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PACKET SPREADING AND EINSTEIN RETARDATION

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Abstract

According to the classical special theory of relativity any nonstationary system moving with velocity v must evolve (e.g., decay) $1/\gamma$ times slower than the system at rest, $\gamma = (1 - v^2)^{-1/2}$ (the Einstein retardation ER). Quantum mechanics allows one to calculate the evolution of both systems separately and to compare them thus verifying ER. It is shown here that ER is not valid for a simple system: the spreading packet of the free spinless particle. Earlier it was shown that ER does not hold for some other systems. So one may state that ER is not a universal kinematic law in quantum mechanics.

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1 INTRODUCTION

Experiments show that moving unstable particles (e.g., μ-mesons, π-mesons) decay slower than the particles at rest. More explicitly, let $N_0(t) = N\exp(-t/\tau_0)$ be the decay law of the particles at rest, τ_0 being life-time. Then the life time of particles moving with velocity $\vec{v}$ is $\tau_v = \tau_0\gamma$, $\gamma = (1 - v^2)^{-1/2}$, and the decay law of moving particles is

$$N_v(t) = N\exp(-t/\tau_v) = N\exp(-t/\tau_0\gamma) = N_0(t/\gamma). \qquad (1)$$

One may rewrite Eq. (1) as

$$N_p(t) = N_0(t/\gamma), \qquad \gamma = \sqrt{p^2 + m^2}/m \qquad (2)$$

using the corresponding momentum $\vec{p}$ instead of velocity $\vec{v}$: $\vec{p} = E\vec{v}$.

The usual theoretical explanation of Eqs. (1), (2) is based on the Einsteinian special relativity theory. It is set forth as follows. A moving clock has a slower course as compared with the clock at rest, namely $\gamma d\tau = dt$, e.g., see [1], Ch. 2, Eqs. (36) or (38). The unstable substance may serve as a clock, see [1], Ch. 2. Being the clock, the moving ensemble of unstable particles must decay slower than the ensemble at rest. This is described by Eq. (1): N_v assumes at the moment t the value which N_0 assumes at the moment t/γ.

This argumentation may be applied to any nonstationary physical system which may serve as a clock. Instead of $N_v(t)$ another time-dependent observable $F_v(t)$ may be considered. As the example, the dispersion $\sigma^2(t)$ of the spreading packet may be examined, see Sect. 2 below. In the same manner as above one may argue that the equation

$$F_v(t) = F_0(t/\gamma) \qquad (3)$$

must hold. Equation (1) is a particular case of Eq. (3). Eq. (3) means that F_v assumes at the moment t the value which F_0 assumes at the earlier moment t/γ. I call relation (3) Einsteinian retardation ER. It is a kinematic law in special relativity, see [1], Ch. 2.

However, clocks considered in special relativity are nonquantum objects: they have simultaneously a definite position (e.g., being in frame's origin) and definite velocity (e.g., zero velocity), see the beginning of Ch. (2.6) in [1]. This is impossible for such quantum objects as μ or π mesons. Therefore, the usual explanation of relations (1), (2), (3) is not valid for quantum objects.

However, one may verify the validity of these relations in quantum mechanics. The number of particles and other time-dependent observables may be considered as quantum observables. Using quantum mechanics one may calculate the observables separately for the moving system and the system at rest. Comparing them one may ascertain whether Eqs. (1), (2), (3) hold. For unstable particles this approach was considered in [2]-[6]. The result may be formulated as follows: ER does not hold exactly but it is valid up to high precision.

Oscillating systems (K_0-$\bar{K}_0$ mesons and oscillating neutrino) were considered in [5], [6]. It is shown that large deviations from ER may exist.

A moving nonstationary system was discussed in refs. [6], [7] which evolves faster than the system at rest: $F_v(t) = F_0(\gamma t)$ holds instead of Eq. (3)!

In this paper I consider in Sect. 2 the simple nonstationary system: the spreading wave packet of the free spinless particle. Packet dispersions (see below Eq. (4)) are used as the time-dependent observables which describe packet spreading. In Sect. 2 I calculate the longitudinal dispersion $\sigma_l^2(\vec{v}, t)$ (dispersion along the packet velocity $\vec{v} = \vec{p}/E$). It is compared in Sect. 3 with the dispersion $\sigma^2(0, t)$ of the packet at rest. The connection $\sigma_l^2(\vec{v}, t) = \sigma^2(0, t/\gamma^3)$ is obtained. So $\sigma_l^2(\vec{v}, t)$ is retarded as compared to $\sigma^2(0, t)$ but Eq. (3) is not valid, i.e. ER fails. The premises of the result are summed up in Sect. 3.

2 Dispersions of Gaussian packet

ER is the kinematic statement on the time evolution of nonstationary physical systems which may be considered as clocks. So the quantum mechanical consideration of ER must deal with time-dependent observables (so that the known S-matrix approach is not relevant).

In the capacity of the nonstationary system let us consider the spreading packet of the scalar particle. Let us consider the packet dispersions $\sigma_1^2(t)$, $\sigma_2^2(t)$, $\sigma_3^2(t)$

$$\sigma_j^2(t) = \int d^3x x_j^2 \rho(\vec{x}, t) - \left[\int d^3x x_j \rho(\vec{x}, t) \right]^2, \quad j = 1, 2, 3 \tag{4}$$

as time-dependent observables. Here $\rho(\vec{x},t)$ is the probability density to find the particle at the point $\vec{x}$ at time t. The density and dispersions may be experimentally measured. In quantum theory ρ is expressed in terms of the packet wave function Ψ, the positive-energy solution of the Klein-Gordon equation

$$i\partial\Psi/\partial t = \hat{E}\Psi, \quad \hat{E} \equiv \left[(-i\partial/\partial\vec{x})^2 + m^2\right]^{1/2}. \tag{5}$$

The known usual expression of ρ is $\rho(\vec{x},t) \sim \hat{E}\Psi^*(\vec{x},t)\Psi(\vec{x},t) + \Psi^*(\vec{x},t)\hat{E}\Psi(\vec{x},t)$, e.g. see [8], Ch. 3. However, the expression is not positive definite function of $\vec{x}$, e.g. see [9], Supplement II. Therefore, it does not suit as a probability density, although $\int \rho(\vec{x},t)d^3x$ is positive and may be normalized to unity.

Here I use Newton-Wigner wave function $\Psi_{\rm NW}$, see [10], Eq. (5). In their representation $\rho(\vec{x},t) = \Psi^*_{\rm NW}(\vec{x},t)\Psi_{\rm NW}(\vec{x},t)$, see [10], Eq. (6). In this equation and in what follows the letter $\vec{x}$ denotes the Newton-Wigner coordinate, see [8], Ch. 3. The function $\Psi_{\rm NW}$ will be denoted by Ψ.

The solution of (5) then may be represented as

$$\Psi(\vec{x},t) = (2\pi)^{-3/2}\int d^3k \exp(i\vec{k}\vec{x})\Phi(\vec{k})\exp(-itE_k), \;\; E_k = \sqrt{k^2+m^2}, \tag{6}$$

see [8], Chs. 7 and 3; [10], Eq. (5). $\Phi(\vec{k})$ is the initial wave function of the packet in momentum representation. For $\Phi(\vec{k})$ let us choose the product of three Gaussian packets

$$\Phi(\vec{k}) = \varphi_1(k_1)\varphi_2(k_2)\varphi_3(k_3), \quad \varphi_j(k_j) = M\exp[-(k_j-p_j)^2\sigma^2]. \tag{7}$$

The functions φ_j, $j = 1, 2, 3$, are normalized to unity

$$\int dk_j|\varphi_j(k_j)|^2 = 1 \tag{8}$$

if $M^2 = \sigma\sqrt{2/\pi}$. Then $\Phi(\vec{k})$ is also normalized: $\int d^3k|\Phi(\vec{k})|^2 = 1$.

It is easy to show that the parameters p_j in Eq. (7) are components of the mean momentum of the packet:

$$\begin{aligned}\int d^3k k_j|\Phi(\vec{k})|^2 &= M^2\int_{-\infty}^{+\infty} dk_j k_j|\varphi_j(k_j)|^2 \\ &= M^2\int dk'_j(p_j+k'_j)\exp[-2(k')^2\sigma^2] = p_j. \end{aligned}\tag{9}$$

The derivation uses the normalization (8), the change of the integration variables $k'_j = k_j - p_j$, the parity of the function $\exp[-2(k')^2\sigma^2]$.

The initial wave function in the coordinate representation $\Psi(\vec{x}, 0)$, see Eq. (6) at $t = 0$, also reduces to the product of three factors

$$\Psi(\vec{x}, 0) = \psi(x_1)\psi(x_2)\psi(x_3).$$

However $\Psi(\vec{x}, t)$, Eq. (5), cannot be represented in such a simple form because of the factor

$$\exp[-it(k_1^2 + k_2^2 + k_3^2 + m^2)^{1/2}]$$

in the integrand of Eq. (6).

The triple integral in Eq. (6) may be calculated approximately if the parameter σ is large enough, e.g. cf. [11], Ch. 3. To show this, let us change the integration variables $\vec{k}' = \vec{p} - \vec{k}$ in Eq. (6):

$$\begin{aligned}\Psi(\vec{x}, t) \quad = \quad & (2\pi)^{-3/2}M^3 \int d^3k' \exp[i(\vec{p} - \vec{k}')\vec{x}] \\ & \times \exp[-(\vec{k}')^2\sigma^2] \exp\{-it[(\vec{p} - \vec{k}')^2 + m^2]^{1/2}\}. \quad (10)\end{aligned}$$

The function $\exp[-(\vec{k}')^2\sigma^2]$ cuts off the values of $(\vec{k}')^2$ which are much larger than $1/\sigma^2$. So one may assume, e.g., $k' < 3/\sigma$. Let σ be much larger than the Compton wave length $\lambda_m = 1/m$, e.g. $\sigma > 3\lambda_m$ or $3/\sigma < m$. It will be shown below that σ^2 is space dispersion of the initial packet (see Eq. (24)). It follows from the inequalities $k' < 3/\sigma$ and $3/\sigma < m$ that $k' \ll m$. Then $k' \ll \sqrt{p^2 + m^2} \equiv E$ all the more. As $k'/E \ll 1$, one may expand

$$\begin{aligned}\sqrt{(\vec{p} - \vec{k}')^2 + m^2} \quad &= \quad \sqrt{\vec{p}^2 + m^2 + (\vec{k}')^2 - 2(\vec{p}\vec{k}')} \\ &= \quad E\sqrt{1 - 2(\vec{p}\vec{k}')/E^2 + (\vec{k}')^2/E^2}\end{aligned}$$

in the series over degrees of k'/E. Using the expansion

$$\sqrt{1+\alpha} = 1 + \alpha/2 - \alpha^2/8 + \ldots, \quad \alpha = -2(\vec{p}\vec{k}')/E^2 + (\vec{k}')^2/E^2$$

and neglecting the term smaller than $(k'/E)^2$ one gets

$$\sqrt{(\vec{p} - \vec{k}')^2 + m^2} \cong E[1 - (\vec{p}\vec{k}')/E^2 + (\vec{k}')^2/2E^2 - (\vec{v}\vec{k}')/2E^2], \ \vec{v} = \vec{p}/E. \quad (11)$$

Let us direct the third axis $\vec{e}_3$ ($\vec{e}_z$) of the coordinate frame along $\vec{p}$ so that $\vec{p} = (0, 0, p)$ and $\vec{v} = (0, 0, v)$ (p denotes $|\vec{p}|$ and v denotes $|\vec{v}|$). In this frame (11) turns into

$$\sqrt{(\vec{p} - \vec{k}')^2 + m^2} \cong E\{1 - pk'_z/E^2 + [(k'_1)^2 + (k'_2)^2 + (1 - v^2)(k'_3)^2]/2E^2\}. \tag{12}$$

Note that no supposition on p value has been assumed so that $0 \leq p < \infty$. Using the approximation (12) in Eq. (10) one gets that the triple integral in Eq. (10) reduces to the product of three single-valued integrals:

$$\Psi(\vec{x}, t) \cong \psi_1(x_1, t)\psi_2(x_2, t)\psi_3(x_3, t),$$

$$\psi_1(x_1, t) = (2\pi)^{-1/2} M I_1(x_1, t);\ \psi_2(x_2, t) = (2\pi)^{-1/2} M I_2(x_2, t); \tag{13}$$

$$\psi_3(x_3, t) = (2\pi)^{-1/2} M \exp[ipx_3 - itE] I_3(x_3, t); \tag{14}$$

$$I_j(x_j, t) = \int_{-\infty}^{+\infty} dk'_j \exp[-ik'_j(x_j - v_j t)] \exp[-i(k'_j)^2 (a_j)^2],\quad j = 1,\ 2,\ 3; \tag{15}$$

$$a_1^2 = a_2^2 = \sigma^2 + it/2E, \quad a_3^2 = \sigma^2 + it(1 - v^2)/2E,$$
$$v_1 = v_2 = 0, \quad v_3 = v. \tag{16}$$

For integrals $I_j(x_j, t)$, Eq. (15), see e.g. [12], Ch. 2.5.36.1:

$$I_j(x_j, t) = \sqrt{\pi/a_j} \exp[-(x_j - v_j t)^2/4a_j^2]. \tag{17}$$

Using other tabular integrals one may verify that $\psi_j(x_j, t)$, Eqs. (13), (14), (15), are normalized:

$$\int dx_j |\psi_j(x_j, t)|^2 = (2\pi)^{-1} M^2 \int_{-\infty}^{+\infty} dx_j |I_j(x_j, t)|^2 = 1. \tag{18}$$

Let us calculate the mean positions $X_n(t)$, $n = 1, 2, 3$, of the moving packet at the moment t:

$$X_n(t) = \iiint d^3x x_n |\Psi(\vec{x}, t)|^2 \cong \int dx_n x_n |\psi_n(x_n, t)|^2$$
$$= (2\pi)^{-1} M^2 \int dx_n x_n |I_n(x_n, t)|^2. \tag{19}$$

Here I use the normalization (18) of the function ψ_j with $j \neq n$ and then use Eqs. (13), (14). Further the change $x'_n = x_n - v_n t$ of the integration variables is used in the last integral in Eq. (19). Finally, the parity of I_n is taken into account: $I_n(x'_n, t) = I_n(-x'_n, t)$. One obtains the result

$$X_1(t) \cong X_2(t) \cong 0, \;\; X_3 \cong vt, \;\; v = |\vec{v}| = v_3 = p/E, \;\; E = \sqrt{p^2 + m^2}. \tag{20}$$

This means that the center of the packet moves along $\vec{p}$ with the velocity $\vec{v} = \vec{p}/E$. In addition, the packet spreads, the spreading being characterized by the packet dispersions σ_1^2, σ_2^2, σ_3^2, see Eq. (4).

Let us name $\sigma_3^2(\vec{p}, t)$ the longitudinal dispersion $\sigma_l^2(\vec{p}, t)$ (dispersion along $\vec{p}$) and σ_1^2, σ_2^2 transversal ones. Using Eqs. (4), (19), (20) one obtains for σ_l^2:

$$\begin{aligned}\sigma_l^2(\vec{p}, t) \;\; &\equiv \;\; \sigma_3^2(\vec{p}, t) = \int dx_3 x_3^2 |\psi_3(x_3, t)|^2 - (vt)^2 \\ &= \;\; (2\pi)^{-1} M^2 \int dx_3 x_3^2 |I_3(x_3, t)|^2 - (vt)^2. \end{aligned} \tag{21}$$

The further derivation of σ_l^2 is more tedious than the calculation of $X_3(t)$. The result is

$$\sigma_l^2(\vec{p}, t) = \sigma^2 + t^2(1 - v^2)^2/4E^2\sigma^2, \quad E = \sqrt{p^2 + m^2} = m\gamma. \tag{22}$$

In the same manner one obtains the transversal dispersions

$$\sigma_1^2(\vec{p}, t) = \sigma_2^2(\vec{p}, t) = \sigma^2 + t^2/4E^2\sigma^2. \tag{23}$$

If the packet is at rest ($v = 0$, $p = 0$, $E = m$) all dispersions are equal:

$$\sigma_l^2(0, t) = \sigma_1^2(0, t) = \sigma_2^2(0, t) = \sigma^2 + t^2/4m^2\sigma^2. \tag{24}$$

When $t = 0$ Eqs. (21), (22) turn into the initial dispersions:

$$\sigma_l^2(\vec{p}, 0) = \sigma_1^2(\vec{p}, 0) = \sigma_2^2(\vec{p}, 0) = \sigma^2.$$

This equation makes clear the physical meaning of the parameter σ in Eq. (7).

Note. The quantities $\sigma_j^2(t)$ occurred in paper [13], see Eq. (24) in App. A. There they played the role of notations, their physical meaning being not revealed. It was shown here that these quantities (denoted as $\sigma_j^2(\vec{p},t)$ here) do have the meaning of packet dispersions at the moment t. Remark also the error in writing the expression for $\sigma_3^2(t)$ in Eq. (24) in [11]: there $(1-v^2)$ must be squared, cf. Eq. (22) here.

3 Discussion

Let us compare the dispersions of the moving packet and the packet at rest. Using the designations

$$E = \sqrt{p^2+m^2} = \gamma m, \quad \gamma = (1-v^2)^{-1/2},$$

rewrite Eqs. (21), (22) in the form

$$\sigma_l^2(\vec{p},t) = \sigma^2 + t^2/\gamma^6 m^2 \sigma^2, \tag{25}$$

$$\sigma_1^2(\vec{p},t) = \sigma_2^2(\vec{p},t) = \sigma^2 + t^2/\gamma^2 m^2 \sigma^2. \tag{26}$$

Comparing with the dispersions $\sigma^2(0,t)$ of the packet at rest, see Eq. (24), one obtains

$$\sigma_l^2(\vec{p},t) = \sigma^2(0, t/\gamma^3), \tag{27}$$

$$\sigma_1^2(\vec{p},t) = \sigma_2^2(\vec{p},t) = \sigma^2(0,t/\gamma). \tag{28}$$

The transversal dispersions of the moving packet evolves slower than the dispersions of the packet at rest. Its slowing down is Einsteinian: the dispersions $\sigma_1^2(\vec{p},t)$ and $\sigma_2^2(\vec{p},t)$ at the moment t assume the value which $\sigma^2(0,t)$ assumes at the earlier moment $1/\gamma$ (see ER definition in the Introduction).

The longitudinal spreading σ_l^2 also grows slower than $\sigma^2(0,t)$, but the retardation of σ_l^2 is not ER, see Eq. (27). So ER fails. I suppose that this result deserves its detailed derivation in Sect. 2. The derivation used the following premises.

The packet of scalar (spinless) particle is described by the wave function $\Psi(\vec{x},t)$ which satisfies relativistic positive-energy Klein-Gordon equation.

The initial packet state is described by the simple Gaussian function of macroscopical space size.

To calculate $\Psi(\vec{x}, t)$, the usual approximation was exploited, see Eq. (12).

Using $\Psi(\vec{x}, t)$ the packet dispersions at the moment t were obtained. Unlike $\Psi(\vec{x}, t)$ the dispersions are the observable nonstationary quantities which can be experimentally measured.

Examples of nonstationary systems for which ER fails were given in [5]-[7]. The considered system complements the examples. So one may state that ER is not a universal (kinematic) law for quantum clocks.

However, in the case of unstable particles quantum mechanics shows that ER holds with high precision [3]-[5]. Experiments also agree with ER, see e.g. the corresponding references in [3]-[6]. As was argued in Introduction, usual explanation of ER (based on Lorentzian transformations of position and time) is nonapplicable for quantum clocks. It is quantum mechanics which provides the suitable theoretical explanation.

References

[1] C. Møller. The theory of Relativity (Clarendon, Oxford, 1972), Ch. 2.6.

[2] P. Exner. "Representations of the Poincare Group Associated with unstable particles". Phys. Rev. D **28**, 2621-2627 (1983).

[3] E. Stefanovich. "Quantum Effects in Relativistic Decays". Int. J. Theor. Phys. **35**, 3534-2554 (1996).

[4] L. Khalfin. "Quantum Theory of Unstable Particles and Relativity". http://www.pdmi.ras.ru/preprint/1997/97-06.html.

[5] M. Shirokov. "Decay Law of Moving Unstable Particles". Int. J. Theor. Phys. **43**, 1541-1553 (2004).

[6] M. Shirokov. "Evolution in Time of Moving Unstable Systems". Concepts Phys. **3**, 193-205 (2006).

[7] M. Shirokov. "Moving System with Speeded-up Evolution". Physics of Particles and Nuclei Letters. **6**, 14-17 (2009).

[8] S. Schweber. An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Co. Evanston, Elmsford, 1961).

[9] D. Blokhintsev. JINR preprint P-2631 (1966).

[10] R. Haag. Suppl. Nuovo Cim. **14**, 134-152 (1959).

[11] M. Goldberger, K. Watson. Collision Theory (Wiley and Sons, New York, 1964).

[12] A. Prudnikov, Yu. Brychkov, O. Marichev. Integrals and Series ("Nauka", Moscow, 1981).

[13] M. Shirokov, V. Naumov. "Time-to-Space Conversion in Neutrino Oscillations". Concepts Phys. **4**, 127-143 (2007).

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Comment on
PACKET SPREADING AND EINSTEIN RETARDATION

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Shirokov considers the dispersion $\sigma(0,t)$ of a positive-energy wave function for a scalar relativistic particle with zero mean momentum. In a moving frame, $\mathbf{v}$=$\mathbf{p}$/m, the dispersion is denoted by $\sigma(\mathbf{p},t)$. Shirokov exhibits an example where $\mathbf{p}=p\mathbf{e}_3$ and where the dispersion, σ_3, for the x_3 direction is given by

$$\sigma_3(\mathbf{p},t)=\sigma_3(0,t/\gamma^3), \qquad \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

He argues that the dispersion may be regarded as a clock and that one should therefore expect an Einsteinian retardation γ/t instead of γ/t^3.

The following observations immediately come to mind.

- The determination of the dispersion requires measurements over all space. Moreover, for large times, the spreading increases to infinity so that the "clock" is spread over all space. In contrast, classical clocks are assumed to be localized, and the usual statements of relativity theory depend on this.

- Shirokov uses the Newton-Wigner position operator to calculate σ. However, it is known that this operator possesses unphysical causality properties [1]. Moreover, these unphysical properties are shared by any conceivable choice of a self-adjoint position operator [2]. Is the t/γ^3 result also true for more general operators of localization?

- The approximation used in Eqs. (11/12) seems to imply a restriction on t because the approximation is performed in a phase and therefore one must have that the error multiplied by t is much less than.

Interesting as the behavior of the dispersion for moving particles may be - even when employing the not very satisfactory Newton-Wigner position operator - in my opinion its interpretation in the context of quantum and relativity theory warrants further discussion.

References

[1] G. N. Fleming. Phys. Rev. B 139, 963 (1965).

[2] G. C. Hegerfeldt. Phys. Rev. Lett. 54, 2395 (1985);
in: Irreversibility and Causality. Edited by A. Bohm, H.-D. Doebner, and P. Kielanowski. Springer Lecture Notes in Physics 504, p. 238 - 245 (1998).