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DIRAC FORMULATION OF FREE OPEN STRING

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Abstract

Dirac formulation of open relativistic strings as systems with constraints is made explicitly. Classical theory is given in the standard light-cone and covariant center-of-mass gauges.

It is mentioned that the well-known result $D = 26$ is affected by using the standard quantization of the mutually independent nonphysical boson creation and annihilation operators. It is shown that in the Dirac formulation these operators are not independent in both the gauges.

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We give some new conditions on these operators and show that the theory is consistent with Poincaré algebra in any dimension D.

1 Hamilton description of classical open string

We will study the Nambu–Goto [1] free open string in dimension D. We assume the sign convention $g_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, where $\mu, \nu = 0, 1, \ldots, D-1$. The string is described by the functions $X^\mu(\tau, \sigma)$, where $\tau \in \mathbb{R}$ and $\sigma \in \langle 0, \pi \rangle$. The classical string is described by Lagrangian

$$L(X) = -\omega \int_0^\pi \mathcal{L}\, \mathrm{d}\sigma,$$

where $\omega > 0$ is a constant and the Lagrangian density $\mathcal{L}(\tau, \sigma)$ is

$$\mathcal{L} = \sqrt{\left(\dot{X}X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}\,.$$

A dot means partial derivation with respect to τ, a dash with respect to σ, and

$$XY = g_{\mu\nu} X^\mu Y^\nu = X_\mu Y^\nu\,.$$

The boundary conditions are $X'_\mu(\tau, 0) = X'_\mu(\tau, \pi) = 0$.

In the Hamiltonian formulation we define momenta

$$P_\mu(\tau, \sigma) = \frac{\delta L}{\delta \dot{X}^\mu(\sigma)} = \omega \frac{\dot{X}_\mu(X'X') - X'_\mu(\dot{X}X')}{\sqrt{\left(\dot{X}X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}}\,. \tag{1}$$

From (1) we obtain the relations

$$\Phi_1 = \tfrac{1}{2}\left(P^2 + \omega^2 \left(X'\right)^2\right) = 0\,, \qquad \Phi_2 = PX' = 0 \tag{2}$$

called constraints. For the Poisson brackets of two functionals $F(X, P)$ and $G(X, P)$ we have

$$\{F, G\} = \int_0^\pi \left(\frac{\delta F}{\delta X^\mu(\sigma)} \frac{\delta G}{\delta P_\mu(\sigma)} - \frac{\delta F}{\delta P_\mu(\sigma)} \frac{\delta G}{\delta X^\mu(\sigma)} \right) \mathrm{d}\sigma\,. \tag{3}$$

In particular, the relation $\left\{X^\mu(\sigma), P^\nu(\sigma')\right\} = g^{\mu\nu}\delta(\sigma - \sigma')$ is valid.

The Hamiltonian of the system with constraints (2) is

$$H = \int_0^\pi \mathcal{H}\, \mathrm{d}\sigma,$$

where the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= P_\mu \dot{X}^\mu - \mathcal{L} + \lambda_1(\sigma)\Phi_1 + \lambda_2(\sigma)\Phi_2 = \lambda_1(\sigma)\Phi_1 + \lambda_2(\sigma)\Phi_2 = \\ &= \tfrac{1}{2}\,\lambda_1(\sigma)\Big(P^2 + \omega^2\big(X'\big)^2\Big) + \lambda_2(\sigma) P X' \,.\end{aligned}$$

The equations of motion of the string are

$$\begin{aligned}\dot{X}^\mu &= \big\{X^\mu, H\big\} = \lambda_1 P^\mu + \lambda_2 (X^\mu)' \,, \\ \dot{P}_\mu &= \big\{P_\mu, H\big\} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(\lambda_1 \omega^2 X'_\mu + \lambda_2 P_\mu\Big) \,, \\ \Phi_1 &= \Phi_2 = 0\,.\end{aligned} \tag{4}$$

Starting from the boundary conditions $X'_\mu(\tau,0) = X'_\mu(\tau,\pi)$, it is possible to extend the functions $X'_\mu(\tau,\sigma)$ continuously in variables σ into odd periodic functions with the period 2π. Therefore, the functions $X^\mu(\tau,\sigma)$ and $P_\mu(\tau,\sigma)$ are even with the period 2π. So we have

$$\begin{aligned}X^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,X^\mu_0(\tau) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} X^\mu_n(\tau)\cos n\sigma \,, \\ P^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,P^\mu_0(\tau) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} P^\mu_n(\tau)\cos n\sigma \,, \\ (X^\mu)'(\tau,\sigma) &= -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} n X^\mu_n(\tau)\sin n\sigma \,,\end{aligned}$$

where

$$\begin{aligned}X^\mu_n(\tau) &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} X^\mu(\tau,\sigma)\cos n\sigma\,\mathrm{d}\sigma \,, \\ X^\mu_0(\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} X^\mu(\tau,\sigma)\,\mathrm{d}\sigma \,, \\ P^\mu_n(\tau) &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} P^\mu(\tau,\sigma)\cos n\sigma\,\mathrm{d}\sigma \,, \\ P^\mu_0(\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} P^\mu(\tau,\sigma)\,\mathrm{d}\sigma \,.\end{aligned}$$

From (3) we obtain the Poisson brackets

$$\big\{X^\mu_m, X^\nu_n\big\} = \big\{P^\mu_m, P^\nu_n\big\} = 0\,, \qquad \big\{X^\mu_m, P^\nu_n\big\} = g^{\mu\nu}\delta_{m,n}\,.$$

By using the variables $X_n^\mu(\tau)$ and $P_n^\mu(\tau)$ it is possible to formulate the problem as the system with the Poisson brackets

$$\{f,g\} = g^{\mu\nu}\sum_{n=0}^{\infty}\left(\frac{\partial f}{\partial X_n^\mu}\frac{\partial g}{\partial P_n^\nu} - \frac{\partial f}{\partial P_n^\nu}\frac{\partial g}{\partial X_n^\mu}\right), \tag{5}$$

with the Hamiltonian

$$H = \lambda_{1,0}\Phi_{1,0} + \sum_{k\in\mathbb{N}}\left(\lambda_{1,k}\Phi_{1,k} + \lambda_{2,k}\Phi_{2,k}\right),$$

where $\lambda_{1,k}$ and $\lambda_{2,k}$ are constants and

$$\begin{aligned}
\Phi_{1,0} &= \tfrac{1}{2}\,g_{\mu\nu}\Big(P_0^\mu P_0^\nu + \sum_{n=1}^{\infty}\left(P_n^\mu P_n^\nu + \omega^2 n^2 X_n^\mu X_n^\nu\right)\Big),\\
\Phi_{1,k} &= \tfrac{1}{2}\,g_{\mu\nu}\Big[\sqrt{2}\,P_0^\mu P_k^\nu + \sum_{n=1}^{\infty}\left(P_{n+k}^\mu P_n^\nu + \omega^2 n(n+k) X_{n+k}^\mu X_n^\nu\right) + \\
&\quad + \tfrac{1}{2}\sum_{n=1}^{k-1}\left(P_{k-n}^\mu P_n^\nu - \omega^2 n(k-n) X_{k-n}^\mu X_n^\nu\right)\Big],\\
\Phi_{2,k} &= -\frac{g_{\mu\nu}}{2}\Big[\sqrt{2}\,kP_0^\nu X_k^\mu + \sum_{n=1}^{\infty}\Big((k+n)P_n^\nu X_{k+n}^\mu - nP_{k+n}^\nu X_n^\mu\Big) + \\
&\quad + \sum_{n=1}^{k-1} nP_{k-n}^\nu X_n^\mu\Big]
\end{aligned}$$

and with the constraints

$$\Phi_{1,0} = \Phi_{1,k} = \Phi_{2,k} = 0\,.$$

The equations of motion of the system are

$$\dot{X}_n^\mu = \left\{X_n^\mu, H\right\}, \qquad \dot{P}_n^\mu = \left\{P_n^\mu, H\right\}.$$

Since

$$\begin{aligned}
\left\{\Phi_{1,k},\Phi_{1,l}\right\} &= \tfrac{1}{2}\,\omega^2\big((k-l)\Phi_{2,k+l} + (k+l)\Phi_{2,k-l}\big)\,, \qquad k>l\,,\\
\left\{\Phi_{1,k},\Phi_{2,l}\right\} &= \tfrac{1}{2}\,\big((l-k)\Phi_{1,k+l} + (l+k)\Phi_{1,|k-l|}\big)\,,\\
\left\{\Phi_{2,k},\Phi_{2,l}\right\} &= \tfrac{1}{2}\,\big((l-k)\Phi_{2,k+l} + (l+k)\Phi_{2,k-l}\big)\,, \qquad k>l\,,
\end{aligned} \tag{6}$$

we have the system with the first class constraints.

As a next step it is useful to define for any $n \geq 1$ complex variables

$$a_{\pm n}^\mu = \tfrac{1}{\sqrt{2}}\left(P_n^\mu \pm \mathrm{i}\omega n X_n^\mu\right)$$

and the functions

$$L_0 = \Phi_{1,0}\,, \quad L_n = \Phi_{1,n} - \mathrm{i}\omega\Psi_{2,n}\,, \quad L_{-n} = \Phi_{1,n} + \mathrm{i}\omega\Psi_{2,n}\,.$$

It is easy to see that $\overline{a_n^\mu} = a_{-n}^\mu$ and $\overline{L_n} = L_{-n}$ hold.

In these variables the Poisson brackets (5) have the form ($\mathbb{Z}_0 = \pm1, \pm2, \ldots$)

$$\{f, g\} = g^{\mu\nu}\left(\frac{\partial f}{\partial X_0^\mu}\frac{\partial g}{\partial P_0^\nu} - \frac{\partial f}{\partial P_0^\mu}\frac{\partial g}{\partial X_0^\nu} + \mathrm{i}\omega\sum_{n\in\mathbb{Z}_0} n\,\frac{\partial f}{\partial a_n^\mu}\frac{\partial g}{\partial a_{-n}^\nu}\right) \quad (7)$$

and for the constraints we obtain

$$\begin{aligned} L_0 &= \tfrac{1}{2}(P_0)^2 + \sum_{n=1}^{\infty} a_n a_{-n} = 0\,, \\ L_k &= P_0 a_k + \sum_{n=1}^{\infty} a_{k+n} a_{-n} + \tfrac{1}{2}\sum_{n=1}^{k-1} a_{k-n} a_n = 0\,, \qquad k \geq 1\,, \\ L_{-k} &= P_0 a_{-k} + \sum_{n=1}^{\infty} a_{-k-n} a_n + \tfrac{1}{2}\sum_{n=1}^{k-1} a_{n-k} a_{-n} = 0\,, \qquad k \geq 1\,. \end{aligned} \quad (8)$$

Now we can write the Hamiltonian in the form

$$H = \sum_{n\in\mathbb{Z}} \kappa_n L_n\,, \quad (9)$$

where $\overline{\kappa}_n = \kappa_{-n}$ is valid for constants κ_n, and equations (6) give

$$\{L_k, L_n\} = \mathrm{i}\omega(n-k)L_{n+k}\,.$$

2 Gauge conditions, Dirac brackets and quantization

First we will briefly recall the main idea of the Dirac formulation of the description of the Hamiltonian systems with the first class constraints [2, 3].

The Hamiltonian of these systems (9) includes the auxiliary constants κ_n. The equations of motion depend on the choice of these constants but time evolution of such physical systems must be independent of their choice. Consequently, physically meaningful are only the functions whose time evolution is independent of the choice of the constants κ_n, i.e. the functions, whose Poisson brackets with constraints vanish on the set defined by $L_n = 0$.

To determine the constants κ_n, we choose the system of additional conditions $R_n = 0$, the so-called gauge conditions. We require the regularity of the matrix

$$\mathbf{C} = \begin{pmatrix} \{L, L\} & \{L, R\} \\ \{R, L\} & \{R, R\} \end{pmatrix}$$

on the set $\mathcal{M}$, where $L_n = R_n = 0$ is valid. The constants κ_n are obtained from the equations

$$\frac{\mathrm{d}R_n}{\mathrm{d}\tau} = \{R_n, H\} + \frac{\partial R_n}{\partial \tau} = 0\,, \tag{10}$$

which give the time conservation of the calibration conditions.

In the Dirac terminology the system of equations $L_n = R_n = 0$ forms the constraints of second class. For these systems the Dirac brackets defined by

$$\{f, g\}_{\mathrm{Dir}} = \{f, g\} + \Big(\{f, L,\},\, \{f, R\}\Big)\mathbf{C}^{-1} \begin{pmatrix} \{g, L\} \\ \{g, R\} \end{pmatrix} \tag{11}$$

play a very important role. The Dirac brackets have on the set $\mathcal{M}$ similar properties as the Poisson brackets, but in addition for any constraints L_n and gauge conditions R_n we have

$$\{f, L_n\}_{\mathrm{Dir}} = \{f, R_n\}_{\mathrm{Dir}} = 0\,.$$

In consequence, for calculation of the Dirac brackets, it makes no difference, contrary to the Poisson brackets, if we use the equations of constraints before or after calculation.

Quantization is understood as assignment to any function f of the operator $\mathbf{f}$, so that

$$\{f, g\}_{\mathrm{Dir}} = h \;\longrightarrow\; [\mathbf{f}, \mathbf{g}] = \mathrm{i}\mathbf{h}\,. \tag{12}$$

The constraint $L = 0$ and gauge condition $R = 0$ give us the conditions between the operators, or we can use these for definition of the physical states $|\,\psi\,\rangle_{\mathrm{phys}}$ by the equations

$$\mathbf{L}\,|\,\psi\,\rangle_{\mathrm{phys}} = \mathbf{R}\,|\,\psi\,\rangle_{\mathrm{phys}} = 0\,.$$

3 Light cone gauge

This gauge is often used in the string theory (see e.g. [4]) but it is not Lorentz covariant. For explicit description of this gauge we define new variables

$$P_0^\pm = \tfrac{1}{\sqrt{2}}\left(P_0^0 \pm P_0^{D-1}\right), \quad X_0^\pm = \tfrac{1}{\sqrt{2}}\left(X_0^0 \pm X_0^{D-1}\right),$$
$$a_k^\pm = \tfrac{1}{\sqrt{2}}\left(a_k^0 \pm a_k^{D-1}\right).$$

We have

$$XY = -X^+Y^- - X^-Y^+ + \sum_{\beta=1}^{D-2} X^\beta Y^\beta$$

and the Poisson brackets (7) take the form

$$\begin{aligned}\{f,g\} = &\\ \sum_{n=0}^{\infty}\Bigg(&-\frac{\partial f}{\partial X_n^+}\frac{\partial g}{\partial P_n^-} - \frac{\partial f}{\partial X_n^-}\frac{\partial g}{\partial P_n^+} + \frac{\partial f}{\partial P_n^+}\frac{\partial g}{\partial X_n^-} + \frac{\partial f}{\partial P_n^-}\frac{\partial g}{\partial X_n^+}\\ &+\sum_{\beta=1}^{D-2}\Big(\frac{\partial f}{\partial X_n^\beta}\,\frac{\partial g}{\partial P_n^\beta} - \frac{\partial f}{\partial P_n^\beta}\,\frac{\partial g}{\partial X_n^\beta}\Big)\Bigg). \qquad (13)\end{aligned}$$

We define the gauge conditions by the equations

$$R_0 = \frac{X_0^+}{P_0^+} - \tau = 0\,, \quad R_k = \frac{a_k^+}{\mathrm{i}\omega k P_0^+} = 0\,. \qquad (14)$$

From the Poisson brackets (13) we obtain on the set $\mathcal{M}$

$$\big\{L_n, L_k\big\} = \big\{R_n, R_k\big\} = 0 \quad \text{and} \quad \big\{L_n, R_k\big\} = -\delta_{n,-k}\,. \qquad (15)$$

Equations (9) and (10) give in this gauge

$$\kappa_n = 0 \ \text{ for } n \neq 0 \ \text{ and } \ \kappa_0 = 1\,, \qquad (16)$$

and equations of motion are

$$\dot{X}_0^\mu = P_0^\mu\,, \quad \dot{P}_0^\mu = 0\,, \quad \dot{a}_n^\mu = \mathrm{i}\omega n a_n^\mu\,. \qquad (17)$$

With respect to (15) the Dirac bracket (11) of the functions f and g is

$$\{f,g\}_{\mathrm{Dir}} = \{f,g\} + \sum_{n\in\mathbb{Z}}\Big(\{f,L_n\}\{g,R_{-n}\} - \{f,R_n\}\{g,L_{-n}\}\Big).$$

If we define new variables by ($\beta = 1, \ldots, D-2$)

$$
\begin{aligned}
&P^+ = P_0^+ \,, && \widehat{X}^- = \frac{X_0^- P_0^+ - X_0^+ P_0^-}{P_0^+} = X_0^- - \tau P_0^- \,, \\
&P^\beta = P_0^\beta \,, && \widehat{X}^\beta = \frac{X_0^\beta P_0^+ - X_0^+ P_0^\beta}{P_0^+} = X_0^\beta - \tau P_0^\beta \,, \\
&\widehat{X}^+ = X_0^+ - \tau P_0^+ \,, && a_k^\beta \,, \quad a_k^+ \,, \quad k \in \mathbb{Z}_0 \,, \\
&\widehat{L}_0 = P_0^+ P_0^- \,, && \widehat{L}_k = P_0^+ a_k^- \,,
\end{aligned}
\tag{18}
$$

we obtain that the only nonzero Dirac brackets are as follows

$$
\begin{aligned}
&\{P^+, \widehat{X}^-\}_{\rm Dir} = 1 \,, && \{P^\beta, \widehat{X}^\gamma\}_{\rm Dir} = -\delta^{\beta\gamma} \,, \\
&\{a_k^\beta, a_n^\gamma\}_{\rm Dir} = \mathrm{i}\omega k \delta^{\beta\gamma}\delta_{k,-n} \,, && \\
&\{\widehat{X}^\beta, \widehat{L}_0\}_{\rm Dir} = P^\beta \,, && \{\widehat{X}^\beta, \widehat{L}_k\}_{\rm Dir} = a_k^\beta \,, \\
&\{a_n^\beta, \widehat{L}_k\}_{\rm Dir} = \mathrm{i}\omega n a_{n+k}^\beta \,, \quad n \neq -k \,, && \{a_k^\beta, \widehat{L}_{-k}\}_{\rm Dir} = \mathrm{i}\omega k P^\beta \,, \\
&\{\widehat{L}_n, \widehat{L}_k\}_{\rm Dir} = \mathrm{i}\omega (n-k) \widehat{L}_{n+k} \,. &&
\end{aligned}
$$

Equations (14) and (8) have in the new variables the following form:

$$
\begin{aligned}
X^+ &= a_k^+ = 0 \,, \\
\widehat{L}_0 &= \tfrac{1}{2}\sum_{\beta=1}^{D-2} \left(P^\beta\right)^2 + \sum_{n=1}^{\infty}\sum_{\beta=1}^{D-2} a_n^\beta a_{-n}^\beta = 0 \,, \\
\widehat{L}_n &= \sum_{\beta=1}^{D-2}\Big(P_0^\beta a_n^\beta + \sum_{k=1}^{\infty} a_{n+k}^\beta a_{-k}^\beta + \tfrac{1}{2}\sum_{k=1}^{n-1} a_{n-k}^\beta a_k^\beta\Big) \,, \qquad n > 0 \\
\widehat{L}_{-n} &= \sum_{\beta=1}^{D-2}\Big(P_0^\beta a_{-n}^\beta + \sum_{k=1}^{\infty} a_{-n-k}^\beta a_k^\beta + \tfrac{1}{2}\sum_{k=1}^{n-1} a_{k-n}^\beta a_{-k}^\beta\Big) \qquad n > 0
\end{aligned}
$$

and for the equations of motion (17) we obtain

$$
\dot{f} = \{f, H^{\rm Dir}\}_{\rm Dir} \,, \tag{19}
$$

where

$$
H^{\rm Dir} = \sum_{n=1}^{\infty}\sum_{\beta=1}^{D-2} a_n a_{-n} = \widehat{L}_0 - \tfrac{1}{2}\sum_{\beta=1}^{D-2}\left(P^\beta\right)^2 = -\tfrac{1}{2}\, P_\mu P^\mu = \tfrac{1}{2}\, M^2 \,.
$$

It results from the quantization principle (12) that we have to assign the operators to functions in such a way that

$$
\begin{aligned}
&\big[\mathbf{P}^+,\widehat{\mathbf{X}}^-\big]=\mathrm{i}\,, && \big[\mathbf{P}^\beta,\widehat{\mathbf{X}}^\gamma\big]=-\mathrm{i}\delta^{\beta\gamma}\,,\\
&\big[\mathbf{a}_k^\beta,\mathbf{a}_n^\gamma\big]=-\omega k\delta^{\beta\gamma}\delta_{k,-n}\,, && \\
&\big[\widehat{\mathbf{X}}^\beta,\widehat{\mathbf{L}}_0\big]=\mathrm{i}\mathbf{P}^\beta\,, && \big[\mathbf{a}_k^\beta,\widehat{\mathbf{L}}_0\big]=-\omega k\mathbf{a}_k^\beta\,,\\
&\big[\widehat{\mathbf{X}}^\beta,\widehat{\mathbf{L}}_k\big]=\mathrm{i}\mathbf{a}_k^\beta\,, && \\
&\big[\mathbf{a}_n^\beta,\widehat{\mathbf{L}}_k\big]=-\omega n\mathbf{a}_{n+k}^\beta\,,\quad n\neq-k\,, && \big[\mathbf{a}_k^\beta,\widehat{\mathbf{L}}_{-k}\big]=-\omega k\mathbf{P}^\beta\,,\\
&\big[\widehat{\mathbf{L}}_n,\widehat{\mathbf{L}}_k\big]=\omega(k-n)\widehat{\mathbf{L}}_{n+k} &&
\end{aligned}
\tag{20}
$$

hold.

To realize the operators we put

$$
\widehat{\mathbf{L}}_n=\sum_{\beta=1}^{D-2}\mathbf{L}_n^\beta\,,\quad \mathbf{L}_0^\beta=\tfrac{1}{2}\,(\mathbf{P}^\beta)^2+\mathbf{E}_0^\beta\,,\quad \mathbf{L}_k^\beta=\mathbf{P}^\beta\mathbf{a}_k^\beta+\mathbf{E}_k^\beta\,,\quad k\neq0\,,
\tag{21}
$$

where

$$
\big[\mathbf{P}^+,\mathbf{E}_n^\beta\big]=\big[\mathbf{P}^\gamma,\mathbf{E}_n^\beta\big]=\big[\mathbf{X}^-,\mathbf{E}_n^\beta\big]=\big[\mathbf{X}^\gamma,\mathbf{E}_n^\beta\big]=0\,.
$$

From (20) we obtain for $\mathbf{a}_k^\beta$,$\mathbf{E}_k^\beta$ the following relations:

$$
\begin{aligned}
&\big[\mathbf{a}_k^\beta,\mathbf{a}_n^\gamma\big]=-k\omega\delta^{\beta\gamma}\delta_{k,-n}\,, && \big[\mathbf{a}_k^\beta,\mathbf{E}_n^\gamma\big]=-k\omega\mathbf{a}_{k+n}^\beta\delta^{\beta\gamma}\,,\\
&\big[\mathbf{E}_k^\beta,\mathbf{E}_n^\gamma\big]=\omega(n-k)\mathbf{E}_{k+n}^\beta\delta^{\beta\gamma}\,, && \mathbf{a}_0^\beta=0\,.
\end{aligned}
$$

These operators $\mathbf{E}_k^\beta$ and $\mathbf{a}_k^\beta$ form an algebra which will be denoted by $\mathcal{A}$.

The gauge conditions $X^+=a_k^+=0$ give $\mathbf{X}^+=\mathbf{a}_k^+=0$ and the representations of the operators $\mathbf{P}^+$, $\widehat{\mathbf{X}}^-$, $\mathbf{P}^\beta$ and $\widehat{\mathbf{X}}^\beta$ (where $\beta=1,\ldots,D-2$) can be obtained in a standard way. In the following, we concentrate on the representation of the algebra $\mathcal{A}$.

3.1 Standard quantization of string

By the standard quantization (see [4]) the main emphasis is placed on the relations

$$
\big[\mathbf{a}_k^\beta,\mathbf{a}_n^\gamma\big]=-k\omega\delta^{\beta\gamma}\delta_{k,-n}\,.
$$

It is well known that these operators are possible to realize on the Fock space $\mathcal{V}$ which is generated by the action of the operators $\mathbf{a}_k^\beta$, $k > 0$ on the vacuum state $|\,0\,\rangle$ which is defined by the conditions

$$\mathbf{a}_k^\beta\,|\,0\,\rangle = 0 \quad \text{for} \quad k < 0\,.$$

The operators $\mathbf{E}_k^\beta$ are then defined by

$$\begin{aligned} \mathbf{E}_0^\beta &= \sum_{n=1}^{\infty} \mathbf{a}_n^\beta \mathbf{a}_{-n}^\beta\,, \\ \mathbf{E}_n^\beta &= \sum_{k=1}^{\infty} \mathbf{a}_{n+k}^\beta \mathbf{a}_{-k}^\beta + \tfrac{1}{2}\sum_{k=1}^{n-1} \mathbf{a}_{n-k}^\beta \mathbf{a}_k^\beta\,, \\ \mathbf{E}_{-n}^\beta &= \sum_{k=1}^{\infty} \mathbf{a}_k^\beta \mathbf{a}_{-n-k}^\beta + \tfrac{1}{2}\sum_{k=1}^{n-1} \mathbf{a}_{k-n}^\beta \mathbf{a}_{-k}^\beta\,. \end{aligned} \tag{22}$$

The normal ordering of the operators guarantees the convergence of operators (22), but if we put

$$\mathbf{E}_k = \sum_{\beta=1}^{D-2} \mathbf{E}_k^\beta\,,$$

we obtain

$$\big[\mathbf{E}_n, \mathbf{E}_k\big] = \omega(k-n)\mathbf{E}_{n+k} - \frac{\omega^2}{12}\,(D-2)n(n^2-1)\delta_{n,-k}\,, \tag{23}$$

which is commonly used in the string theory .

We see that this choice of the representation of the operators $\mathbf{E}_k$ is in contradiction with the relation

$$\big[\mathbf{E}_n, \mathbf{E}_k\big] = \omega(k-n)\mathbf{E}_{n+k}\,. \tag{24}$$

which follows from the standard classical theory.

The consequence of the change of Dirac commutation relations is that the desired commutation relations are disturbed for the same important operators. For example, the operators of angular momenta

$\mathbf{J}^{\mu\nu}$ are given by

$$\begin{aligned}
\mathbf{J}^{01} &= \tfrac{1}{2}\left(\mathbf{P}^{+}\widehat{\mathbf{X}}^{-}+\widehat{\mathbf{X}}^{-}\mathbf{P}^{+}\right)=\mathbf{P}^{+}\widehat{\mathbf{X}}^{-}-\tfrac{1}{2}\,\mathrm{i}\,,\\
\mathbf{J}^{+\beta} &= -\mathbf{P}^{+}\widehat{\mathbf{X}}^{\beta}\,,\\
\mathbf{J}^{\mu\nu} &= \mathbf{P}^{\nu}\widehat{\mathbf{X}}^{\mu}-\mathbf{P}^{\mu}\widehat{\mathbf{X}}^{\nu}-\frac{\mathrm{i}}{\omega}\sum_{n=1}^{\infty}\frac{\mathbf{a}_n^{\mu}\mathbf{a}_{-n}^{\nu}-\mathbf{a}_n^{\nu}\mathbf{a}_{-n}^{\mu}}{n}\,,\\
\mathbf{J}^{-\beta} &= \frac{1}{\mathbf{P}^{+}}\Bigg(\mathbf{P}^{\beta}\mathbf{J}^{01}+\mathbf{P}^{-}\mathbf{J}^{+\beta}-\\
&\quad -\frac{\mathrm{i}}{\omega}\Big(\sum_{\alpha=1}^{D-2}\sum_{n=1}^{\infty}\mathbf{P}^{\alpha}\,\frac{\mathbf{a}_n^{\alpha}\mathbf{a}_{-n}^{\beta}-\mathbf{a}_n^{\beta}\mathbf{a}_{-n}^{\alpha}}{n}+\sum_{n=1}^{\infty}\frac{\mathbf{E}_n\mathbf{a}_{-n}^{\beta}-\mathbf{a}_n^{\beta}\mathbf{E}_{-n}}{n}\Big)\Bigg),
\end{aligned}$$

where $\mathbf{P}^{-}=\left(\mathbf{P}^{+}\right)^{-1}\Big(\frac{1}{2}\sum_{\beta=1}^{D-2}\mathbf{P}^{\beta}\mathbf{P}^{\beta}+\mathbf{E}_0\Big)$. The relation

$$\left[\mathbf{J}^{-\beta},\mathbf{J}^{-\gamma}\right]=0 \tag{25}$$

is violated since from the commutation relation we obtain

$$\begin{aligned}
\left[\mathbf{J}^{-\beta},\mathbf{J}^{-\gamma}\right] &= \sum_{n=1}^{\infty}\Bigg(\frac{\omega(D-26)n}{12}\left(\mathbf{a}_n^{\beta}\mathbf{a}_{-n}^{\gamma}-\mathbf{a}_n^{\gamma}\mathbf{a}_{-n}^{\beta}\right)-\\
&-\frac{2}{n}\Big(\mathbf{P}^{+}\mathbf{P}^{-}-\frac{1}{2}\sum_{\alpha=1}^{D-2}\mathbf{P}^{\alpha}\mathbf{P}^{\alpha}-\mathbf{E}_0-\frac{\omega(D-2)}{24}\Big)\left(\mathbf{a}_n^{\beta}\mathbf{a}_{-n}^{\gamma}-\mathbf{a}_n^{\gamma}\mathbf{a}_{-n}^{\beta}\right)\Bigg).
\end{aligned}$$

Therefore to obtain relation (25) we have to put $D=26$ and change the constraint, which defines $\mathbf{P}^{-}$, to

$$\mathbf{P}^{+}\mathbf{P}^{-}=\tfrac{1}{2}\sum_{\beta=1}^{D-2}\mathbf{P}^{\beta}\mathbf{P}^{\beta}+\mathbf{E}_0+\omega\,.$$

3.2 Dirac quantization

First let us consider in more detail the problem which leads in Section 3.1 to the change of the commutation relation (24) to (23).

If M_+, $M_- > n$, the direct calculation leads to the formula

$$\left[\sum_{k=1}^{M_+}\mathbf{a}^{\beta}_{n+k}\mathbf{a}^{\beta}_{-k}+\tfrac{1}{2}\sum_{k=1}^{n-1}\mathbf{a}^{\beta}_{n-k}\mathbf{a}^{\beta}_{k},\sum_{k=1}^{M_-}\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-n-k}+\tfrac{1}{2}\sum_{k=1}^{n-1}\mathbf{a}^{\beta}_{k-n}\mathbf{a}^{\beta}_{-k}\right]=$$
$$=\omega\sum_{k=M+1}^{M+n}(k+n)\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}-2\omega n\sum_{k=n}^{M+n}\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}-\omega\sum_{k=1}^{n-1}(n+k)\mathbf{a}_{k}\mathbf{a}_{-k}-$$
$$-\frac{\omega}{2}\sum_{k=1}^{n-1}(n-k)\big(\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}+\mathbf{a}^{\beta}_{-k}\mathbf{a}^{\beta}_{k}\big)\,, \tag{26}$$

where $M=\min\big(M_+,M_-\big)$. The statement (26) can be rewritten in a different way. For the normal ordering $\mathbf{a}^{\beta}_{n}\mathbf{a}^{\beta}_{-n}$, which we used in 3.1, we have

$$\omega\sum_{k=M+1}^{M+n}(k+n)\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}-2\omega n\sum_{k=1}^{M+n}\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}-\frac{\omega^2}{12}\,n\big(n^2-1\big) \tag{27}$$

and in the symmetric ordering we obtain

$$\tfrac{1}{2}\,\omega\sum_{k=M+1}^{M+n}(k+n)\big(\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}+\mathbf{a}^{\beta}_{-k}\mathbf{a}^{\beta}_{k}\big)-\omega n\sum_{k=1}^{M+n}\big(\mathbf{a}^{\beta}_{k}\mathbf{a}^{\beta}_{-k}+\mathbf{a}^{\beta}_{-k}\mathbf{a}^{\beta}_{k}\big)\,. \tag{28}$$

Formula (27) was used in 3.1. We see that the first term is zero on the Fock space but the second changes commutation relation (24) to (23).

In the Dirac quantization we have to use formula (28). However, in this case, we cannot use the representation of operators $\mathbf{a}^{\beta}_{n}$ on the Fock space because the series

$$\tfrac{1}{2}\sum_{\beta=1}^{D-2}\sum_{n=1}^{\infty}\big(\mathbf{a}^{\beta}_{n}\mathbf{a}^{\beta}_{-n}+\mathbf{a}^{\beta}_{-n}\mathbf{a}^{\beta}_{n}\big)\,|\,0\,\rangle$$

is divergent.

So we must find other representations of $\mathbf{a}^{\beta}_{k}$ for which

$$\lim_{m\to\infty} m\sum_{\beta=1}^{D-2}\sum_{r=1}^{n}\Big(\mathbf{a}^{\beta}_{k+m+r}\mathbf{a}^{\beta}_{-m-r}+\mathbf{a}^{\beta}_{-m-r}\mathbf{a}^{\beta}_{k+m+r}\Big)=0$$

for any $n>0$ and $k\in\mathbb{Z}$.

In the limits the first term of (28) will be zero and for definition of the operators $\mathbf{E}_k$ we can use

$$\begin{aligned}
\mathbf{E}_0^\beta &= \tfrac{1}{2} \lim_{N\to\infty} \sum_{n=1}^{N} \left(\mathbf{a}_n^\beta \mathbf{a}_{-n}^\beta + \mathbf{a}_{-n}^\beta \mathbf{a}_n^\beta\right), \\
\mathbf{E}_k^\beta &= \lim_{N\to\infty} \left(\sum_{k=1}^{N} \mathbf{a}_{n+k}^\beta \mathbf{a}_{-k}^\beta + \tfrac{1}{2} \sum_{k=1}^{n-1} \mathbf{a}_{n-k}^\beta \mathbf{a}_k^\beta \right), \\
\mathbf{E}_{-k}^\beta &= \lim_{N\to\infty} \left(\sum_{k=1}^{N} \mathbf{a}_k^\beta \mathbf{a}_{-n-k}^\beta + \tfrac{1}{2} \sum_{k=1}^{n-1} \mathbf{a}_{k-n}^\beta \mathbf{a}_{-k}^\beta \right).
\end{aligned}$$

The second, alternative, possibility is to start from any representation of the algebra $\mathcal{A}$ and try to define the physical states as the vectors $|\,\psi\,\rangle_{\rm phys}$, for which

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{\beta=1}^{D-2} \Big(\mathbf{E}_0^\beta - \tfrac{1}{2} \sum_{n=1}^{N} \left(\mathbf{a}_n^\beta \mathbf{a}_{-n}^\beta + \mathbf{a}_{-n}^\beta \mathbf{a}_n^\beta\right) \Big) |\,\psi\,\rangle_{\rm phys} = 0\,, \\
&\lim_{N\to\infty} \sum_{\beta=1}^{D-2} \Big(\mathbf{E}_n^\beta - \sum_{k=1}^{N} \mathbf{a}_{n+k}^\beta \mathbf{a}_{-k}^\beta - \tfrac{1}{2} \sum_{k=1}^{n-1} \mathbf{a}_{n-k}^\beta \mathbf{a}_k^\beta \Big) |\,\psi\,\rangle_{\rm phys} = 0\,, \\
&\lim_{N\to\infty} \sum_{\beta=1}^{D-2} \Big(\mathbf{E}_{-n}^\beta - \sum_{k=1}^{N} \mathbf{a}_k^\beta \mathbf{a}_{-n-k}^\beta - \tfrac{1}{2} \sum_{k=1}^{n-1} \mathbf{a}_{k-n}^\beta \mathbf{a}_{-k}^\beta \Big) |\,\psi\,\rangle_{\rm phys} = 0\,.
\end{aligned}$$

In both cases the commutator (24) and the Lorentz covariance will no longer be broken, but the explicit realization of this programme is an open question.

4 Lorentz covariant gauge

The gauge conditions, which we used in Section 3, are not Lorentz covariant. In this part, we will study the Lorentz covariant case. This gauge was partially used by Rohrlich in [5]. Specifically, we will take gauge conditions of the form

$$R_0 = \frac{P_0 X_0}{P_0^2} - \tau = 0\,, \qquad R_k = \frac{P_0 a_k}{\mathrm{i}\omega k P_0^2} = 0\,, \quad k \in \mathbb{Z}_0\,, \tag{29}$$

where

$$XY = g_{\mu\nu} X^\mu Y^\nu \quad \text{and} \quad P_0^2 = P_0 P_0 = g_{\mu\nu} P_0^\mu P_0^\nu = -M^2 < 0\,.$$

For the Poisson brackets (7) of these functions and constraints (8) on the set $\mathcal{M}$ (where the equations are valid (8) and (29)) we obtain

$$\big\{L_n, L_k\big\} = 0\,, \quad \big\{L_n, R_k\big\} = -\delta_{n,-k}\,, \quad \big\{R_n, R_k\big\} = \frac{\mathrm{i}\delta_{n,-k}}{\omega n P_0^2}\,.$$

For the constant from equations (10) κ_n we have again the conditions (16) and the equations of motion (17).

In this case, the Dirac brackets (11) are

$$\begin{aligned}\big\{f,g\big\}_{\mathrm{Dir}} \;=\; &\big\{f,g\big\} + \sum_{n\in\mathbb{Z}}\Big(\big\{f,L_n\big\}\big\{g,R_{-n}\big\} - \big\{f,R_n\big\}\big\{g,L_{-n}\big\}\Big) - \\ &-\frac{\mathrm{i}}{\omega P_0^2}\sum_{k\in\mathbb{Z}_0}\frac{\big\{f,L_k\big\}\big\{g,L_{-k}\big\}}{k}\,.\end{aligned}$$

Particularly, for the Dirac brackets between the variables we have

$$\begin{aligned}
&\big\{P_0^\mu, P_0^\nu\big\}_{\mathrm{Dir}} = 0\,, \qquad \big\{P_0^\mu, X_0^\nu\big\}_{\mathrm{Dir}} = -g^{\mu\nu} + \frac{P_0^\mu P_0^\nu}{P_0^2}\,,\\
&\big\{X_0^\mu, X_0^\nu\big\}_{\mathrm{Dir}} = \frac{1}{P_0^2}\Big(P_0^\mu X_0^\nu - P_0^\nu X_0^\mu - \frac{\mathrm{i}}{\omega}\sum_{n=1}^{\infty}\frac{a_n^\nu a_{-n}^\mu - a_n^\mu a_{-n}^\nu}{n}\Big) = \\
&\qquad\qquad = -\frac{J^{\mu\nu}}{P_0^2}\,,\\
&\big\{P_0^\mu, a_k^\nu\big\}_{\mathrm{Dir}} = \mathrm{i}\omega k\,\frac{P_0^\mu a_k^\nu}{P_0^2}\,,\\
&\big\{a_k^\mu, X_0^\nu\big\}_{\mathrm{Dir}} = \frac{1}{P_0^2}\Big(P_0^\mu a_k^\nu + \mathrm{i}\omega k\big(X_0^\nu - 2\tau P_0^\nu\big)a_k^\mu\Big),\\
&\big\{a_k^\mu, a_n^\nu\big\}_{\mathrm{Dir}} = \frac{\mathrm{i}\omega k n}{P_0^2}\sum_{s\neq 0,-k,n}\frac{a_{k+s}^\mu a_{n-s}^\nu}{s}\,, \qquad n+k\neq 0\,,\\
&\big\{a_k^\mu, a_{-k}^\nu\big\}_{\mathrm{Dir}} = \mathrm{i}\omega k\Big(g^{\mu\nu} - \frac{P_0^\mu P_0^\nu}{P_0^2}\Big) + \frac{\mathrm{i}\omega k^2}{P_0^2}\sum_{n\in\mathbb{Z}_k}\frac{a_{k-n}^\mu a_{n-k}^\nu}{n}\,,
\end{aligned}$$

in which in agreement with the constraints we put $\tau = \dfrac{P_0 X_0}{P_0^2}$.

The equations of motion can be written in the form (19), where

$$H^{\mathrm{Dir}} = -\tfrac{1}{2}\,g_{\mu\nu}P_0^\mu P_0^\nu = \tfrac{1}{2}\,M^2\,.$$

If we put

$$\mathcal{P}^\mu = \frac{P_0^\mu}{M}\,, \quad \mathcal{Q}^\mu = M\big(X_0^\mu - \tau P_0^\mu\big)\,, \quad a_0^\mu = 0\,,$$

we obtain the Dirac brackets in the form

$$
\begin{aligned}
\{\mathcal{P}^\mu,\mathcal{P}^\nu\}_{\rm Dir} &= 0\,, \qquad \{\mathcal{Q}^\mu,\mathcal{P}^\nu\}_{\rm Dir} = g^{\mu\nu}+\mathcal{P}^\mu\mathcal{P}^\nu\,,\\
\{\mathcal{Q}^\mu,\mathcal{Q}^\nu\}_{\rm Dir} &= \mathcal{Q}^\mu\mathcal{P}^\nu-\mathcal{Q}^\nu\mathcal{P}^\mu-\frac{\rm i}{\omega}\sum_{n=1}^{\infty}\frac{a_n^\mu a_{-n}^\nu-a_n^\nu a_{-n}^\mu}{n}=J^{\mu\nu}\,,\\
\{\mathcal{P}^\mu,a_k^\nu\}_{\rm Dir} &= 0\,, \qquad \{\mathcal{Q}^\mu,a_k^\nu\}_{\rm Dir}=\mathcal{P}^\nu a_k^\mu\,,\\
\{a_k^\mu,a_n^\nu\}_{\rm Dir} &= -\frac{{\rm i}\omega kn}{M^2}\sum_{s\in\mathbb{Z}+0}\frac{a_{k+s}^\mu a_{n-s}^\nu}{s}\,, \qquad n+k\neq 0\,, \qquad (30)\\
\{a_k^\mu,a_{-k}^\nu\}_{\rm Dir} &= {\rm i}\omega k\big(g^{\mu\nu}+\mathcal{P}^\mu\mathcal{P}^\nu\big)-\frac{{\rm i}\omega k^2}{M^2}\sum_{n\in\mathbb{Z}_0}\frac{a_{k-n}^\mu a_{n-k}^\nu}{n}\,,\\
\{M^2,\mathcal{P}^\mu\}_{\rm Dir} &= \{M^2,\mathcal{Q}^\mu\}_{\rm Dir}=0\,, \qquad \{M^2,a_k^\mu\}_{\rm Dir}=-2{\rm i}\omega k a_k^\mu\,.
\end{aligned}
$$

The constraints (8) and the gauge conditions (29) have in these variables the form

$$
\begin{aligned}
&\mathcal{P}\mathcal{Q}=\mathcal{P}a_k=\mathcal{P}\mathcal{P}+1=0\,,\\
&\sum_{n\in\mathbb{Z}}a_n a_{-n}=M^2\,,\\
&\sum_{n\in\mathbb{Z}}a_n a_{k-n}=0\,, \quad \text{for} \quad k\neq 0\,.
\end{aligned}
$$

By the quantization of the first five equations in (30), we obtain

$$
\begin{aligned}
\big[\mathbf{P}^\mu,\mathbf{P}^\nu\big] &= 0\,, \qquad \big[\mathbf{Q}^\mu,\mathbf{P}^\nu\big]={\rm i}\big(g^{\mu\nu}+\mathbf{P}^\mu\mathbf{P}^\nu\big)\,,\\
\big[\mathbf{Q}^\mu,\mathbf{Q}^\nu\big] &= {\rm i}\Big(\mathbf{P}^\nu\mathbf{Q}^\mu-\mathbf{P}^\mu\mathbf{Q}^\nu-\frac{\rm i}{\omega}\sum_{n=1}^{\infty}\frac{\mathbf{a}_n^\mu\mathbf{a}_{-n}^\nu-\mathbf{a}_n^\nu\mathbf{a}_{-n}^\mu}{n}\Big)={\rm i}\mathbf{J}^{\mu\nu}\,,\\
\big[\mathbf{P}^\mu,\mathbf{a}_k^\nu\big] &= 0\,, \qquad \big[\mathbf{Q}^\mu,\mathbf{a}_k^\nu\big]={\rm i}\mathbf{P}^\nu\mathbf{a}_k^\mu\,.
\end{aligned}
$$

Directly from these commutation relations it follows that

$$
\begin{aligned}
\Big[\mathbf{P}^\beta,\big[\mathbf{Q}^\mu,\mathbf{Q}^\mu\big]\Big] &= {\rm i}g^{\beta\mu}\mathbf{P}^\nu-{\rm i}g^{\beta\nu}\mathbf{P}^\mu\,,\\
\Big[\mathbf{Q}^\rho,\big[\mathbf{Q}^\mu,\mathbf{Q}^\mu\big]\Big] &= g^{\mu\rho}\mathbf{Q}^\nu-g^{\nu\rho}\mathbf{Q}^\mu
\end{aligned}
$$

and by using the Jacobi identities we have

$$
\begin{aligned}
\big[\mathbf{J}^{\mu\nu},\mathbf{J}^{\rho\sigma}\big] &= -\Big[\big[\mathbf{Q}^\mu,\mathbf{Q}^\nu\big],\big[\mathbf{Q}^\rho,\mathbf{Q}^\sigma\big]\Big]=\\
&= \Big[\mathbf{Q}^\rho,\big[\mathbf{Q}^\sigma,[\mathbf{Q}^\mu,\mathbf{Q}^\nu]\big]\Big]-\Big[\mathbf{Q}^\sigma,\big[\mathbf{Q}^\rho,[\mathbf{Q}^\mu,\mathbf{Q}^\nu]\big]\Big]=\\
&= {\rm i}g^{\mu\rho}\mathbf{J}^{\nu\sigma}+{\rm i}g^{\nu\sigma}\mathbf{J}^{\mu\rho}-{\rm i}g^{\mu\sigma}\mathbf{J}^{\nu\rho}-{\rm i}g^{\nu\rho}\mathbf{J}^{\mu\sigma}\,,
\end{aligned}
$$

which are the quantum relations for the operators of angular momenta $\mathbf{J}^{\mu\nu}$.

Again, a big problem is how to find explicit realization of the operators $\mathbf{a}_k^\beta$.

5 Conclusion

We have studied the Dirac theory of a free open string. We gave the explicit formulae for the Dirac brackets in the light-cone and Lorentz calibrations. We should like to emphasize that standard quantization, which is used now in modern string theory, changes a classical system by quantization.

Physically, the main difference between these two approaches is that in the standard method of quantization we suppose mutual independence of the oscillators corresponding to the operators $\mathbf{a}_k^\beta$ and in the Dirac method these oscillators are strongly dependent.

We gave more attention to the ideas of Dirac quantization and formulated some new problem for quantization of the classical open string theory. The change of the classical system by quantization will not be needed and solution will be possible in any dimension D.

The principal matter for solution of the problems of Dirac quantization of the string is to study the representations of the operators $\mathbf{a}_k^\beta$ which are not of Fock type.

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Comment on DIRAC FORMULATION OF FREE OPEN STRING

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Quantization of constrained physical systems is one among the most important and, simultaneously, the most difficult problems of contemporary theoretical physics. In fact field theorists badly need to know how to quantize gauge field models, string theories and Einsteinian gravity which, according to our best knowledge, all are examples of field theories with constraints for which simple standard methods of fields quantization do not work. Although many scientists are working on the problem of quantization of constrained systems and although to find its universal solution is really urgent and necessary for further development of quantum field theory the only self-consistent and manageable approach to the problem is the method of quantization invented more than 40 years ago by P.A.M Dirac. Despite of its successes and mathematical beauty the Dirac method leaves many questions open and in order to understand its properties, as well as its physical implications, it is useful to analyse carefully some particularly chosen models. Burdik and Navratil do this by investigating the free open string in dimension D. They develop step by step classical canonical formalism for such a model and arrive at sets of Dirac brackets for two choices of gauge conditions: the so-called light cone gauge being Lorentz noncovariant and Lorentz covariant gauge generalizing the condition proposed by Rohrlich about thirty years ago. Obviously, to get classical Dirac brackets is only a primary step if one wants to investigate a quantum system. The real challenge is to find a representation of an operator algebra obtained by formal replacement of Dirac brackets by commutators. Trying to solve this problem Burdik and Navratil are partially successful - for the light cone gauge they find a Fock-type representation which for $D = 2$ and $D = 26$ is consistent with results following from standard classical theory. Unfortunately, they do not give an explanation whether these specific values of D have any physical meaning or are purely incidental. Investigating the Lorentz covariant gauge the authors arrive at results which for me seem more intriguing. For this case the quantization of Dirac brackets leads to the algebra in which canonical commutation rules between coordinate Q^μ and momentum P^ν operators are modified by terms proportional to $P^\mu P^\nu$, momenta commute and commutators between coordinates are proportional to the components of angular momentum. Such an algebra coincides with noncanonical algebras found in investigations rooted in quantum gravity and leading to theories with fundamental lenght [1] and for the so-called Wigner quantum systems [2], [3]. More detailed investigation of such a coincidence seems to be an interesting problem as well as it would be extremely useful to find representations of this algebra. This, I believe, will be the subject of Burdik and Navratil's further research.

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