,\beta)}_{n,m}(r)+\left[(n-\frac{m}{2})\beta-\frac{m}{2}(\ \alpha+\frac{m}{2})\frac{1}{r}\right]L^{\alpha,\beta}_{n,m}(r)=0, \tag{17}$$

where $\alpha,\beta\leq -1$ and $o\leq m\leq n$. By comparing two equation (16) and (17) to each other one can obtain following expression [11-18].

$$U(r)=\frac{1}{r}r^{\lambda(\alpha+1)}e^{-\beta r}L^{\alpha,\beta}_{n,m}(r). \tag{18}$$

The general solution for the three dimensional non-linear oscillator $\Psi(r,\theta\ \phi)=R(r)Y_l^m(\theta,\phi)$ is,

$$\Psi(r,\theta,\phi)=Cr^{\lambda(\alpha+1)}e^{-\beta r}L^{\alpha,\beta}_{n,m}(r)Y_l^m(\theta,\phi). \tag{19}$$

So, Here we have seen that the solution of wave function from $sl(2)$ algebra can be obtained by the generator of $P_+(r)$. So, the complication of redial part of equation give motivation to us to employed the information from $sl(2)$ algebra and used the associated polynomial as Laguerre equation [8-18]. Finally, we could manage three dimensional non-linear oscillator with a position dependent mass and obtained the energy spectrum and full wave function.

3 Conclusion

In this paper, first we introduced the three dimensional non-linear oscillator with a position dependent mass. Note that the wave function depend on three spatial variables and the usual process of variable separation in spherical coordinates renders wave function as a product of radial and angular solution. We could easily solve the angular part of equation but the redial part of equation was complicated. In order to solve such complication we took advantage from generalized $sl(2)$ algebra and wrote the corresponding equation in terms of $P_+(r)$, $P_-(r)$ and $P_0(r)$. The commutation relations between these generators of generalized $sl(2)$ algebra helped us to obtain the energy spectrum. Also, we use the raising operators on the highest state and achieved the second order equation as (16). We connected the equation (16) to associated Laguerre equation and received the radial part of equation. Finally, we accounted the angular and radial part of solutions we obtained the general form of wave function for the corresponding system.

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