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Insertion of A Baire-one Function

MAJID MIRMIRAN

Department of Mathematics, University of Isfahan, Isfahan , 81746-73441, Iran
*Corresponding Author: mirmir@sci.ui.ac.ir

Abstract A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a Baire-one function between two comparable real-valued functions on the topological spaces that $\Lambda-$sets are $G_\delta-$sets.

Keywords Insertion, Strong binary relation, Baire-one function, $\Lambda-$sets, Lower cut set.

1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [5]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [5].

Results of Katětov [2] , [3] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a necessary and sufficient condition for the insertion of a Baire-one function between two comparable real-valued functions on the topological spaces that $\Lambda-$sets are $G_\delta-$sets.

A real-valued function f defined on a topological space X is called *Baire-one* if the preimage of every open subset of R is a $F_\sigma-$set in X.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [4].

A property P defined relative to a real-valued function on a topological space is a B_1-*property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-one function also has property P. If P_1 and P_2 are B_1-properties, the following terminology is used: (i) A space X has the *weak* B_1-*insertion property for* (P_1, P_2) iff for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a Baire-one function h such that $g \leq h \leq f$. (ii) A space X has the B_1-*insertion property for* (P_1, P_2) iff for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a Baire-one function h such that $g < h < f$.

In this paper, for a topological space that $\Lambda-$sets are $G_\delta-$sets, is given a sufficient condition for the weak B_1-insertion property. Also for a space with the weak B_1-insertion property, we give a necessary and sufficient condition for the space to have the B_1-insertion property. Several insertion theorems are obtained as corollaries of these results.

2 The main results

Before giving a sufficient condition for insertability of a Baire-one function, the necessary definitions and terminology are stated.

Definition. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
A^Λ is called *kernel* of A.

The following first two definitions are modifications of conditions considered in [2], [3].

Definition. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ \nu$ implies $x\ \rho\ \nu$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If $A\ \rho\ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, that $\Lambda-$sets in X are $G_\delta-$sets, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a Baire-one function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [3] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is a $F_\sigma-$set in X, i.e., h is a Baire-one function on X.

The above proof used the technique of theorem 1 of[2].

Theorem 2.2. Let P_1 and P_2 be B_1-property and X be a space that satisfies the weak B_1-insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the B_1-insertion property for (P_1,P_2) iff there exist lower cut sets $A(f-g,3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separated by Baire-one functions.

Proof. Assume that X has the weak B_1-insertion property for (P_1,P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f-g,3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each $n, X \setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separated by Baire-one functions. Let k_n be a Baire-one function such that $k_n = 0$ on $A(f-g,3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties of Baire-one, the function k is a Baire-one function. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f-g,3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f-g,1)$ or for some n,

$$x \in A(f-g,3^{-n+1}) - A(f-g,3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are B_1-properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak B_1-insertion property for (P_1,P_2), then there exists a Baire-one function h such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that X satisfies the B_1-insertion property for (P_1,P_2). (The technique of this proof is by Katětov[3]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a Baire-one function h such that $g < h < f$.We follow an idea contained in Lane [4]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the B_1-insertion property for (P_1,P_2), then there exists a Baire-one function k such that $0 < k < f - h$. Let $A(f-g,3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Since

$$A(f-g,3^{-n+1}) \subseteq \{x \in X : (f-g)(x) \leq 3^{-n+1}\} \subseteq$$

$$\{x \in X : k(x) \leq 3^{-n+1}\}$$

and since $\{x \in X : k(x) \leq 3^{-n+1}\}$ and $\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$ are completely separated by $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$ that is a Baire-one function, it follows that for each $n, A(f-g,3^{-n+1})$ and $X \setminus D_n$ are completely separated by Baire-one functions.

2.1 Applications

Definition. A real-valued function f defined on a space X is called *upper semi-Baire-one* (resp. *lower semi-Baire-one*) if $f^{-1}(-\infty,t)$ (resp. $f^{-1}(t,+\infty)$) is a $F_\sigma-$set for any real number t.

The abbreviations usc, lsc, usB_1 and lsB_1 are used for upper semicontinuous, lower semicontinuous, upper semi-Baire-one, and lower semi-Baire-one, respectively.

Remark 1. [2], [3]. A space X has the weak $c-$insertion property for (usc, lsc) iff X is normal.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space that $\Lambda-$sets are $G_\delta-$sets.

Corollary 3.1. For each pair of disjoint $G_\delta-$sets G_1, G_2, there are two $F_\sigma-$sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ iff X has the weak B_1-insertion property for (usB_1, lsB_1).

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $A^\Lambda \subseteq B^V$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq$$

$$\{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $G_\delta-$set and since $\{x \in X : g(x) < t_2\}$ is a $F_\sigma-$set, it follows that $A(f, t_1)^\Lambda \subseteq A(g, t_2)^V$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2. 1.

On the other hand, let G_1 and G_2 are disjoint $G_\delta-$sets. Set $f = \chi_{G_1^c}$ and $g = \chi_{G_2}$, then f is lsB_1, g is usB_1, and $g \leq f$. Thus there exists Baire-one function h such that $g \leq h \leq f$. Set $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then F_1 and F_2 are disjoint $F_\sigma-$sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$.

Before stating the consequences of theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.2. The following conditions on the space X are equivalent:

(i) Every two disjoint $G_\delta-$sets of X can be separated by $F_\sigma-$sets of X.

(ii) If G is a $G_\delta-$set of X which is contained in a $F_\sigma-$set F, then there exists a $F_\sigma-$set H such that $G \subseteq H \subseteq H^\Lambda \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are $G_\delta-$set and $F_\sigma-$set of X, respectively. Hence, F^c is a $G_\delta-$set and $G \cap F^c = \emptyset$.

By (i) there exists two disjoint $F_\sigma-$sets F_1, F_2 s.t. $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is a $G_\delta-$set containing F_1 we conclude that $F_1^\Lambda \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^\Lambda \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint $G_\delta-$sets of X.

This implies that $G_1 \subseteq G_2^c$ and G_2^c is a $F_\sigma-$set. Hence by (ii) there exists a $F_\sigma-$set H s.t., $G_1 \subseteq H \subseteq H^\Lambda \subseteq G_2^c$. But

$$H \subseteq H^\Lambda \Rightarrow H \cap (H^\Lambda)^c = \emptyset$$

and

$$H^\Lambda \subseteq G_2^c \Rightarrow G_2 \subseteq (H^\Lambda)^c.$$

Furthermore, $(H^\Lambda)^c$ is a $F_\sigma-$set of X. Hence $G_1 \subseteq H, G_2 \subseteq (H^\Lambda)^c$ and $H \cap (H^\Lambda)^c = \emptyset$. This means that condition (i) holds.

Lemma 3.3. Suppose that X is the topological space s.t. we can separate every two disjoint $G_\delta-$sets by $F_\sigma-$sets. If G_1 and G_2 are two disjoint $G_\delta-$sets of X, then there exists a Baire-one function $h : X \to R$ s.t. $h(G_1) = \{0\}$ and $h(G_2) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint $G_\delta-$sets of X. Since $G_1 \cap G_2 = \emptyset$, hence $G_1 \subseteq G_2^c$. In particular, since G_2^c is a $F_\sigma-$set of X containing G_1, by Lemma 3.2, there exists a $F_\sigma-$set $H_{1/2}$ s.t.,

$$G_1 \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq G_2^c.$$

Note that $H_{1/2}$ is a $F_\sigma-$set and contains G_1, and G_2^c is a $F_\sigma-$set and contains $H_{1/2}^\Lambda$. Hence, by Lemma 3.2, there exists $F_\sigma-$sets $H_{1/4}$ and $H_{3/4}$ s.t.,

$$G_1 \subseteq H_{1/4} \subseteq H_{1/4}^\Lambda \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq H_{3/4} \subseteq H_{3/4}^\Lambda \subseteq G_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $F_\sigma-$sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the real-valued function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_2$ and $h(x) = 1$ for $x \in G_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$. Also, we note that for any $t \in D, G_1 \subseteq H_t$; hence $h(G_1) = \{0\}$. Furthermore, by definition, $h(G_2) = \{1\}$. It remains only to prove that h is a Baire-one function on X. For every $\alpha \in R$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are $F_\sigma-$sets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^\Lambda)^c : t > \alpha\}$ hence, every of them is a $F_\sigma-$set. Consequently h is a Baire-one function.

Lemma 3.4. Suppose that X is the topological space such that every two disjoint $G_\delta-$sets can be separated by $F_\sigma-$sets. The following conditions are equivalent:

(i) Every countable convering of $F_\sigma-$sets of X has a refinement consisting of $F_\sigma-$sets s.t., for every $x \in X$, there exists a $F_\sigma-$set containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of $G_\delta-$sets with empty intersection there exists a decreasing sequence $\{F_n\}$ of $F_\sigma-$sets s.t., $\bigcap_{n=1}^\infty F_n = \emptyset$ and for every $n \in N, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii). suppose that $\{G_n\}$ be a decreasing sequence of $G_\delta-$sets with empty intersection. Then $\{G_n^c : n \in N\}$ is a countable covering of $F_\sigma-$sets. By hypothesis (i) and Lemma 3.2, this covering has a refinement $\{V_n : n \in N\}$ s.t. every V_n is a $F_\sigma-$set and $V_n^\Lambda \subseteq G_n^c$. By setting $F_n = (V_n^\Lambda)^c$, we obtain a decreasing sequence of $F_\sigma-$sets with the required properties.

(ii) $\Rightarrow$ (i). Now if $\{H_n : n \in N\}$ is a countable covering of $F_\sigma-$sets, we set for $n \in N, G_n = (\bigcup_{i=1}^n H_i)^c$. Then $\{G_n\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of $F_\sigma-$sets s.t., $\bigcap_{n=1}^\infty F_n = \emptyset$ and for every $n \in N, G_n \subseteq F_n$. Now we define the subsets W_n of X in the following manner:

W_1 is a $F_\sigma-$set of X s.t. $F_1^c \subseteq W_1$ and $W_1^\Lambda \cap G_1 = \emptyset$.

W_2 is a $F_\sigma-$set of X s.t. $W_1^\Lambda \cup F_2^c \subseteq W_2$ and $W_2^\Lambda \cap G_2 = \emptyset$, and so on. (By Lemma 3.2, W_n exists).

Then since $\{F_n^c : n \in N\}$ is a covering for X, hence $\{W_n : n \in N\}$ is a covering for X consisting of $F_\sigma-$sets. Moreover, we have

(i) $W_n^\Lambda \subseteq W_{n+1}$

(ii) $F_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.
Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus W_{n-1}^{\Lambda}$.

Then since $W_{n-1}^{\Lambda} \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in N\}$ consists of $F_\sigma-$sets and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ iff $|i-j| \leq 1$. Finally, consider the following sets:

$S_1 \cap H_1, \quad S_1 \cap H_2$
$S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3$
$S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4$

and continue ad infinitum. These sets are $F_\sigma-$sets, cover X and refine $\{H_n : n \in N\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $F_\sigma-$set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in N, j = 1, \dots, i+1\}$ refines $\{H_n : n \in N\}$ s.t. its elements are $F_\sigma-$sets, and for every point in X we can find a $F_\sigma-$set containing the point that intersects only finitely many elements of that refinement.

Remark 2. [2] , [3]. A space X has the $c-$insertion property for (usc, lsc) iff X is normal and countably paracompact.

Corollary 3.5. X has the B_1-insertion property for (usB_1, lsB_1) iff every two disjoint $G_\delta-$sets of X can be separated by $F_\sigma-$sets, and in addition, every countable covering of $F_\sigma-$sets has a refinement that consists of $F_\sigma-$sets s.t., for every point of X we can find a $F_\sigma-$set containing that point s.t., it intersects only a finite number of refining members.

Proof. Suppose that G_1 and G_2 are disjoint $G_\delta-$sets . Since $G_1 \cap G_2 = \emptyset$, it follows that $G_2 \subseteq G_1^c$. We set $f(x) = 2$ for $x \in G_1^c$, $f(x) = \frac{1}{2}$ for $x \notin G_1^c$, and $g = \chi_{G_2}$.

Since G_2 is a $G_\delta-$set, and G_1^c is a $F_\sigma-$set, therefore g is usB_1,f is lsB_1 and furthermore $g < f$. Hence by hypothesis there exists a Baire-one function h s.t., $g < h < f$. Now by setting $F_1 = \{x \in X : h(x) < 1\}$ and $F_2 = \{x \in X : h(x) > 1\}$. We can say that F_1 and F_2 are disjoint $F_\sigma-$sets that contain G_1 and G_2, respectively. Now suppose that $\{G_n\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection. Set $G_0 = X$ and define for every $x \in G_n \setminus G_{n+1}$, $f(x) = \frac{1}{n+1}$. Since $\bigcap_{n=0}^{\infty} G_n = \emptyset$ and for every $x \in X$, there exists $n \in N$, s.t., $x \in G_n \setminus G_{n+1}, f$ is well defined. Furthermore, for every $r \in R$, if $r \leq 0$ then $\{x \in X : f(x) > r\} = X$ is a $F_\sigma-$set and if $r > 0$ then by Archimedean property of R, we can find $i \in N$ s.t. $\frac{1}{i+1} \leq r$. Now suppose that k is the least natural number s.t. $\frac{1}{k+1} \leq r$. Hence $\frac{1}{k} > r$ and consequently, $\{x \in X : f(x) > r\} = X \setminus G_k$ is a $F_\sigma-$set. Therefore, f is lsB_1. By setting $g = 0$, we have g is usB_1 and $g < f$. Hence by hypothesis there exists a Baire-one function h on X s.t., $g < h < f$.

By setting $F_n = \{x \in X : h(x) < \frac{1}{n+1}\}$, we have F_n is a $F_\sigma-$set. But for every $x \in G_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$ therefore $0 < h(x) < \frac{1}{n+1}$, i.e., $x \in F_n$ therefore $G_n \subseteq F_n$ and since $h > 0$ it follows that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Hence by Lemma 3.4, the conditions holds.

On the other hand, since every two disjoint $G_\delta-$sets can be separated by $F_\sigma-$sets, therefore by Corollary 3.1, X has the weak B_1-insertion property for (usB_1, lsB_1). Now suppose that f and g are real-valued functions on X with $g < f$, s.t., g is usB_1 and f is lsB_1. For every $n \in N$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since g is usB_1, and f is lsB_1, therefore $f-g$ is lsB_1. Hence $A(f-g, 3^{-n+1})$ is a $G_\delta-$set of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $G_\delta-$sets and furthermore since $0 < f-g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \emptyset$. Now by Lemma 3.4, there exists a decreasing sequence $\{D_n\}$ of $F_\sigma-$sets s.t. $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 3.3, $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of $G_\delta-$sets can be completely separated by Baire-one functions. Hence by Theorem 2.2, there exists a Baire-one function h defined on X s.t., $g < h < f$, i.e., X has the B_1-insertion property for (usB_1, lsB_1).

Remark 3. [6]. A space X has the weak $c-$insertion property for (lsc, usc) iff X is extremally disconnected.

Corollary 3.6. For every F of $F_\sigma-$set, F^{Λ} is a $F_\sigma-$set iff X has the weak B_1-insertion property for (lsB_1, usB_1).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.7. The following conditions on the space X are equivalent:

(i) For every F of $F_\sigma-$set we have F^{Λ} is a $F_\sigma-$set.

(ii) For each pair of disjoint $F_\sigma-$sets as F_1 and F_2 we have $F_1^{\Lambda} \cap F_2^{\Lambda} = \emptyset$.

The proof of lemma 3.7 is a direct consequence of the definition $\Lambda-$sets.
We now give the proof of corollary 3.6.

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1,g is usB_1, and $f \leq g$.If a binary relation ρ is defined by $A \ \rho \ B$ in case $A^{\Lambda} \subseteq F \subseteq F^{\Lambda} \subseteq B^{V}$ for some $F_\sigma-$set F in X, then by hypothesis and Lemma 3.7 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\};$$

$$= A(f, t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a $F_\sigma-$set and since $\{x \in X : f(x) \leq t_2\}$ is a $G_\delta-$set, by hypothesis it follows that $A(g, t_1) \ \rho \ A(f, t_2)$. The proof follows from Theorem 2.1.

On the other hand, Let F_1 and F_2 are disjoint $F_\sigma-$sets. Set $f = \chi_{F_2}$ and $g = \chi_{F_1^c}$, then f is lsB_1,g is usB_1, and $f \leq g$.

Thus there exists Baire-one function h such that $f \leq h \leq g$. Set $G_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $G_2 = \{x \in X : h(x) \geq 2/3\}$ then G_1 and G_2 are disjoint $G_\delta-$sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. Hence $F_1^{\Lambda} \cap F_2^{\Lambda} = \emptyset$.

Remark 4. [4]. A space X has the $c-$insertion property for (lsc, usc) iff X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection such that $G_n \subseteq F_n$ for each n.

Corollary 3.8. For every F of $F_\sigma-$set, F^Λ is a $F_\sigma-$set and in addition for every decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection, there exists a decreasing sequence $\{G_n\}$ of $G_\delta-$sets with empty intersection s.t. for every $n \in N, F_n \subseteq G_n$ iff X has the B_1-insertion property for (lsB_1, usB_1).

Proof. Since for every F of $F_\sigma-$set, F^Λ is a $F_\sigma-$set, therefore by corollary 3.6, X has the weak B_1-insertion property for (lsB_1, usB_1). Now suppose that f and g are real-valued functions defined on X with $g < f, g$ is lsB_1, and f is usB_1. Set $A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) < 3^{-n+1}\}$. Then since $f-g$ is usB_1, hence $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. By hypothesis, there exists a decreasing sequence $\{D_n\}$ of $G_\delta-$sets with empty intersection s.t., for every $n \in N, A(f-g, 3^{-n+1}) \subseteq D_n$. Hence $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are two disjoint $F_\sigma-$sets and therefore by Lemma 3.7, we have

$$A(f-g, 3^{-n+1})^\Lambda \cap (X \setminus D_n)^\Lambda = \emptyset$$

and therefore by Lemma 3.3, $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separable by Baire-one functions. Therefore by Theorem 2.2, there exists a Baire-one function h on X s.t., $g < h < f$, i.e., X has the B_1-insertion property for (lsB_1, usB_1).

On the other hand, suppose that F_1 and F_2 be two disjoint $F_\sigma-$sets. Since $F_1 \cap F_2 = \emptyset$. We have $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$ and $g = \chi_{F_2}$.

Then since F_2 is a $F_\sigma-$set and F_1^c is a $G_\delta-$set, we conclude that g is lsB_1 and f is usB_1 and furthermore $g < f$. By hypothesis, there exists a Baire-one function h on X s.t., $g < h < f$. Now we set $G_1 = \{x \in X : h(x) \leq \frac{3}{4}\}$ and $G_2 = \{x \in X : h(x) \geq 1\}$. Then G_1 and G_2 are two disjoint $G_\delta-$sets contain F_1 and F_2, respectively. Hence $F_1^\Lambda \subseteq G_1$ and $F_2^\Lambda \subseteq G_2$ and consequently $F_1^\Lambda \cap F_2^\Lambda = \emptyset$. By Lemma 3.7, for every F of $F_\sigma-$set, the set F^Λ is a $F_\sigma-$set.

Now suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection.

We set $F_0 = X$ and $f(x) = \frac{1}{n+1}$ for $x \in F_n \setminus F_{n+1}$. Since $\bigcap_{n=0}^\infty F_n = \emptyset$ and for every $n \in N$ there exists $x \in F_n \setminus F_{n+1}, f$ is well-defined. Furthermore, for every $r \in R$, if $r \leq 0$ then $\{x \in X : f(x) < r\} = \emptyset$ is a $F_\sigma-$set and if $r > 0$ then by Archimedean property of R, there exists $i \in N$ s.t. $\frac{1}{i+1} \leq r$. Suppose that k is the least natural number with this property. Hence $\frac{1}{k} > r$. Now if $\frac{1}{k+1} < r$ then $\{x \in X : f(x) < r\} = F_k$ is a $F_\sigma-$set and if $\frac{1}{k+1} = r$ then $\{x \in X : f(x) < r\} = F_{k+1}$ is a $F_\sigma-$set. Hence f is a usB_1 on X. By setting $g = 0$, we have conclude that g is lsB_1 on X and in addition $g < f$. By hypothesis there exists a Baire-one function h on X s.t., $g < h < f$.

Set $G_n = \{x \in X : h(x) \leq \frac{1}{n+1}\}$. This set is a $G_\delta-$set. But for every $x \in F_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$ thus $h(x) < \frac{1}{n+1}$, this means that $x \in G_n$ and consequently $F_n \subseteq G_n$.

By definition of $G_n, \{G_n\}$ is a decreasing sequence of $G_\delta-$sets and since $h > 0, \bigcap_{n=1}^\infty G_n = \emptyset$. Thus the conditions holds.

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