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DEMONSTRATION OF THE SPIN-STATISTICS CONNECTION IN ELEMENTARY QUANTUM MECHANICS

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(Received 21 August 2007; accepted 21 December 2007)

Abstract

A simple demonstration of the spin-statistics connection is presented. The effect of exchange and space inversion operators on two-particle states is reviewed. The connection follows directly from successive application of these operations to the two-particle wave function for identical particles in an s-state, evaluated at spatial coordinates $\pm\mathbf{x}$, but at equal time, *i.e.*, at spacelike interval.

DOI: 10.2478/v10005-007-0035-6

1 Introduction

The connection between spin and statistics, first conjectured by Pauli, and subsquently proved by Pauli [1], Burgoyne [2], Lüders and Zumino [3], and others, has an understandable appeal to students of physics as an example of a phenomenon arising from quantum mechanics and relativity that has palpable consequences in the realm of everyday experience [4, 5, 6]. This paper presents a demonstration of the spin-statistics connection by an simple argument involving symmetry of two-particle wave functions under the combined operations of exchange and parity. It is intended to be accessible to final-year undergraduate students of quantum mechanics, who will have had exposure to simple angular momentum theory, the Pauli principle, and the concept of parity.

It is well-known that, while relativistic quantum theory supplies sufficient conditions for validity of the spin-statistics connection, the question of just how weak the necessary conditions can be remains open. That question is not addressed here. The demonstration given here renders in (largely) elementary language a proof originally devised for $(j, 0)$ or $(0, j)$ irreducible representations of the Poincaré group, sometimes called Weinberg fields [7, 8, 9, 10]. It exploits the properties of two-particle states constructed from identical noninteracting states of massive particles corresponding to Weinberg fields. These single-particle states have the simple, definite symmetries required for the following argument: They are irreducible representations of the rotation group, possess definite intrinsic parity, and satisfy local commutativity.

2 Background

In the following, a (single-particle) state may be described by the ket vector $|\phi\rangle$ or by the wave function $\phi(\mathbf{x}, t) = \langle \mathbf{x}, t|\phi\rangle$. The proof which follows concerns symmetries of two-particle wave functions evaluated at a pair of spacetime positions lying at spacelike interval from one another. It is possible, therefore, to specify a Lorentz frame in which the coordinates occur at equal time. The dependence upon time will usually not be shown.

2.1 Quantum states of higher spin

We start by reciting results from the theory of angular momentum in quantum mechanics that find use in the following. The total angular momentum operators J_i give rise to infinitesimal rotations of a state about the x_i axes [11]. Eigenstates of total angular momentum $\hbar j$ can take on a range of values for the z-projection of angular momentum [12],

$$\langle m|J_z|\phi_j\rangle = m\hbar\langle m|\phi_j\rangle, \tag{1}$$

where the magnetic quantum number m has values in the range [13]

$$-j \leq m \leq j. \tag{2}$$

Coupling of two single-particle states to a state of specified angular momentum is accomplished with a unitary transformation whose matrix elements are Clebsch-Gordan coefficients [14]. The Clebsch-Gordan coefficient coupling two states with total and magnetic angular momentum quantum numbers (j_a, m_a) and (j_b, m_b), respectively, to a state with quantum numbers (J, M) is denoted $\langle j_a m_a j_b m_b|JM\rangle$. Thus,

$$|JM\rangle = \sum_{-j_a\leq m_a\leq j_a;-j_b\leq m_b\leq j_b} \langle j_a m_a j_b m_b|JM\rangle|j_a m_a; j_b m_b\rangle. \tag{3}$$

2.2 The exchange operator

The exchange operator $\mathcal{X}$ acting on the state

$$|\psi\rangle = |\phi(1)\rangle|\phi(2)\rangle \tag{4}$$

gives [15, 16]

$$\mathcal{X}|\phi(1)\rangle|\phi(2)\rangle = |\phi(2)\rangle|\phi(1)\rangle. \tag{5}$$

It is assumed [17] the state $|\psi\rangle$ is either symmetric (*bosonic*) or antisymmetric (*fermionic*) under exchange of $|\phi(1)\rangle$ and $|\phi(2)\rangle$,

$$\mathcal{X}|\psi\rangle = \pm|\psi\rangle. \tag{6}$$

Consider now the inverse to $\mathcal{X}$. Given $|\psi\rangle$ and another two-particle state $|\xi\rangle$, their matrix element $\langle\xi|\psi\rangle$ should be left unchanged by application of $\mathcal{X}$ to both states:

$$\mathcal{X}\langle\xi|\mathcal{X}|\psi\rangle = \langle\xi|\psi\rangle, \tag{7}$$

which is readily seen to be the same as

$$\langle \xi | \mathcal{X}^\dagger \mathcal{X} | \psi \rangle = \langle \xi | \psi \rangle, \tag{8}$$

or

$$\mathcal{X}^{-1} = \mathcal{X}^\dagger. \tag{9}$$

2.3 The parity operator

The result of the space inversion, or parity, operation on a spinless state $|\xi_0\rangle$ is [18]

$$\langle \mathbf{x}, t | \mathcal{P} | \xi_0 \rangle) = \langle -\mathbf{x}, t | \xi_0 \rangle. \tag{10}$$

Parity acting on position and momentum variables gives

$$\begin{aligned} \mathbf{x} &\Rightarrow -\mathbf{x}, \\ \mathbf{p} &\Rightarrow -\mathbf{p}. \end{aligned} \tag{11}$$

It follows that (orbital) angular momentum is unaltered by the parity operator:

$$\mathbf{x} \times \mathbf{p} \equiv \mathbf{L} \Rightarrow \mathbf{L}, \tag{12}$$

and that the $\mathcal{P}$ operation commutes with rotations. In order that $\mathcal{P}$, which may be regarded as a passive coordinate transformation, not alter the total angular momentum of a wave function possessing both orbital and spin angular momentum degrees of freedom, its effect on components of a state with definite, nonzero spin must likewise be diagonal, allowing us to write

$$\langle \mathbf{x}, m | \mathcal{P} | \xi_j \rangle = \langle -\mathbf{x}, m | \xi_j \rangle. \tag{13}$$

An eigenstate of parity obeys

$$\langle \mathbf{x}, t | \mathcal{P} | \psi \rangle = \eta \langle \mathbf{x}, t | \psi \rangle. \tag{14}$$

As two successive applications of the parity operation give the identity [20],

$$\mathcal{P}^2 = 1, \tag{15}$$

which implies

$$\eta^2 = 1; \tag{16}$$

$$\eta = \pm 1 \tag{17}$$

for a state of definite parity. States with $\eta = +1$ are symmetric under space inversion (*even* parity), while states with $\eta = -1$ are antisymmetric (*odd* parity). Parity is a unitary operator, so we also have

$$\mathcal{P}^{\dagger} = \mathcal{P}^{-1} = \mathcal{P}, \tag{18}$$

analogous to (9).

If the states $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively have parities η_1 and η_2, then the combined state $|\psi_1\rangle|\psi_2\rangle$ has parity

$$\eta_{12} = \eta_1\eta_2. \tag{19}$$

3 The connection between spin and statistics

The connection is proved with the aid of a wave function that is the amplitude for the particles in a two-particle state $|\psi\rangle$ to be a relative s-state:

$$\langle \mathbf{r}_1, \mathbf{r}_2; 00|\psi\rangle = \sum_m \langle jmj-m|00\rangle\langle \mathbf{r}_1 m; \mathbf{r}_2 - m|\psi\rangle. \tag{20}$$

Given (20), we are at liberty to evaluate it at $\mathbf{r}_1 = \mathbf{x}$ and $\mathbf{r}_2 = -\mathbf{x}$:

$$\langle \mathbf{x}, -\mathbf{x}; 00|\psi\rangle = \sum_m \langle jmj-m|00\rangle\langle \mathbf{x}m; -\mathbf{x} - m|\psi\rangle. \tag{21}$$

Consider the effect of exchange and space inversion operations on the wave functions appearing in equation (21). We have

$$\mathcal{X}|\psi\rangle = \pm|\psi\rangle \tag{22}$$

as the particles obey Bose (+) or Fermi (−) exchange symmetry, [21] and

$$\mathcal{X}|\mathbf{x}m; -\mathbf{x} - m\rangle = |-\mathbf{x} - m; \mathbf{x}m\rangle \tag{23}$$

so that

$$\langle \mathbf{x}m; -\mathbf{x} - m|\psi\rangle = \langle \mathbf{x}m; -\mathbf{x} - m|\mathcal{X}^{-1}\mathcal{X}|\psi\rangle = \pm\langle -\mathbf{x} - m; \mathbf{x}m|\psi\rangle. \tag{24}$$

Next, apply the parity operator to the wave function appearing on the RHS of (24). The state $|\psi\rangle$ is composed of products of identical single-particle states $|\phi_m\rangle$. According to (19) the parity of such a product must be even,

$$\mathcal{P}|\psi\rangle = |\psi\rangle \tag{25}$$

with

$$\mathcal{P}|-\mathbf{x}-m;\mathbf{x}m\rangle = |\mathbf{x}-m;-\mathbf{x}m\rangle \tag{26}$$

leading to

$$\langle -\mathbf{x}-m;\mathbf{x}m|\mathcal{P}^{-1}\mathcal{P}|\psi\rangle = \langle \mathbf{x}-m;-\mathbf{x}m|\psi\rangle. \tag{27}$$

Inserting (27) into (24) gives

$$\langle \mathbf{x}m;-\mathbf{x}-m|\psi\rangle = \pm\langle \mathbf{x}-m;-\mathbf{x}m|\psi\rangle. \tag{28}$$

Upon substituting (28) into (21),

$$\langle \mathbf{x},-\mathbf{x};00|\psi\rangle = \pm\sum_m \langle jmj-m|00\rangle\langle \mathbf{x},-m;-\mathbf{x},m|\psi\rangle. \tag{29}$$

We may invert the order of summation by replacing m with $-m'$ to get

$$\langle \mathbf{x},-\mathbf{x};00|\psi\rangle = \pm\sum_{m'} \langle j-m'jm'|00\rangle\langle \mathbf{x}m';-\mathbf{x}-m'|\psi\rangle. \tag{30}$$

At this point it is advantageous to rewrite (30) in a suggestive way. The Clebsch-Gordan coefficient appearing in (30) is [22]

$$\langle j-m'jm'|00\rangle = \frac{(-1)^{(j+m')}}{\sqrt{2j+1}}. \tag{31}$$

Note that the quantity $j-m'$ is always an integer, and $2j-2m'$ an even integer. We may write

$$(-1)^{m'} = (-1)^{m'}(-1)^{2j-2m'} = (-1)^{2j}(-1)^{-m'} \tag{32}$$

and conclude

$$\langle j-m'jm'|00\rangle = (-1)^{2j}\langle jm'j-m'|00\rangle. \tag{33}$$

Employing this relation in (30) and recalling (21) gives us

$$\langle \mathbf{x},-\mathbf{x};00|\psi\rangle = \pm(-1)^{2j}\langle \mathbf{x},-\mathbf{x};00|\psi\rangle. \tag{34}$$

The singlet wave function appearing in (34) is nonvanishing if the individual wave functions from which it is constructed are themselves nonvanishing. A proof of this assertion appears in the Appendix. If we can assume the matrix element on both sides of (34) does not vanish, we immediately have

$$1 = \pm(-1)^{2j}. \tag{35}$$

According to (35), states $|\mathbf{x}, -\mathbf{x}\rangle$ with $2j$ *even* necessarily have *Bose* exchange symmetry, while those with $2j$ *odd* necessarily have *Fermi* symmetry. This is the connection between spin and statistics.

4 Discussion

The demonstration just presented is neither so simple nor rigorous as the formal proof in relativistic field theory given by Burgoyne [2]. On the other hand it does rely, for the most part, upon concepts and methods taken from elementary quantum mechanics. Apart from the material appearing in the appendix, it depends on nothing that cannot readily be obtained (or at a minimum, motivated) starting from pertinent discussions in the Feynman Lectures. It may appear that the proof as given in Section 3 could be accomplished without any reliance upon relativistic quantum mechanics. However, at certain points the argument rests upon assumptions that flow in a natural and unforced way from requirements of relativistic symmetry, but which would arguably enter a truly nonrelativistic exposition in neither fashion.

An instance is the symmetry of the wave function in (24), which is a disguised statement of an equal-time commutation relation. Exhibiting the dependence upon t, (24) becomes

$$\langle(\mathbf{x}, t)m; (-\mathbf{x}, t) - m|\psi\rangle = \pm\langle(-\mathbf{x}, t) - m; (\mathbf{x}, t)m|\psi\rangle. \tag{36}$$

In (36), the wave functions that give the probability amplitude for the particles (1) and (2) at spatial position $\pm\mathbf{x}$ are evaluated at equal time t. Put another way,

$$|\mathbf{x}_2 - \mathbf{x}_1|^2 - (t_2 - t_1)^2 > 0 \tag{37}$$

The statement that a relation holds between two points separated by nonzero distance at equal time has no unambiguous meaning in special relativity [23]. Equation (37), however, is an invariant statement under arbitrary Lorentz transformations. In proofs of the spin-statistics relation, the exchange symmetry that appears in (24) is normally stipulated subject to (37). One says that wave functions of identical particles commute or anticommute outside the light cone [24].

Moreover, it was assumed that massive particle states exist with certain simple, conjoined symmetries with respect to the operations of parity and rotation. As noted earlier, the assumed symmetries of the states are those of an irreducible representation of the Poincaré group. [9] Thus, elements of the present demonstration that would enter a genuinely nonrelativistic proof as distinct hypotheses all follow from the single requirement of Poincaré invariance in an explicitly relativistic treatment. Granted this observation, the nonrelativistic view does not appear to be the parsimonious one, even should it be possible to construct a completely nonrelativistic proof.

5 Appendix

We apologize for the fact that we cannot give you an elementary explanation.

-R. P. Feynman, Ref. [11], Vol. III, p. 4-3

In the following it will be convenient to write two-particle wave functions in factored form so that, *e. g.,* the wave function in (21) is written as

$$\langle \mathbf{x}m; -\mathbf{x} - m|\psi\rangle = \langle \mathbf{x}m|\phi_j(1)\rangle\langle -\mathbf{x} - m|\phi_j(2)\rangle. \tag{38}$$

From single-particle wave functions for spin j, which may be assumed to belong to an irreducible representation of the rotation group, form

$$(\xi_j, \phi_j) \equiv (-1)^{-j} \sum_m \int d^3x \, \langle jmj - m|00\rangle \langle \mathbf{x}m|\xi_j\rangle \langle \mathbf{x}m|\phi_j\rangle^*. \tag{39}$$

This quantity serves as an inner product in the Hilbert space of wave functions on $\mathbf{R}_3$ [8]. In

$$(\phi_j, \phi_j) = (-1)^{-j} \sum_m \int d^3x \, \langle jmj - m|00\rangle \langle \mathbf{x}m|\phi_j\rangle \langle \mathbf{x}m|\phi_j\rangle^*. \tag{40}$$

we may write

$$\langle \mathbf{x}m|\phi_j\rangle = f_j(r)\mathcal{Y}_{jm}(\Omega) \tag{41}$$

at radius r. Here the function $\mathcal{Y}_{jm}(\Omega)$ is a suitable angular momemtum eigenfuncton that generalizes the properties of spherical harmonics to include half-integral as well as integral angular momenta [25, 26]. It may be defined so as to share with ordinary spherical harmonics $Y_{lm}(\Omega)$ the conjugation property

$$\mathcal{Y}^*_{jm} = (-1)^m \mathcal{Y}_{j\,-m}. \tag{42}$$

We also have

$$\int d\Omega \mathcal{Y}^*_{jm'}\mathcal{Y}_{jm} = \int d\Omega \mathcal{Y}_{jm'}\mathcal{Y}^*_{jm} = \delta_{m'm}. \tag{43}$$

The angular momentum ladder operators $J_\pm$ are defined by

$$J_\pm = J_x \pm iJ_y \tag{44}$$

and have the effect of raising and lowering m:

$$\langle m|J_\pm|\phi_j\rangle = -i\hbar\sqrt{(j \mp m)(j \pm m + 1)}\langle m \pm 1|\phi_j\rangle \tag{45}$$

The $J_\pm$ are differential operators that act on orbital and spin degrees of freedom *only* [25]. This observation means that the $J_\pm$ raise and lower m in $\mathcal{Y}_{jm}(\Omega)$ and have no effect upon $f_j(r)$. The radial weight $f_j(r)$ can, therefore, have no dependence upon m [27]. Recalling the definition of the Clebsch appearing in (40) (*vide.* (31)), we find

$$(\phi_j, \phi_j) = \int r^2 dr\, f_j(r) f^*_j(r) \geq 0, \tag{46}$$

with equality iff $f_j(r)$ vanishes everywhere. Should

$$\sum_m \langle jmj-m|00\rangle\langle \mathbf{x}m|\phi_j\rangle\langle \mathbf{x}m|\phi_j\rangle^* = 0, \forall\, \mathbf{x} \tag{47}$$

then (ϕ_j, ϕ_j) will vanish. But $(\phi_j, \phi_j) = 0$ iff $\langle \mathbf{x}m|\phi_j\rangle$ vanishes, as well.

Assume $|\zeta_j\rangle$ is a state of a spin j particle such that

$$\langle \mathbf{x}m|\zeta_j\rangle \neq 0. \tag{48}$$

From $|\zeta_j\rangle$ form

$$\langle \mathbf{x}m|\phi_j\rangle \equiv \langle \mathbf{x}m|\zeta_j\rangle^* \pm \langle -\mathbf{x}-m|\zeta_j\rangle. \tag{49}$$

Then

$$\begin{aligned} &\sum_m \langle jmj-m|00\rangle\langle \mathbf{x}m|\phi_j\rangle\langle -\mathbf{x}-m|\phi_j\rangle = \\ &\quad = \pm\sum_m \langle jmj-m|00\rangle\langle \mathbf{x}m|\phi_j\rangle\langle \mathbf{x}m|\phi_j\rangle^*. \end{aligned} \tag{50}$$

As a general rule, the wave function $\langle \mathbf{x}m|\phi_j\rangle$ will have nonvanishing norm and the RHS of (50) will differ from zero. But suppose that for one choice of sign in (49), $\langle \mathbf{x}m|\phi_j\rangle$ were to vanish $\forall\,\mathbf{x}$. In that event $\langle \mathbf{x}m|\phi_j\rangle$, and hence (50), cannot vanish for the other choice. We suppose in the main text that the appropriate choice of sign has been made, if necessary, and that (21) is therefore nonvanishing on some open set of $\mathbf{x}$.

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Comment

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Comment on DEMONSTRATION OF THE SPIN-STATISTICS CONNECTION IN ELEMENTARY QUANTUM MECHANICS

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The spin-statistics theorem, as well as the PCT theorem, is usually formulated in the framework of the relativistic quantum field theory and subsequently proved using all the machinery of this theory with special emphasis put on the locality principle [1]. In this context the spin-statistics theorem is considered as a consequence of basic axioms underlying the relativistic quantum field theory. Moreover, it is believed to be provable only within the formalism of the relativistic quantum field theory which differs its status from other properties of physical systems usually considered as consequences of relativistic invariance but in fact shown to be correctly described also within consistently formulated Galilean covariant field-theoretical models [2]. As examples of such prejudices one can mention the spin and the gyromagnetic ratio of the electron, or the existence of antiparticles. Being dominated, or even suppressed, by both the formalism of the relativistic quantum field theory and requirements of its mathematical rigor we used to accept such an approach and in fact we have not paid attention to the fact that the spin-statistics connection is valid for physical systems independently from the choice of their correct description, relativistic or nonrelativistic. But if it is so, there should exist consistent explanations (not necessary "proofs" in a rigorous mathematical sense) of the spin-statistics connection which use notions taken from nonrelativistic quantum physics and, eventually, possible to be formulated in the language of standard quantum mechanics. It is my feeling that J.A.Morgan's work follows such a way of thinking and provides the reader, hopefully also a student who passed a course of elementary quantum mechanics and who is still interested in learning and understanding quantum physics, an example that the spin-statistics connection is deeply built-in into physics and that we should not treat its appearence in the relativistic quantum field theory a mysterous *deus ex machina* event.

As claimed by the author "the most of the paper rely upon concepts and methods taken from elementary quantum mechanics" and I share his opinion. Reading the technical part of the paper reminds how useful can be analysis of the angular momentum, the Clebsch-Gordan coefficients and the parity operation - everything seems to be clear. The situation changes when one comes to the discussion presented in the Section 4. The author links his nonrelativistic considerations to relativistic physics and here I face the impression that this is unnecessary and done, in a sense, by force. For me the concept of changing the particles is purely nonrelativistic and we should restrict ourselves to this statement - relating it to commuting field operators localized in space-like separated regions, even if close to intuition, does not give any profit to the pedagogically oriented paper.

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