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MOTION OF AN ELECTRON IN THE FIELD OF A BINOMIAL POTENTIAL OF A PROTON

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Abstract

We propose a binomial form of the interaction of an electron and a proton and study the classical solution of the Kepler problem and the scattering of electrons by protons. The derived formulas allow one to calculate the deflection angles and the trajectories of motion of electrons with energies from several eV to hundreds of MeV with impact parameters up to 10^{-13} cm.

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1 Introduction

The relation between the classical orbits of motion of an electron in the hydrogen atom and the experimental data was the central problem of the old quantum theory [1] and is an object of the intent attention for the almost century in the course of the development of quantum theory. The modern interest in this theme can be explained particularly by the constant desire to attain a more complete quantum-classical comprehension of the structure of the microworld and by the attempts to develop new and more precise semiclassical approximations [2]. It is worth noting that a great attention was concentrated on the aspects of the classical pattern of the motion of an electron in the field of the Coulomb potential after the experimental observation of wave packets [3]. These experiments showed unambiguously that the explanation of the motion of an electron should be seek for with regard for Kepler's classical orbits [4].

In the present work, we continue the search for the more acceptable classical explanation of the phenomena of atomic physics on the basis of a binomial interaction potential between an electron and a proton [5, 6, 7]. By the example of a hydrogen atom, the authors of the last cited works bring readers to the thought that the problems of quantum mechanics are, beginning from its origination, a result of our incomplete representation about the character of forces acting between an electron and a proton.

Therefore, we will demonstrate below how the electron-proton scattering occurs in the case of a binomial potential.

2 Substantiation of the binomial form of the electron-proton interaction

We introduce the binomial potential for the interaction of an electron with a proton in view of the indisputable fact that an atomic electron is constantly located at some distance from the nucleus. Not concerning the physical essence of this phenomenon, we present the electron-proton interaction potential in a hydrogen atom as the following function:

$$V = -\frac{e^2}{r} + \frac{\Gamma}{r^x} \, . \tag{1}$$

Here, the first term on the right-hand side is the Coulomb interaction, e is the electron charge, and the second term is a hypothetical interaction which counteracts the Coulomb attraction.

Generally, expression (1) was considered earlier in solid-state theory to describe the interaction of atoms in molecules and crystals [8]. For the atomic electrons, it has not received a recognition for some reason. Here, we will show that formula (1) is also suitable for modeling the atomic structure.

In expressions of the form (1), the constant Γ and the exponent x are unknown, as a rule. Their numerical values are found by solving the system of two algebraic equations corresponding to a certain state of the system under study [8]. For a hydrogen atom, we consider its ground state as such a state for the sake of definiteness. In this case, we have all the necessary experimental data and may construct the following system of two algebraic equations [8]:

$$-\frac{e^2}{r_0}+\frac{\Gamma}{r_0^x}=E_0\ ,\ \frac{e^2}{r_0^2}-\frac{x\ \Gamma}{r_0^{x+1}}=0\ . \tag{2}$$

Here, $_0$ is the energy of the ground state of a hydrogen atom, r_0 is its equilibrium radius and e is the electron charge. The second equation in (2) represents the sum of the forces acting on the electron in the ground state. From system (2), we get

$$\frac{1}{x}-1=\frac{E_0\ r_0}{e^2}\ . \tag{3}$$

Substituting the numerical values of E_0, r_0, and e^2, we find $x=2$ and, in view of the second relation in (2),

$$\Gamma=\frac{e^2\ r_0}{2}=6.10276\times10^{-28}\ CGSE\ units. \tag{4}$$

Let's pay attention to some features of these values. First, we emphasize the smallness of the constant Γ as compared to $e^2(23.06112\times 10^{-20})$. This causes the insignificant contribution of the positive additive of expression (1) to the total interaction energy already at distances within the limits of 2 equilibrium radii. Secondly, the exponent equals exactly 2. This property will allow us to carry out the integration of the relevant equations with this potential without

any difficulties in analytic form. Thirdly, by its numerical value, the constant

$$\Gamma = \frac{\hbar^2}{2m} , \tag{5}$$

where $\hbar$ is the Planck's constant, and m is the electron mass.

It is necessary also to note that expression (1) coincides with the potential derived by Weisskopf [9]. But Weisskopf arrived at this formula basing on the quantum-mechanical premises of the Schrödinger theory and taking the incompressibility of atoms into account. Such an approach assumes the existence of the repulsion forces between an electron and a proton. Hence, two essentially different approaches (classical [5, 6] and quantum-mechanical ones) have resulted in the same function of the interaction of an electron with a proton. This coincidence can be considered as a manifestation of the unity of classical and quantum-mechanical theories.

We note that V. Weisskopf used the binomial potential to find the clear estimates for some basic results of quantum theory. In this case, V. Weisskopf did not assume that the binomial potential can be interpreted as a real object for the electron-proton interaction. We also mention the other attempts [10] to change the Coulomb potential in order to simplify the comprehension of intraatomic phenomena, but all they consider this problem from the viewpoint of the quasiclassical approximation. We will try to show that the solution of this problem involving the binomial potential is possible also in the purely classical version. Moreover, the binomial potential allows one to make calculations of multielectron systems [6] and to explain many other atomic phenomena.

In Fig.1, we present the interaction energy of an electron with a proton versus the distance between them. It is obvious that, at distances of about 2 equilibrium radii ($\sim 1 \times 10^{-8} cm$), the dominant component in the total interaction energy is the Coulomb component. But starting approximately from a half of the equilibrium radius ($\sim 0.25 \times 10^{-8} cm$), the positive additive in formula (1) becomes dominant at small distances and hence represents the short-range force. Apparently, only in such a way, it is possible to explain why no deviations from the Coulomb law were observed at the macrodistances.

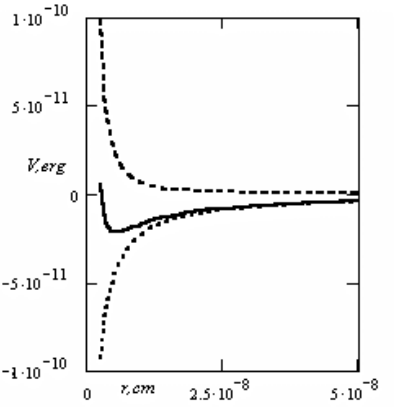

Figure 1: Electron-proton interaction energy versus the distance between particles for the binomial potential (1) ——, for the Coulomb potential - - - , and for their sum – – –.

3 Kepler problem with binomial potential

Consider the problem of the motion of an electron with mass m in the central field of a proton with regard for the above-introduced potential. The energy of the electron E can be presented as

$$E = \frac{m\dot{r}^2}{2} + \frac{M^2}{2mr^2} - \frac{e^2}{r} + \frac{\Gamma}{r^2}\,, \tag{6}$$

where r is the distance between an electron and a proton; M is the angular momentum; and $\dot{r}$ is the derivative with respect to time.

The solution of Eq. (6) [5, 6] yields the following expression defining the motion trajectory of an electron:

$$r = \frac{P}{1 - \varepsilon \sin(k\varphi)}\,, \tag{7}$$

where

$$\varepsilon = \sqrt{1 + \frac{2E\,(2m\Gamma + M^2)}{me^4}}\,,\ P = \frac{2m\Gamma + M^2}{me^2}\,,\ k = \frac{\sqrt{2m\Gamma + M^2}}{M}\,. \tag{8}$$

In the general case, the required function (7) represents two types of motion, finite and infinite ones, depending on the energy sign in Eq. (6). The first is the motion of a bound electron in a hydrogen atom, and the second is related to the scattering of an electron by a proton.

3.1 Atom of hydrogen

First, we consider a finite motion. The solution of (7) differs from the well-known solution of the Kepler problem with the Coulomb potential only by the presence of the coefficient k of the argument φ. In the general case, the presence of this coefficient leads to the appearance of a motion of the perihelion, namely its circular motion. It follows from expression (8) that if the positive term in (1) is absent, then $k = 1$ and the trajectory is a conic section with the focus at the coordinate origin. The presence of the mentioned term makes the coefficient k not equal to 1. This means that the motion trajectory will be unclosed in the general case, and only its separate values will satisfy the conditions of closedness. In other words, in the Coulomb potential field, an electron possesing any energy can move only along a closed orbit. But, in the field with potential (1), its motion along a closed orbit can be realized only at strictly definite energies.

Fig. 2 shows the motion trajectories of an electron with the coefficient k equal to 1, 2, and 3.

3.2 Scattering of electrons by protons

If the total energy (6) is positive, then the motion trajectory of an electron in the central field of the proton is an unclosed curve, whose beginning and end are at infinity. Since we consider the infinite motion in this case, it is convenient to introduce the so-called impact parameter ρ instead of a constant momentum M. Therefore, we can write

$$M = \sqrt{2mE\rho^2} \, . \tag{9}$$

To derive the motion trajectory of an electron scattered by a proton, it is necessary to substitute (9) in relations (8) and pass to the Cartesian coordinates

$$x = r \, \cos\varphi \, , \qquad y = r \, \sin\varphi \, , \tag{10}$$

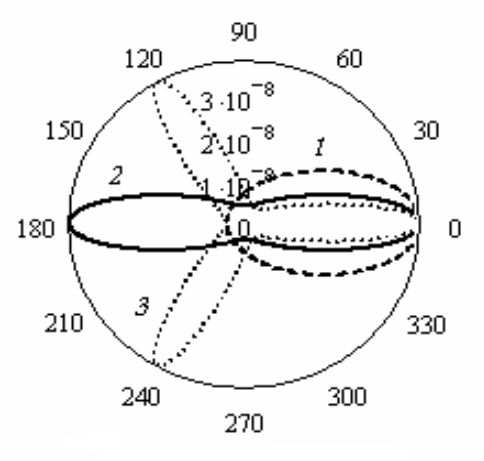

Figure 2: Trajectory of the movement of electron in the field of a binomial potential of a proton.

where values of r should be determined by formula (7).

In Fig. 3, we present the motion trajectories of scattered electrons with energies 400, 188, and 40 MeV calculated by (10). It is seen that if the motion occurs in the purely Coulomb field, the scattering of electrons is not realized.

The deflection angle upon the passage of a particle near the scattering center is presented in the form

$$\chi = \pi - 2\varphi_0 \,, \tag{11}$$

where φ_0 is defined by the integral

$$\varphi_0 = \int\limits_{r_{\min}}^{\infty} Mr^{-2} \left(2m[E - U(r)] - \frac{M^2}{r^2} \right)^{-1/2} dr \tag{12}$$

taken between the positions of the particle which are nearest to the center and infinitely remote [11]. The lower limit of the integration $r_{\min}$ is equal to the smallest distance of the scattered particle from the center [11] and is the root of the radicand in (12).

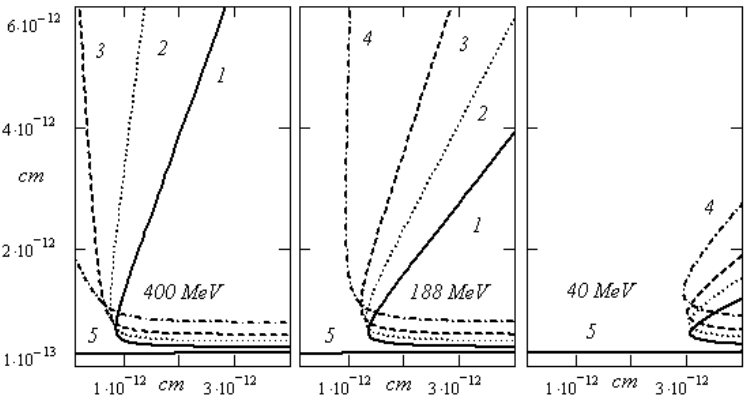

Figure 3: Trajectories of movement of electrons with energies of 400, 188, and 40 MeV near a proton calculated by formula (10) as functions of the impact parameter ρ: 1 - 4×10^{-13} , 2 - 5×10^{-13} , 3 - 6×10^{-13} , 4 - 8×10^{-13} cm. 5 - The trajectory of an electron in the Coulomb field.

With regard for (9), formula (12) reads

$$\varphi_0 = \int\limits_{r_{\min}}^{\infty} r^{-2} \left(\frac{1}{\rho^2} \left[1 - \frac{U(r)}{E} \right] - \frac{1}{r^2} \right)^{-1/2} dr \, . \qquad (13)$$

If we substitute potential (1) in expression (13) and integrate, then we get

$$\varphi_{0b} = \sqrt{\frac{E\rho^2}{E\rho^2 + \Gamma}} \arccos \left[\left(1 + \frac{4E(E\rho^2 + \Gamma)}{e^4} \right)^{-1/2} \right] \, . \qquad (14)$$

Thus, we have obtained the analytical formula allowing the determination of the angle χ as a function of the electron energy E and the impact parameter ρ under the assumption that the particles interact through the binomial potential (1). If we put $\Gamma = 0$ in formula (14), that corresponds to the interaction of an electron with a proton by

the Coulomb law [11], we get

$$\varphi_{0k} = \arccos\left[\left(1 + \frac{4E^2\rho^2)}{e^4}\right)^{-1/2}\right] . \tag{15}$$

Hence, the difference in the formulas describing the electron-proton scattering derived with the help of the Coulomb potential and the binomial one is mainly defined by the factor

$$k = \sqrt{\frac{E\rho^2 + \Gamma}{E\rho^2}} . \tag{16}$$

In the general case, relations (14) and (15) together with (11) give the possibility to calculate the deflection angle χ of a separately scattered electron as a function of its kinetic energy and impact parameter. In the first case, the electron and the proton interact by the binomial law (1). In the second case, they interact by the Coulomb law.

In Fig. 4, we give the results of calculations by formulas (14) and (15).

It is obvious from Fig. 4 - 1 that, in the case of small energies of an electron, its scattering by the Coulomb and binomial potentials occurs identically. The dependences of the deflection angles on the impact parameter remain very close in both cases. The significant difference of the deflection angles χ is observed only at energies of above hundreds of eV. We note that the scattering is completely absent in the relativistic region of energies (Fig. 4 - 2).

But, upon the experimental investigation of the process of scattering, the so-called effective differential scattering cross-section $d\sigma$ is usually used. With regard for the one-to-one relation (14) between the deflection angle χ and the impact parameter ρ, the effective differential scattering cross-section referred to the solid-angle element $d\theta$ can be written as [11]

$$d\sigma = \frac{\rho(\chi)}{\sin(\chi)}\frac{d\rho}{d\chi}d\theta . \tag{17}$$

Then, with regard for (11) $k\varphi_0 = (\pi - \chi)/2$, we get

$$\rho^2 = \frac{e^4}{4E^2}\cot^2\left(\frac{\chi}{2}\right) - \frac{\Gamma}{E} . \tag{18}$$

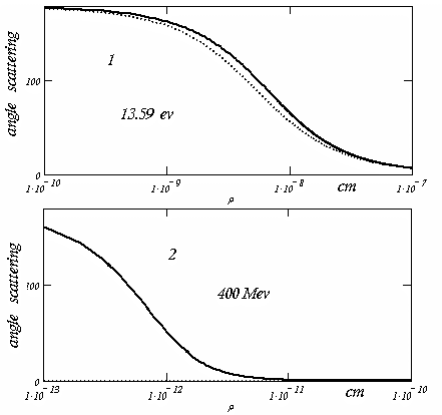

Figure 4: Deflection angle χ versus the impact parameter ρ under the interaction through the binomial potential —— or the Coulomb potential - - - - with energies of electrons of 400 MeV (2) and 13.59 eV (1).

Differentiating relation (18) with respect to χ and substituting the result in formula (17) for the effective scattering cross-section, we get

$$d\sigma = \frac{e^4}{16E^2} \frac{1}{\sin^4\left(\frac{\chi}{2}\right)} d\theta \; . \tag{19}$$

By its view, formula (19) repeats exactly the Rutherford one [11]. But there is the fundamental difference between them which is revealed in the definition of the angle χ with regard for coefficient (16):

$$\chi = \pi - 2k\varphi_0 \; , \tag{20}$$

Generally saying, it is the unexpected result, because it is commonly accepted now [12] that the Rutherford formula is not valid in the relativistic region. However, we have shown above that the model with the binomial potential does give the same formula for the effective scattering cross-section after the execution of all mathematical operations made by Rutherford to derive his formula. We

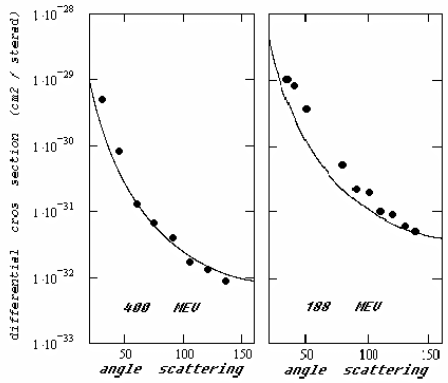

Figure 5: Comparison of the effective scattering cross-sections of electrons with energies of 400 and 188 MeV calculated by formula (19) (the continuous curve) with the experimental data ($\cdots$) taken from [12].

can only conjecture why the author of work [12] did not directly verify the Rutherford formula at first by using his experimental results and made verification only after the refinement of it by relativistic corrections and a form-factor.

In Fig. 5, we compare the results of calculations of the effective scattering cross-sections for electron energies of 400 and 188 eV by formula (19) with the experimental data taken from work [12].

4 Conclusions

The purpose of our study was to demonstrate the potentialities of the binomial potential in the classical problem of the motion of an electron in the field of a proton. The above-presented consideration allows us to draw the following conclusions.

1. With potential (1), we have received a practical possibility to solve the classical problem of the motion of an electron in the central field of a proton, to explain the conditions of the appearance of stationary orbits.

2. With the binomial potential, we have also got the new possibility to describe the process of scattering of electrons by protons. At small energies, the calculations of the scattering angles with the Coulomb and binomial potentials give the same result. This is seen in Fig. 4 - 1. Formulas (14) and (15) give the very close scattering angles only at energies of electrons equal to several tenths of eV. With increase in the energy of an electron, the scattering angles derived with the help of formulas (14) and (15) differ more and more from each other. Fig. 4 - 2 testifies to that the theory with the Coulomb potential has failed already for a $400 - MeV$ electron. We observe no scattering and no deflection of a particle from the initial direction. But the theory with the binomial potential describes well the impact parameters and the scattering angles of electrons possessing relativistic energies.

3. The comparison of the theoretical and experimental data (Fig. 5) yields that the introduction of the binomial potential gives the possibility to calculate the effective scattering cross-section of electrons in a wide energy interval, as well as to construct the motion trajectories of an electron near the proton at distances up to $10^{-13}cm$ (Fig. 3). These results are essentially new, because the available theories based on other potentials assert that one cannot say about the trajectory of a relativistic electron at such small distances. Thus, the binomial potential allows one to calculate the scattering angles and the motion trajectory of an electron both in the nonrelativistic and relativistic regions.

4. At this point, we will explain why the Rutherford formula based on the binomial potential leads to proper results in the relativistic region despite the common opinion.

First of all, we note that the positive additional term in (1) should not be interpreted as a centrifugal one. If we will adhere to the position that some centrifugal forces or the forces related to "some quantum-mechanical effect" keep the electron from "the fall" on the nucleus, we will meet difficulties by trying to construct a physical model. But all things are simple with the binomial potential (a least hypothetically) [9]: a scattered electron interacts with two fields counteracting each other.

As shown in Fig. 1, the Coulomb component in the total potential energy dominates up to distances $\sim 0.3 \times 10^{-8}cm$. At smaller

distances, our hypothetical additional term begins to prevail. Just the limit of the action of the Coulomb component defines the non-relativistic energy, with which the electron can approach the proton at this distance. Electrons with greater (relativistic) energies can be at distances from the proton less than $10^{-9}cm$, where they interact already by the law defined by the positive additional term in (1) (at such distances, it dominates). Let us imagine that the positive additional term is absent. Then the electron will move to the proton along the straight line, which is seen in Fig. 3 and Fig. 4 - 2. Thus, the relativistic corrections to the Rutherford formula [12] were generated by the search for the explanation to the fact that an electron with nonzero kinetic energy in the Coulomb field of a proton remains at some distance from it or is scattered.

5. The following question should be mentioned: How does the binomial potential agree with the positions of quantum electrodynamics as for the motion of an electron in a central field [13, 14]? We note that the binomial potential does not contradict the conclusions of quantum electrodynamics, by possessing the same properties at infinity as the Coulomb potential. Moreover, since the binomial potential has the additional zero point (at the intersection of the potential curve and the abscissa axis), the solutions of the Dirac radial equations with the binomial potential will possess a number of new interesting properties. Generally saying, we believe that the binomial potential can be justified with the help of the Dirac theory. But this question is outside of the scope of this work.

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Comment

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Comment on MOTION OF AN ELECTRON IN THE FIELD OF A BINOMIAL POTENTIAL OF A PROTON

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It has been shown that the binomial potential

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^2}, \qquad \text{with} \qquad \Gamma = \frac{\hbar^2}{2m_e}$$

for proton-electron interaction cannot be accepted from physical reasons and that it leads to predictions that are in definite contradiction with experiment of atomic physics.

1 Preliminary remarks

V.K. Gudym and E.V. Andreeva, in their paper *Motion of an electron in the field of a binomial potential of a proton*, assumed a binomial potential of the form

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^\gamma}, \qquad \gamma = 2, \quad \Gamma = \frac{\hbar^2}{2m}$$

to describe electron–proton interaction. They discussed in the framework of Newtonian mechanics its effect on shapes of electron's bounded

classical orbits as well as on scattering of high energy electrons (with energies $40-400MeV$). The particular values of γ and Γ were specified by the requirement that the potential should have its minimum numerically equal to energy of the ground state of Bohr's atom $E_B \approx -13.6eV$ and be located at the Bohr's radius $r_B \approx 5.3 \times 10^{-9}cm$, where, as usual,

$$E_B = -\frac{1}{2}\alpha^2 m_e c^2, \qquad r_B = \alpha^{-1}\frac{\hbar}{m_e c}, \qquad \alpha = \frac{e^2}{\hbar c}.$$

Their requirement had originated, as they wrote, from *'the indisputable fact that an atomic electron is constantly located at some distance from the nucleus'*. Unfortunately, the Authors did not explain in more detail the mysterious statement. Therefore, to clarify that point, note that the shape of the binomial potential's well suggest one should expect the electron's ground state wave function to be spherically symmetric with respect to the centre and concentrated mainly in the vicinity of a spherical shell of radius r_B. Recall that electron's kinetic energy in the ground state must be bounded from zero, thus quantum-mechanically the ground state electron is not at rest at all, in contrast to what the Authors expected. We shall came to the issue in the next section in which we solve Schrödinger equation with the binomial potential.

In their paper the Authors focused mainly on examining the classical orbits of electrons in the binomial potential in the framework of Newtonian mechanics. Calculation of classical orbits of trial bodies in such potential (with arbitrary constants in place of Γ and e) is a very old (celestial mechanics) and now standard graduate textbook problem, thus the Author's calculations introduce nothing new, the more that, as it will be shown, the particular binomial potential would lead to effects that would be not observable in atomic physics. Moreover, the Author's using of Newtonian mechanics for scattering of ultra-relativistic electrons with energy $40-400MeV$, thus $80-800$ times greater than the electron's rest mass, cannot be accepted. For such electrons the relativistic dynamics must be necessarily used.

Another weak point is that the Authors have not explained the nature of the Γ term. With no such information given, one can only surmise from the fact of using atomic quantities in specifying the term, that it should have electromagnetic origin, yet it would be an

exaggeration to expect the term could be caused by gravitational or even short range forces such as weak interactions, in the latter case the potential would have to be damped exponentially, however. In what follows, we shall therefore be assuming that the Γ term has electromagnetic origin.

The next deficiency of the paper is a lack of physical justification for such a fundamental modification of the proton-electron interaction. Various kinds of corrections to the Coulomb field are interpreted as effective potentials arising in the presence of other charges in atomic interiors or in ambient space of, say, crystals. Effective potentials of this kind have no fundamental nature and vanish for isolated proton-electron pairs. The proposed Γ term is spherically symmetric and has a dipole-like falloff. Such term, however, cannot originate from proton's internal structure because dipolar potential is angle dependent – in spherical coordinates it is proportional to $r^{-2}\cos\theta$ – and due to a complicated internal proton's dynamics its multipolar structure is rather a zero mean value fluctuation field. Higher multipolar momenta falloff quicker so they cannot produce the Γ term, as well. It cannot be excluded a possibility that a Γ-like term, possibly more complicated than merely r^{-2}, still could be considered as an attempt to a naive and very approximate description of an effective correction to the Coulomb field at larger radii due to, say, internal electromagnetic structure of the proton. However, such a correction should have only a very tiny effect on energy structure of the hydrogen, and could be derived only in the framework of quantum field theory.

The other weak point is the Authors have not specified the region of validity of the Γ term (e.g. by giving a cutoff radius). Their calculations suggest they assumed validity of the binomial potential globally, moreover, they claimed the term was suitable also for modelling internal atomic structure. However, the term cannot be correct in whole space. Note that the charge density $-(4\pi)^{-1}\vec{\nabla}^2 U$ corresponding to the Γ term with $U = -(\Gamma/e)r^{-2}$ does not make sense as a global charge distribution. Electric charge outside a sphere of radius a equals $\Gamma/(ae)$, while total charge enclosed by the sphere is the opposite. Both charges sum up to zero, and in the limit $a \to 0$ are infinite (so in the centre there is concentrated infinite negative charge). Another way of seeing this (this shows also distributional

character of the charge density corresponding to the Γ term) can be inferred by inspection of a regularized profile $-(\Gamma/e)\left(r^2+\varepsilon^2\right)^{-1}$ of the Γ term, which leads to a smooth charge density

$$\frac{\Gamma\,\left(r^2-3\,\varepsilon^2\right)}{2\,e\,\pi\,\left(r^2+\varepsilon^2\right)^3}$$

negative for $0<r<\sqrt{3}\varepsilon$ and positive elsewhere. Contribution to total charge from each region is $\mp 3\sqrt{3}\Gamma\,(8e\varepsilon)^{-1}$, respectively, and infinite in the limit $\varepsilon\to 0$. This shows the Γ term should be somehow modified for small radii, note however, that outside a sphere of radius a there would still reside er_B/a of positive charge (where it would come from then?).

A serious omission is that the Authors have not assessed the effect of the Γ term on energy of photons emitted by hydrogen, which is the question that naturally arises and that allows one to decide whether or not such correction to the Coulomb field is physically viable in atomic interiors. By construction, the Coulomb and Γ terms of the binomial potential are comparable in the vicinity of Bohr's radius. One should therefore expect significant reorganization of energy structure of hydrogen as wave functions must penetrate also regions where the Γ term is dominant and strongly repulsive. That such assessment requires quantum-mechanical arguments is best seen from the following observation: in classical picture the lowest energy electron in the field of the binomial potential, which has the global minimum, is at rest at Bohr's radius and has energy $-13.6\,eV$, thus one would expect $13.6\,eV$ for ionization energy, the same as for ordinary hydrogen, and this is what the Authors expected. Quantum mechanically, however, it suffices only $5.13\,eV$ for ionization! Accidentally, the semiclassical Bohr's model with pure Coulomb field yielded the same energy spectrum as the one predicted quantum mechanically! For the binomial potential such correspondence of the two pictures is no longer valid. As an aside, we remark that the Bohr's quantization condition $m_e v_r r=n\hbar$ for circular orbits in the binomial potential gives

$$E_n=\frac{E_B}{n^2+\frac{2m\Gamma}{\hbar^2}}$$

for energy levels, and $n=0$ is also allowable if $\Gamma\neq 0$. This simplified

calculation shows that if Γ was much less than $\hbar^2/(2m_e)$ one would expect energy levels to be only slightly different from those of ordinary hydrogen for $n \neq 0$, while one should expect large departures from the energy spectrum for $\Gamma = \hbar^2/(2m_e)$. As we shall see in the next section the correct non-relativistic quantum mechanical calculation yields much more complicated energy spectrum than the one above, and also that the modification of proton-electron interaction by the Γ term, the Authors have proposed, leads to predictions that are in severe contradiction with experiment.

2 Eigenvalue problem for the binomial potential

In what follows we shall find bound states of electron in the binomial potential and the corresponding energy spectrum. We shall refer to the hydrogen atom with the modified Coulomb potential as 'Γ-hydrogen'.

The stationary Schrödinger equation $\hat{H}\Psi = E\Psi$ with the binomial potential reduces to

$$\left(-\partial_x^2 - \frac{2}{x}\partial_x + \frac{1}{x^2}\hat{L}^2 - \frac{2}{x} + \frac{1}{x^2} + \epsilon\right)\Psi = 0,$$

with

$$\hat{L}^2 = -\left(\partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right),$$

where we have defined dimensionless radius x and dimensionless energy ϵ

$$r = r_B \cdot x, \qquad E = E_B \cdot \epsilon.$$

As the differential operators $\hat{H}$, $\hat{L}^2$ and $\hat{L}_z = -i\partial_\phi$ commute with each other, the general solution reads

$$\Psi = R(x)Y_{lm}(\theta,\phi), \qquad l = 0, 1, 2, \ldots, \qquad m = 0, \pm 1, \cdots \pm l,$$

where $Y_{lm}(\theta,\phi)$ are spherical harmonics and the radial function $R(x)$ is a corresponding solution of equation

$$R_{,xx} + \frac{2}{x}R_{,x} - \frac{1 + l(l+1)}{x^2}R + \frac{2}{x}R - \epsilon R = 0.$$

For bound states we assume $\epsilon > 0$ (then $E < 0$). By setting $\beta_l = \frac{1}{2}\left(\sqrt{5+4l(l+1)}-1\right)$ and $R(x) = x^{\beta_l}e^{-x\sqrt{\epsilon}}Q(2x\sqrt{\epsilon})$ we obtain the Laguerre equation

$$\xi\ddot{Q}(\xi) + (1+a-\xi)\,\dot{Q}(\xi) + n_r Q(\xi) = 0,$$
$$x = \frac{\xi}{2\sqrt{\epsilon}}, \quad a = 1+2\beta, \quad n_r = \frac{1}{\sqrt{\epsilon}} - (1+\beta),$$

of which solutions read $Q(\xi) = L_{n_r}^{1+2\beta_l}(\xi)$ (L's are Laguerre's functions) and are normalizable only if $n_r = 0, 1, 2, \ldots$. Consequently, we obtain a discrete spectrum of energy levels

$$\epsilon_{l,n_r} = \frac{1}{(1+\beta_l+n_r)^2}, \qquad \beta_l = l + \frac{2}{1+2\,l+\sqrt{5+4\,l\,(l+1)}},$$
$$n_r = 0, 1, 2, \ldots$$

together with the corresponding eigenfunctions

$$\Psi_{(m,l,n_r)}(x,\theta,\phi) =$$
$$= C_{(m,l,n_r)}x^{\beta_l}e^{-x\sqrt{\epsilon_{l,n_r}}}L_{n_r}^{1+2\beta_l}(2x\sqrt{\epsilon_{l,n_r}})Y_{lm}(\theta,\phi)e^{im\phi}$$

(C's are normalization constants).

2.1 Implications for the binomial potential

As follows from the energy spectrum found in the previous section, the presence of the Γ term removes the characteristic for the pure Coulomb field accidental degeneracy of energy levels of Bohr's atom and, due to the huge value of $\Gamma = \hbar^2/(2m_e)$, drastically affects energy structure, such that this change could be easily detected experimentally. In particular, for the ground state, which is spherically symmetric

$$\Psi = C\left(xe^{-x}\right)^{1/\chi},$$
$$\epsilon = \frac{1}{\chi^2}, \qquad C = \frac{1}{4\sqrt{\pi\,\Gamma(2\chi)}}\left(\frac{2}{\chi}\right)^{\chi+1}, \quad \chi = \frac{1+\sqrt{5}}{2}$$

(χ is the famous 'golden ratio'). Consequently, the ionization energy of the Γ-hydrogen is only $5.13\,eV$ while ionization energy of Bohr's

atom is $13.6\,eV$. The ground state of the electron in the Γ-hydrogen, in contrast to what the Authors assumed while determining the binomial potential, is different from the first energy level of the Bohr's model. The reason is that the wave function of the electron in the ground state in the particular binomial potential, penetrates the region of small radii where the repulsive Γ term dominates the attractive Coulomb interaction. Note, that the corresponding probability density has its maximum located on the sphere of one Bohr's radius ($x = 1$), (this in a sense agrees with the Author's intention of having the electron *'constantly located'* at the Bohr's radius, however the electron is not at rest at all as the expectation value of kinetic energy in the ground state is nonzero) and that the probability density is zero in the centre reflecting the dominant repulsive effect of the Γ term (to compare with, the ground state wave function of the ordinary hydrogen is $e^{-x}/\sqrt{\pi}$ and has its maximum located in the centre).

As concerns the removed degeneracy of energy levels, it predicts, in particular, the transition $2S$ ($n_r = 1, l = 0$) $\rightarrow$ $2P$ ($n_r = 0, l = 1$) with energy

$$\frac{2\hbar^2}{m_e e^4}\,(E_{2S} - E_{2P}) = \frac{4}{\left(1+\sqrt{13}\right)^2} - \frac{4}{\left(3+\sqrt{5}\right)^2} \approx 0.0427$$

(absent for ordinary hydrogen). The eigenfunctions $2S$ and $2P$ overlap in the limit $\Gamma \rightarrow 0$ with eigenfunctions of ordinary hydrogen (the subtle structure of $2S$ and $2P$, that is, $2^2S_{1/2}$, $2^2P_{1/2}$ and $2^2P_{3/2}$, is not seen in non-relativistic description, of course). Photon transitions between $2S$ and $2P$ with such large energy are not observed.

Both the above examples emphatically prove that the value of Γ the Authors assumed in the binomial potential is far too large, and the effect of the Γ term could be easily detected if it really existed.

3 Summary

It has been shown in the previous sections that the binomial potential

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^2}, \qquad \text{with} \qquad \Gamma = \frac{\hbar^2}{2m_e}$$

for proton-electron interaction cannot be accepted from mathematical reasons and that it is definitely excluded by experiment.