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Inequalities for the sth Derivative of A Polynomial with Prescribed Zeros

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Abstract Let $p(z)$ be a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$ Bidkham and Dewan [J. Math. Anal. Appl. 166 (1992), 191–193] proved

$$\max_{|z|=R} |p'(z)| \geq \frac{n(R+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)| \ .$$

In this paper we shall present several intersting generalizations and a refinement of this result which includes some results due to Malik, Govil and others. We also present a refinement of some other results.

Keywords Derivative of a polynomial, Zeros, Inequalities

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1 Introduction

For $p \in p_n$ where p_n be class of polynomials $p(z)$ of degree atmost n, we define

$$\|p\|_\gamma := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma \right\}^{\frac{1}{\gamma}}, \quad 1 \leq \gamma < \infty$$

$$\|p\|_\infty := \max_{|z|=1} |p(z)| \qquad \text{and}$$

$$m := \min_{|z|=k} |p(z)| \,.$$

Let $p(z)$ be a polynomial of degree n, then according to a famous result known as Bernstein inequality (for reference see [13] or [12])

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \,. \tag{1.1}$$

The result is best possible and equality holds for the polynomial having all its zeros at origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max |p(z)| \,. \tag{1.2}$$

As an extension of (1.2) Malik [8] verified that if $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)| \,. \tag{1.3}$$

Equality holds for $p(z) = (z+k)^n$.

Bidkham and Dewan [2] obtained a generalization of (1.3) for the same class of polynomials by proving

$$\max_{|z|=R} |p'(z)| \geq \frac{n(R+k)^{n-1}}{1+k} \max_{|z|=1} |p(z)| \,. \tag{1.4}$$

The result is best possible and equality holds for $p(z) = (z+k)^n$.

In the reverse direction it was proved by Turan [9] that if $p(z)$ does not vanish in $|z| > 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| \,. \tag{1.5}$$

Inequality (1.5) was refined by Aziz and Dawood [1] by showing that under the same hypothesis that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}. \tag{1.6}$$

Both the inequalities (1.5) and (1.6) are sharp and equality holds for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. As an extension of (1.5), Govil [3] proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)| \,. \tag{1.7}$$

Equality holds for $p(z) = (z+k)^n$, $k \leq 1$.

Whereas if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$ with s-fold zeros at the origin, then Aziz and Shah [4] proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n+sk}{1+k} \max_{|z|=1} |p(z)| \,. \tag{1.8}$$

The result is sharp and extremal polynomial is $p(z) = z^s(z+k)^{n-s}$, $0 < s \leq n$.

Govil and Mctume [6] generalized inequality (1.6) of Aziz and Dawood [1] by proving that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k}\left\{\max_{|z|=1} |p(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)|\right\}. \tag{1.9}$$

The result is best possible and equality holds for the polynomial $p(z) = (z+k)^n$.

2 Main Results

In this paper we shall first present the following generalization as well as an improvement of (1.4) by considering the sth derivative of $P(z)$. More precisely, we have

Theorem 1. *For $p \in p_n$ and If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ having no zeros in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$, and $1 \leq s \leq n$*

$$\max_{|z|=R} |p^s(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{R^\mu + k^\mu}\left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} \left\{\max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)|\right\}. \tag{2.1}$$

The result is best possible for $s = 1$ and equality holds for $p(z) = (z+k)^n$.

We now present some integral inequalities in the reverse direction for polynomials having a zero of order s at the origin. More precisely, we prove

Theorem 2. *Let $p \in p_n$ and $p(z)$ having all its zeros in $|z| \leq k$, $k \leq 1$ with a zero of order s at $z = 0$, then for β with $|\beta| < k^{n-s}$ and $\gamma \geq 1$*

$$\left\| p' - \frac{sm}{k^n}\bar{\beta} z^{s-1}\right\|_\gamma \geq \left\{ n - (n-s)C_\gamma^\mu \left\| p + \frac{m}{k^n}\bar{\beta} z^s \right\|_\gamma \right\} \tag{2.2}$$

Where $C_\gamma^\mu = \left\| \frac{k^\mu}{1+k^\mu z}\right\|_\gamma$.

By letting $\gamma \to \infty$ in place of Theorem 2, we obtain

Corollary 1. *Let $p \in p_n$ and $p(z)$ having all its zeros in $|z| \leq k$, $k \leq 1$ with a zero of order s at $z = 0$, then for β with $|\beta| < k^{n-s}$*

$$\left\| p'(z) + \frac{sm}{k^n}\bar{\beta} z^{s-1}\right\|_\infty \geq \left(\frac{n + sk^\mu}{1+k^\mu}\right) \left\| p(z) + \frac{m}{k^n}\bar{\beta} z^s \right\|_\infty. \tag{2.3}$$

For $\beta = 0$ in inequality (2.3), we get

$$\|p'\|_\infty \geq \left(\frac{n+sk^\mu}{1+k^\mu}\right)\|p\|_\infty \tag{2.4}$$

Next, we prove the following result which yields refinements of both the inequalities (1.8) and (1.9) as special cases.

Theorem 3. *If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ having all its zeros in $|z| \leq k \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$\max_{|z|=1} |p'(z)| \geq \left(\frac{n + k^\mu s}{1+k^\mu}\right) \max_{|z|=1} |p(z)| + \frac{(n-s)}{k^{n-\mu}(1+k^\mu)} \min_{|z|=k} |p(z)|. \tag{2.5}$$

The result is best possible and equality holds for $p(z) = z^s(z+k)^{n-s}$, $0 \leq s \leq n$.

3 Lemmas

For the proof of these theorems, we need the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1} |p^s(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \left\{\max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)|\right\}. \tag{3.1}$$

Lemma 1 is due to Govil [7].

Lemma 2. *If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, the for $0 \leq r \leq R \leq k$,*

$$\max_{|z|=R} |p(z)| \leq \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} \max_{|z|=1} |p(z)| + \left\{1 - \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}\right\} \min_{|z|=k} |p(z)|. \tag{3.2}$$

The above lemma is due to Dewan, Yadav and Pukhta [5].

Lemma 3. *If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree n having all its zeros in $|z| \geq k \geq 1$ and $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$,*

$$k^\mu |p'(z)| \leq |q'(z)|. \tag{3.3}$$

The above lemma is due to Chan and Malik [10]. By applying Lemma 3 to the polynomial one can deduce:

Lemma 4. *If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$ and $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$,*

$$k^\mu |p'(z)| \geq |q'(z)|. \tag{3.4}$$

Lemma 5. *Let $p \in p_n$ and $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ having no zeros in $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$ and for each $\gamma > 0$,*

$$\left\| p'(z) + \frac{mn}{1+k^\mu}\beta \right\|_\gamma \leq C_\gamma^\mu \|p\|_\gamma. \tag{3.5}$$

Where $C^{\mu}_{\gamma} = \left\| \frac{k^{\mu}}{1+k^{\mu}z} \right\|_{\gamma}$.

The above lemma is due to Shah [11].

4 Proof of Theorems

Proof of Theorem 1. If $p(z)$ has no zeros in $|z| < k$, $k > 0$ and if $0 < r \le R \le k$, then $G(z) = P(Rz)$ has no zeros in $|z| < \frac{k}{R}$, $\frac{k}{R} > 1$.

Thus applying Lemma 1 to $G(z)$, we get

$$\max_{|z|=1} |G^s(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\left(\frac{k}{R}\right)^s} \times \left\{ \max_{|z|=1} |G(z)| - \min_{|z|=\frac{k}{R}} |G(z)| \right\}$$

which implies

$$R^s \max_{|z|=1} |p^s(Rz)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\frac{k^s}{R^s}} \times \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=\frac{k}{R}} |p(Rz)| \right\}$$

which is equivalent to

$$\max_{|z|=R} |p'(z)| \le \frac{n(n-1)\cdots(n-s+1)}{R^s+k^s} \times \left\{ \max_{|z|=R} |p(z)| - \min_{|z|=k} |p(Rz)| \right\} \quad (4.1)$$

inequality (4.1) in conjunction with Lemma 2 yields,

$$\max_{|z|=R} |p^s(z)| \le \frac{n(n-1)\cdots(n-s+1)}{R^s+k^s} \left(\frac{k^{\mu}+R^{\mu}}{k^{\mu}+r^{\mu}} \right) \times \left\{ \max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)| \right\}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. We have $p(z) = z^s \phi(z)$, where $\phi(z)$ is a polynomial of degree $n-s$, with the property that $\phi(0) \neq 0$, then

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = z^{n-s} \overline{\phi\left(\frac{1}{\bar{z}}\right)},$$

is a polynomial of degree $n-s$ with no zeros in $|z| < \frac{1}{k}$.
Now if $m_0 = \min_{|z|=\frac{1}{k}} |q(z)| = \frac{m}{k^n}$.

By Rouche's theorem, the polynomial

$$T(z) = q(z) + m_0 \beta z^{n-s}, \quad |\beta| < k^{n-s}$$

of degree $n-s$, will also have no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \ge 1$. Hence by Lemma 5, we have for $\gamma \ge 1$ and $|\beta| < k^{n-s}$

$$\|T'\|_{\gamma} \le (n-s) C^{\mu}_{\gamma} \|T\|_{\gamma}$$

or

$$\left\| q'(z) + \frac{(n-s)m}{k^n} \beta z^{(n-s-1)} \right\|_{\gamma} \le (n-s) C^{\mu}_{\gamma} \left\| q(z) + \frac{m}{k^n} \beta z^{n-s} \right\|_{\gamma}$$

i.e.,

$$\left\| np(z) - zp'(z) + \frac{(n-s)m}{k^n} \bar{\beta} z^s \right\|_{\gamma} \le (n-s) C^{\mu}_{\gamma} \left\| p(z) + \frac{m}{k^n} \bar{\beta} z^s \right\|_{\gamma}.$$

Now by Minkowski inequality, we have for $\gamma \ge 1$ and $|\beta| < k^{n-s}$,

$$\begin{aligned} n\left\| p(z) + \frac{m}{k^n} \bar{\beta} z^s \right\|_{\gamma} &\le \left\| np(z) - zp'(z) + \frac{(n-s)m}{k^n} \bar{\beta} z^s \right\|_{\gamma} + \left\| zp'(z) + \frac{mS}{k^n} \bar{\beta} z^s \right\|_{\gamma} \\ &\le (n-s) C^{\mu}_{\gamma} \left\| p(z) + \frac{m}{k^n} \bar{\beta} z^s \right\|_{\gamma} + \left\| zp'(z) + \frac{mS}{k^n} \bar{\beta} z^s \right\|_{\gamma} \end{aligned}$$

which implies

$$\left\| p'(z) + \frac{sm}{k^n} \bar{\beta} z^{s-1} \right\| \ge \left\{ n - (n-s) C^{\mu}_{\nu} \left\| p + \frac{m}{k^n} \bar{\beta} z^s \right\|_{\gamma} \right\}.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Since all the zeros of $p(z)$ lie in $|z| \le k \le 1$, with s-fold zeros at the origin, therefore, for every complex number β such that $|\beta| < 1$, it follows by Rouche's theorem for $m > 0$ that the polynomial $F(z) = p(z) - \frac{m}{k^n} \beta z^n$ has all its zero in $|z| \le k$, $k \le 1$. It can be easily verified that if $H(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)} = z^{n-s} \overline{G\left(\frac{1}{\bar{z}}\right)}$, then

$$|H'(z)| = |nF(z) - zF'(z)|, \quad \text{for} \quad |z| = 1.$$

Applying Lemma 4 to the polynomial $F(z)$, we get for $|z| = 1$,

$$k^{\mu} |F'(z)| \ge |H'(z)| = |nF(z) - zF'(z)|$$

Again applying inequality (2.4) to the polynomial $F(z)$, we get

$$|F'(z)| \ge \frac{n + sk^{\mu}}{1+k^n} |F(z)|, \quad \text{for} \quad |z| = 1. \quad (4.2)$$

Replacing $F(z)$ by $p(z) - \frac{\alpha m}{k^n} z^n$ in (4.2), we get

$$\left| p'(z) - \frac{\alpha nm}{k^n} z^{n-1} \right| \ge \frac{n + sk^{\mu}}{1+k^{\mu}} \left| p(z) - \frac{\alpha m}{k^n} z^n \right|, \quad \text{for} \quad |z| = 1 \quad (4.3)$$

for every α with $|\alpha| < 1$. Choosing the argument of α such that

$$\left|p(z) - \frac{\alpha m}{k^n} z^n\right| = |p(z)| - |\alpha| \frac{m}{k^n}, \quad \text{for } |z| = 1\,.$$

It follows from (3.3) that

$$|p'(z)| - \frac{n|\alpha|m}{k^n} \geq \frac{n + sk^\mu}{1 + k^n}\left\{|p(z)| - \frac{|\alpha|m}{k^n}\right\}, \qquad \text{for } \ |z| = 1\,.$$

Letting $|\alpha| \to 1$, we obtain

$$|p'(z)| \geq \frac{n + sk^\mu}{1 + k^\mu}|p(z)| + \left\{n - \frac{n + sk^\mu}{1 + k^\mu}\right\}\frac{m}{k^n}$$

this implies

$$\max_{|z|=1}|p'(z)| \geq \frac{n + sk^\mu}{1 + k^\mu}|p(z)| + \frac{(n - s)}{k^{n-\mu}(1 + k^\mu)} \min_{|z|=k}|p(z)|\,.$$

This completes the proof of Theorem 3. □

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