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BOGOLIUBOV TRANSFORM AND TOMOGRAPHY OF NONCLASSICAL STATES OF TRAPPED ION

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Abstract

Bogoliubov transforms are discussed in some problems of quantum optics. Squeezed and rotated quadrature of an ion in a Paul trap is discussed in connection with reconstructing its quantum state using symplectic-tomography method. Symplectic and photon-number tomograms of the quadrature for squeezed and correlated states, for nonlinear coherent states and for odd/even coherent states of a trapped ion which are tomographic symbols of density operators of the states are reviewed.

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1 Introduction

In seminal work [1] Bogoliubov discussed the canonical transforms of creation and annihilation operators. These transforms play important role in quantum optics.The canonical transforms of photon quadratures or trapped ion position and momentum creating squeezing phenomena are just one of the realizations of Bogoliubov transforms. Recently, such nonclassical states of a trapped ion as even and odd coherent states [2] (or Schrödinger cat states [3]) were realized experimentally [4] and their properties were discussed in [5, 6]. For light modes, these states were produced in high-Q cavities [7]. The experiments with reproducible measurements of squeezed vacuum state of light generated by an optical parametric oscillator were performed in Ref. [8]. Resonance fluorescence was proposed to reconstruct the quantum-mechanical state of a trapped ion [9].

Endoscopy method of measuring the nonclassical states, in particular, of a trapped ion was suggested in Ref. [10] and tomography method for studying the Schrödinger cat states of an ion in a Paul trap was discussed in Ref. [11]. In Refs. [12], the linear integrals of motion of the parametric oscillator [13, 14], which models the motion of a trapped ion, were used to study the quantum states of the system. The linear integrals of motion in this problem were obtained using the Bogoliubov transforms.

New type of nonclassical states, namely, nonlinear coherent states of an ion and the method of creation of these states, as stationary states of the center-of-mass motion of a trapped and bichromatically laser-driven ion, were suggested in Ref. [15]. The nonlinear states are the particular case of f-coherent states introduced in Ref. [16] to describe a nonlinear quantum oscillator, for which the phase of vibrations depends on the energy of the vibrations. In the linear limit, these nonclassical states, which have the properties of squeezing and correlation of quadratures, become the coherent states of harmonic oscillator [17]. Thus, the problem of experimental reconstructing the nonclassical state in terms of Wigner function or density matrix (in other representations) for the nonlinear coherent state of a trapped ion is actual problem.

Recently, the symplectic tomography method was discussed for measuring a quantum state [18]. The method (extended as well for multimode case [19]) uses Fourier transform of marginal distribution for measurable squeezed and rotated quadrature instead of Radon transform [20], which is used in optical tomography to reconstruct the Wigner function, in this context, the symplectic tomography is similar to the strength-field method of Ref. [21]. The marginal distribution for squeezed and rotated quadrature (called symplectic tomogram) determines completely the quantum state and measuring this distribution implies reconstructing the quantum state. It satisfies the classical-like evolution equation introduced in Ref. [22] (see, also review [23]).

The aim of this work is to review in context of using the Bogoliubov transform the symplectic tomography scheme to measure nonlinear coherent states of a trapped ion, following the approach considered in Ref. [11]. Our goal is to construct explicitly the tomograms for nonlinear coherent state, odd/even coherent states, squeezed correlated states of an ion in a Paul trap. As well, the marginal distribution for discrete oscillator levels will be considered, corresponding to photon-number tomography of Refs. [24, 25, 26], as another procedure for measuring the quantum state of the trapped ion.

2 Nonlinear coherent states of a trapped ion

Since an ion in a Paul trap is described by the model of a parametric oscillator [12, 27] in this section folowing [11],[28]-[30] we review its properties. For a parametric oscillator with an arbitrary time dependence of the frequency and the Hamiltonian

$$H = -\frac{\partial^2}{2\,\partial q^2} + \frac{\omega^2\,(t)\,q^2}{2}\,, \tag{1}$$

where we put $\hbar = m = \omega\,(0) = 1$ and used expressions for the position and momentum operators in the coordinate representation, there is the time-dependent integral of motion found in Ref. [14] :

$$A = \frac{i}{\sqrt{2}}\,[\,\varepsilon\,(t)\,\hat{p} - \dot{\varepsilon}\,(t)\,\hat{q}\,]\,, \tag{2}$$

where

$$\ddot{\varepsilon}\,(t) + \omega^2\,(t)\,\varepsilon\,(t) = 0\,; \qquad \varepsilon\,(0) = 1\,; \qquad \dot{\varepsilon}\,(0) = i\,, \tag{3}$$

which satisfies the commutation relation

$$[A,\, A^{\dagger}] = 1\,. \tag{4}$$

The integral of motion (2) is connected with Bogoliubov transform of creation and annihilation operators [1]. For the trapped ion, the time dependence of the frequency is taken to be periodic [12]:

$$\omega^2\,(t) = 1 + \kappa^2 \sin^2 \Omega t\,. \tag{5}$$

It is easy to show that Gaussian packet solutions to the Schrödinger equation may be introduced and interpreted as coherent states [14], since they are eigenstates of the operator A (2), of the form

$$\Psi_\alpha\,(q,\,t) = \Psi_0\,(q,\,t)\exp\left\{-\frac{|\alpha|^2}{2} - \frac{\alpha^2\varepsilon^*(t)}{2\varepsilon(t)} + \frac{\sqrt{2}\alpha q}{\varepsilon}\right\}, \tag{6}$$

where

$$\Psi_0\,(q,\,t) = \pi^{-1/4}\,[\varepsilon(t)]^{-1/2}\exp\left[\frac{i\dot{\varepsilon}(t)q^2}{2\varepsilon(t)}\right] \tag{7}$$

is an analog of the ground state of the oscillator and α is a complex number. The variances of the position and momentum of the parametric oscillator in the state (6) are

$$\sigma_{q^2} = \frac{|\varepsilon\,(t)|^2}{2}\,; \qquad \sigma_{p^2} = \frac{|\dot{\varepsilon}\,(t)|^2}{2}\,, \tag{8}$$

and the covariance of the position and momentum is

$$\sigma_{qp} = \frac{1}{2}\sqrt{|\dot{\varepsilon}\,(t)\varepsilon\,(t)|^2 - 1}. \tag{9}$$

The correlation coefficient r of the position and momentum has a value corresponding to minimization of the Schrödinger uncertainty relation [31]:

$$\sigma_{q^2}\,\sigma_{p^2} = \frac{1}{4}\,\frac{1}{1-r^2}\,; \qquad r = \frac{\sigma_{pq}}{\sqrt{\sigma_{q^2}\,\sigma_{p^2}}}\,. \tag{10}$$

If $\sigma_{q^2} < 1/2 \;\; (\sigma_{p^2} < 1/2)$, we have squeezing in quadrature components.

Analogs of an orthogonal and complete system of number states, which are excited states of an ion in a Paul trap, are obtained by expansion of (6) into a power series in α. We have

$$\Psi_m\left(q,\,t\right)=\left(\frac{\varepsilon^*(t)}{2\varepsilon(t)}\right)^{m/2}\frac{1}{\sqrt{m!}}\,\Psi_0\left(q,\,t\right)H_m\left(\frac{q}{|\varepsilon(t)|}\right),\qquad(11)$$

and these squeezed and correlated number states are eigenstates of the invariant $A^\dagger A$.

The coherent state (6), which is squeezed and correlated state for quadratures, is the superposition of number states (11)

$$\Psi_\alpha\left(q,\,t\right)=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,\Psi_m\left(q,\,t\right).\qquad(12)$$

There exist the integrals of motion

$$B=A\,f\left(A^\dagger A\right);\qquad B^\dagger=f\left(A^\dagger A\right)A^\dagger\,,\qquad(13)$$

which are determined by a function f of the invariants (2). These integrals of motion satisfy the commutation relations

$$\left[B,B^\dagger\right]=F\left(A^\dagger A\right),\qquad(14)$$

where

$$F\left(A^\dagger A\right)=\left(A^\dagger A+1\right)f^2\left(A^\dagger A+1\right)-A^\dagger A\,f^2\left(A^\dagger A\right).\qquad(15)$$

Generalizing notion of coherent states to the case of the operator, which is nonlinearly transformed annihilation operator, we introduce the eigenfunctions of the invariant B

$$B\,\Psi_\beta\left(q,\,t\right)=\beta\,\Psi_\beta\left(q,\,t\right),\qquad(16)$$

which are the nonlinear coherent states. Such construction of the states, called f-coherent states, was suggested in Ref. [16]. For the function

$$f(y)=L^1_{y+1}(\eta^2)\left[yL^0_{y+1}(\eta^2)\right]^{-1},\qquad(17)$$

where $L^n_m(\eta^2)$ are assosiated Laguerre polynomials and η is Lamb–Dicke parameter, the f-coherent states (nonlinear coherent states) have been considered in Ref. [15].

Using general scheme of constructing the normalized nonlinear coherent states of Refs. [15, 16] one can obtain the function $\Psi_\beta(q,t)$ in the form of series

$$\Psi_\beta\,(q,\,t) = \left(\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{n!\,|[f(n)]!|^2}\right)^{-1/2}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}\,[f(m)]!}\,\Psi_m\,(q,\,t)\,, \tag{18}$$

in which we denote, e.g., $[f(m)]! = f(0)f(1)\cdots f(m)$.

For $f(m) = 1$, the wave function of the nonlinear coherent state (18) becomes the wave function of the coherent state (6), in which the parameter $\beta = \alpha$. The Wigner function of the nonlinear coherent states (18) has the form [16]

$$\begin{aligned} W_\beta\,(q,\,p) &= 2\left(\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{n!\,|[f(n)]!|^2}\right)^{-1} e^{-(q^2+p^2)} \\ &\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,[f(m)]!}\,\frac{1}{[f(n)]!}\,(-\beta)^n\beta^{*m} \\ &\times \left(\sqrt{2}\,[q-ip]\right)^{m-n} L_n^{m-n}\left(2\left[q^2+p^2\right]\right), \end{aligned} \tag{19}$$

where L_m^n denotes associated Laguerre polynomial. For the particular case of the function f given by (17), one has the Wigner function studied in Ref. [15].

Another normalized solutions to the Schrödinger equation are a squeezed even coherent state and a squeezed odd coherent state [2] (squeezed Schrödinger cat states)

$$\begin{aligned} \Psi_\alpha^{(+)}(q,t) &= 2N^{(+)}\Psi_0(q,t)\exp\left\{-\frac{|\alpha|^2}{2}-\frac{\varepsilon^*(t)\alpha^2}{2\varepsilon(t)}\right\}\cosh\frac{\sqrt{2}\alpha q}{\varepsilon(t)} \\ N^{(+)} &= \frac{\exp(|\alpha|^2/2)}{2\sqrt{\cosh|\alpha|^2}}\,, \end{aligned} \tag{20}$$

$$\begin{aligned} \Psi_\alpha^{(-)}(q,t) &= 2N^{(-)}\Psi_0(q,t)\exp\left\{-\frac{|\alpha|^2}{2}-\frac{\varepsilon^*(t)\alpha^2}{2\varepsilon(t)}\right\}\sinh\frac{\sqrt{2}\alpha q}{\varepsilon(t)} \\ N^{(-)} &= \frac{\exp(|\alpha|^2/2)}{2\sqrt{\sinh|\alpha|^2}}\,. \end{aligned} \tag{21}$$

3 Tomography of a trapped ion

It was shown [18] that for the generic linear combination of quadratures, which is a measurable observable ($\hbar = 1$),

$$\widehat{X} = \mu\hat{q} + \nu\hat{p}\,, \tag{22}$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum, respectively, the symplectic tomogram $w\,(X,\,\mu,\,\nu)$ (normalized with respect to the variable X), depending on the two extra real parameters μ and ν, is related to the state of the quantum system expressed in terms of its Wigner function $W(q,\,p)$ as follows

$$w\,(X,\,\mu,\,\nu) = \int \exp\left[-ik(X - \mu q - \nu p)\right] W(q,\,p)\,\frac{dk\,dq\,dp}{(2\pi)^2}\,. \tag{23}$$

It is worthy noting that the transform (22) is just the canonical Bogoliubov transform expressed in terms of quadratures. The physical meaning of the parameters μ and ν is that they describe an ensemble of rotated and scaled reference frames, in which the position X is measured. For $\mu = \cos\,\varphi$; $\nu = \sin\,\varphi$, the symplectic tomogram (23) is the distribution for homodyne output variable used in optical tomography [20]. Formula (23) can be inverted and the Wigner function of the state can be expressed in terms of the symplectic tomogram [18]:

$$W(q,\,p) = \frac{1}{2\pi}\int w\,(X,\,\mu,\,\nu)\exp\left[-i\left(\mu q + \nu p - X\right)\right]\,d\mu\,d\nu\,dX\,. \tag{24}$$

It was shown [22] that for systems with the Hamiltonian of the form $\hat{H} = (\hat{p}^2/2) + V(\hat{q})$ the symplectic tomogram satisfies quantum time-evolution equation. For a trapped ion, the evolution equation takes the form

$$\dot{w} - \mu\,\frac{\partial}{\partial\nu}\,w + \omega^2\,(t)\,\nu\,\frac{\partial}{\partial\mu}\,w = 0\,. \tag{25}$$

If one uses the constrain $\mu = \cos\,\varphi$; $\nu = \sin\,\varphi$, this equation becomes the equation for marginal distribution of optical tomography [20]. Thus, measuring symplectic tomogram for scaled and rotated quadrature one can reconstruct the Wigner function of a trapped ion using the Fourier transform (24).

In quantum mechanics the states are described by density operators acting in Hilbert space of states. In order to consider the states and observables as functions in a phase space we describe first a general conctruction and general relations and properties of a map of the operators onto the functions following [32, 33, 34] without concrete realisation of the map. Given a Hilbert space H and a density operator $\hat{\rho}$ acting in this space. Let us suppose that we have a set of operators $\hat{\mathcal{U}}(\mathbf{x})$ acting in the Hilbert space H, where the n-dimensional vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ labels the particular operator in the set. We construct the c-number function $w_\rho(\mathbf{x})$ (we call it the tomographic symbol of density operator $\hat{\rho}$ or tomogram) using the definition

$$w_{\hat{\rho}}(\mathbf{x}) = \mathrm{Tr}\,(\hat{\rho}\hat{\mathcal{U}}(\mathbf{x})). \tag{26}$$

Let us suppose that this relation has the inverse, i.e., there exists the set of operators $\hat{\mathcal{D}}(\mathbf{x})$ acting in the Hilbert space such that

$$\hat{\rho} = \int w_{\hat{\rho}}(\mathbf{x})\hat{\mathcal{D}}(\mathbf{x})\; d\mathbf{x}. \tag{27}$$

Then we will call the relations (26) and (27) as relations determining the invertable map of the density operator $\hat{\rho}$ onto function $w_{\hat{\rho}}(\mathbf{x})$. In fact we could consider the relations of the form

$$\hat{\rho} \rightarrow w_{\hat{\rho}}(\mathbf{x})$$

and

$$w_{\hat{\rho}}(\mathbf{x}) \rightarrow \hat{\rho}$$

with the properties to be described below as defining the map. The most important property is the existence of associative product (star-product) of functions. The star-product of two tomographic symbols $w_{\hat{\rho}_1}(\mathbf{x})$ and $w_{\hat{\rho}_2}(\mathbf{x})$ corresponding to two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ by the relations:

$$w_{\hat{\rho}_1\hat{\rho}_2}(\mathbf{x}) = w_{\hat{\rho}_1}(\mathbf{x}) * w_{\hat{\rho}_2}(\mathbf{x}) := \mathrm{Tr}\left[\hat{\rho}_1\hat{\rho}_2\hat{\mathcal{U}}(\mathbf{x})\right]. \tag{28}$$

One can write down a composition rule for two tomograms $w_{\hat{\rho}_1}(\mathbf{x})$ and $w_{\hat{\rho}_2}(\mathbf{x})$, which determines the star-product of these symbols. The composition rule is described by the formula

$$w_{\hat{\rho}_1}(\mathbf{x}) * w_{\hat{\rho}_2}(\mathbf{x}) = \int w_{\hat{\rho}_1}(\mathbf{x}'')w_{\hat{\rho}_2}(\mathbf{x}')\mathcal{K}(\mathbf{x}'', \mathbf{x}', \mathbf{x})\, d\mathbf{x}'\, d\mathbf{x}''. \tag{29}$$

The kernel in the integral is determined by the trace of product of the basic operators, which we use to construct the map

$$\mathcal{K}(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \mathrm{Tr}\left[\hat{\mathcal{D}}(\mathbf{x}'')\hat{\mathcal{D}}(\mathbf{x}')\hat{\mathcal{U}}(\mathbf{x})\right]. \tag{30}$$

In [32, 33, 34] it was shown that symplectic tomography scheme is an example of quantization scheme based on star-product of functions and was obtained explicit form of operators $\hat{\mathcal{D}}(\mathbf{x})$, $\hat{\mathcal{U}}(\mathbf{x})$ and kernel of star-product for symplectic tomography scheme. The operators $\hat{\mathcal{D}}(\mathbf{x})$, $\hat{\mathcal{U}}(\mathbf{x})$ which determine the relation between symplectic tomogram (tomographic symbol in symplectic tomography scheme) and density operator are of the form

$$\begin{aligned} \hat{\mathcal{U}}(\mathbf{x}) &\equiv \hat{\mathcal{U}}(X, \mu, \nu) = \delta(X - \mu\hat{q} - \nu\hat{p}), \\ \hat{\mathcal{D}}(\mathbf{x}) &\equiv \hat{\mathcal{D}}(X, \mu, \nu) = \frac{1}{2\pi}\exp\left(iX - i\nu\hat{p} - i\mu\hat{q}\right). \end{aligned} \tag{31}$$

4 Symplectic tomogram for squeezed and correlated states

First we discuss symplectic tomogram for squeezed and correlated state (6) of a trapped ion. For these states, the Wigner function has Gaussian form [35]. Consequently, the Fourier transform (23) of the Gaussian Wigner function determining the symplectic tomogram yields the Gaussian form of this symplectic tomogram

$$w_\alpha\left(X,\, \mu,\, \nu,\, t\right) = \frac{1}{\sqrt{2\pi\sigma_X(t)}}\exp\left\{-\frac{(X-\bar{X})^2}{2\sigma_X(t)}\right\}, \tag{32}$$

where the dispersion of the symplectic observable X and the mean value of the symplectic observable depend on the time and the parameters as follows:

$$\sigma_X(t) = \frac{1}{2}\left(\mu^2 \mid \varepsilon \mid^2 + \nu^2 \mid \dot{\varepsilon} \mid^2 + 2\mu\nu\sqrt{\mid \varepsilon\dot{\varepsilon} \mid^2 + 1}\right); \tag{33}$$

$$\overline{X} = \frac{\alpha}{\sqrt{2}}\left(\mu\varepsilon^* + \nu\dot{\varepsilon}^*\right) + \frac{\alpha^*}{\sqrt{2}}\left(\mu\varepsilon + \nu\dot{\varepsilon}\right). \tag{34}$$

It is easy to check that the symplectic tomogram for the nonclassical state (32) satisfies a classical-like evolution equation for the density matrix (25) introduced in symplectic tomography scheme.

5 Symplectic tomogram for nonlinear coherent states

Using the decomposition of wave function of nonlinear coherent state into series (18) one can obtain by standard method the Wigner function of the trapped ion in the form

$$\begin{aligned} W_\beta\left(q,\,p,\,t\right) = 2\left(\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{[n]!}\right)^{-1} \exp\left(-\left[q^2(t)+p^2(t)\right]\right) \\ \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,[f(m)]!}\,\frac{1}{[f(n)]!}(-\beta)^n\beta^{*m} \\ \times\left(\sqrt{2}\left[q(t)-ip(t)\right]\right)^{m-n} L_n^{m-n}\left(2\left[q^2(t)+p^2(t)\right]\right), \end{aligned} \qquad (35)$$

where

$$p\,(t) = \frac{\varepsilon+\varepsilon^*}{2}\,p-\frac{\dot\varepsilon+\dot\varepsilon^*}{2}\,q\,; \qquad (36)$$

$$q\,(t) = -\frac{\varepsilon-\varepsilon^*}{2i}\,p+\frac{\dot\varepsilon-\dot\varepsilon^*}{2i}\,q\,. \qquad (37)$$

Then calculating the Fourier integral (23) one obtains the symplectic tomogram of the trapped ion in nonlinear coherent state in the form

$$\begin{aligned} w_\beta\left(X,\,\mu,\,\nu,\,t\right) = \left(\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{[n]!}\right)^{-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^n\beta^{*m}}{\sqrt{n!}\,\sqrt{m!}\,[f(n)]!\,[f(m)]!} \\ \times w_{nm}\left(X,\,\mu,\,\nu,\,t\right). \end{aligned} \qquad (38)$$

Here $w_{nm}\left(X,\,\mu,\,\nu,\,t\right)$ is

$$\begin{aligned} w_{nm}\left(X,\,\mu,\,\nu,\,t\right) = \left(\pi\left[\mu^2(t)+\nu^2(t)\right]\right)^{-1/2}\,2^{-(n/2+m/2)}\,(n!\,m!)^{-1/2} \\ \times\frac{\left[\nu(t)+i\mu(t)\right]^n\,\left[\nu(t)-i\mu(t)\right]^m}{\left[\mu^2(t)+\nu^2(t)\right]^{n/2+m/2}}\exp\left(-\frac{X^2}{\mu^2(t)+\nu^2(t)}\right) \\ \times H_n\left(\frac{X}{\sqrt{\mu^2(t)+\nu^2(t)}}\right)H_m\left(\frac{X}{\sqrt{\mu^2(t)+\nu^2(t)}}\right), \end{aligned} \qquad (39)$$

where

$$\nu\,(t) \;=\; \frac{\dot{\varepsilon}-\dot{\varepsilon}^{*}}{2i}\,\nu+\frac{\varepsilon-\varepsilon^{*}}{2i}\,\mu\,; \tag{40}$$

$$\mu\,(t) \;=\; \frac{\dot{\varepsilon}+\dot{\varepsilon}^{*}}{2}\,\nu+\frac{\varepsilon+\varepsilon^{*}}{2}\,\mu\,. \tag{41}$$

One can check that the symplectic tomogram (38) is the solution to quantum evolution equation (25). For $f\,(n)=1$ and $\beta=\alpha$, the symplectic tomogram (38) coincides with the Gaussian distribution (32).

The method of photon number tomography was discussed in Ref. [24]-[26]. In this method, the measurable photon-number tomogram $w\,(n,\,\gamma)$ depends on the discrete number of photons n and on the complex amplitude γ of the local field oscillator, that may be scanned. The photon tomography scheme is an example of quantization scheme based on star-product of functions (photon-number tomograms) and explicit form of operators $\hat{\mathcal{D}}(\mathbf{x})$, $\hat{\mathcal{U}}(\mathbf{x})$ is

$$\hat{\mathcal{U}}(\mathbf{x})=\hat{D}(\gamma)|n\rangle\langle n|\hat{D}^{-1}(\gamma),\quad \mathbf{x}=(n,\gamma),$$
$$\hat{\mathcal{D}}(\mathbf{x})=\frac{4}{\pi(1-s^{2})}\left(\frac{s-1}{s+1}\right)^{(\hat{a}^{\dagger}+\gamma^{*})(\hat{a}+\gamma)-n},$$

where s is ordering parameter and

$$\hat{D}(\gamma)=\exp\left(\gamma\hat{a}^{\dagger}-\gamma^{*}\hat{a}\right).$$

The photon-number tomogram of the squeezed correlated states of the ion in Paul trap is of the form

$$\begin{aligned}\omega_{sq}(n,\gamma)=\frac{\mathrm{th}^{n}r}{n!\,2^{n}\mathrm{ch}r}\exp\Big[&\mathrm{th}r\sin\theta\Big(\langle p\rangle+\sqrt{2}\,\mathrm{Im}\,\gamma\Big)\Big(\langle q\rangle+\sqrt{2}\,\mathrm{Re}\,\gamma\Big)\\&-\frac{1}{2}\Big(\langle p\rangle+\sqrt{2}\,\mathrm{Im}\,\gamma\Big)^{2}(1-\cos\theta\mathrm{th}r)\\&-\frac{1}{2}\Big(\langle q\rangle+\sqrt{2}\,\mathrm{Re}\,\gamma\Big)^{2}(1+\cos\theta\mathrm{th}r)\Big]\\\times\Big|H_{n}\Big\{\frac{1}{2}e^{-i\theta/2}\sqrt{\mathrm{th}r}\Big[&\langle q\rangle-i\langle p\rangle+\sqrt{2}\,\gamma^{*}\\&+e^{i\theta}\mathrm{cth}r\Big(\langle q\rangle+i\langle p\rangle+\sqrt{2}\,\gamma\Big)\Big]\Big\}\Big|^{2}.\end{aligned}$$

where

$$\sin\theta = \frac{2\,\sigma_{pq}}{\sqrt{(\sigma_{p^2}+\sigma_{q^2})^2-1}}, \qquad \mathrm{ch}\,2r = \sigma_{p^2}+\sigma_{q^2}.$$

Using the general relation of symplectic tomogram to the distribution of the discrete photon number [36], one can obtain the photon-number tomogram $w_\beta\,(n,\,\gamma,\,t)$ for an ion in a Paul trap, which is modeled by a parametric oscillator, in terms of symplectic tomogram (23) for the nonlinear coherent states in the form

$$\begin{aligned} w_\beta\,(n,\,\gamma,\,t) &= \frac{1}{2\pi}\int w_\beta\,(X,\,\mu,\,\nu,\,t) \\ &\times \exp\left\{iX - \frac{\mu^2+\nu^2}{4} + \frac{\gamma(\nu+i\mu)}{\sqrt{2}} - \frac{\gamma^*(\nu-i\mu)}{\sqrt{2}}\right\} \\ &\times L_n\left(\frac{\mu^2+\nu^2}{2}\right)\,dx\,d\mu\,d\nu\,, \end{aligned} \tag{42}$$

where $w_\beta\,(X,\,\mu,\,\nu,\,t)$ is given by (38).

6 Symplectic and photon-number tomograms for odd and even coherent states

Now we discuss the tomograms for nonclassical states of the parametric oscillator, namely, even and odd coherent states [2]. The Wigner function for even and odd coherent states is

$$\begin{aligned} W_\pm &= 4\mid N^\pm\mid^2 \exp\left\{-p^2\mid\varepsilon\mid^2 - \mid\dot\varepsilon\mid^2 q^2 + (\dot\varepsilon\varepsilon^* + \varepsilon\dot\varepsilon^*)pq\right\} \\ &\otimes \left\{e^{-2|\alpha|^2}\cosh\left(2\sqrt{2}\,[p\,\mathrm{Im}\,(\alpha\varepsilon^*) - q\mathrm{Im}\,(\alpha\dot\varepsilon^*)]\right)\right. \\ &\pm \left.\cos\left(2\sqrt{2}\,[q\,\mathrm{Re}\,(\alpha\dot\varepsilon^*) - p\mathrm{Re}\,(\alpha\varepsilon^*)]\right)\right\}\,. \end{aligned} \tag{43}$$

The symplectic tomogram of a trapped ion in even/odd coherent states is

$$\begin{aligned} w_\pm &= \mid N^\pm\mid^2 \pi^{-1/2}\left[\mid\dot\varepsilon\mid^2\nu^2 + \mid\varepsilon\mid^2\mu^2 + 2\mu\nu\mathrm{Re}\,(\dot\varepsilon\varepsilon^*)\right]^{-1/2} \\ &\times \{w_1 + w_2 \pm w_3 \pm w_4\}\,, \end{aligned} \tag{44}$$

where

$$
\begin{aligned}
w_1 &= \exp\left[-\frac{\{X-\delta+2\sqrt{2}\mathrm{Re}\,(\alpha[\varepsilon^*\mu+\nu\dot{\varepsilon}^*])\}^2}{|\,\dot{\varepsilon}\,|^2\,\nu^2+|\,\varepsilon\,|^2\,\mu^2+2\mu\nu\mathrm{Re}\,(\dot{\varepsilon}\varepsilon^*)}\right]\,;\\
w_2 &= \exp\left[-\frac{\{X-\delta-2\sqrt{2}\mathrm{Re}\,(\alpha[\varepsilon^*\mu+\nu\dot{\varepsilon}^*])\}^2}{|\,\dot{\varepsilon}\,|^2\,\nu^2+|\,\varepsilon\,|^2\,\mu^2+2\mu\nu\mathrm{Re}\,(\dot{\varepsilon}\varepsilon^*)}\right]\,;\\
w_3 &= \exp\left[-2\,|\,\alpha\,|^2-\frac{\{X-\delta+2i\sqrt{2}\mathrm{Im}\,(\alpha[\varepsilon^*\mu+\nu\dot{\varepsilon}^*])\}^2}{|\,\dot{\varepsilon}\,|^2\,\nu^2+|\,\varepsilon\,|^2\,\mu^2+2\mu\nu\mathrm{Re}\,(\dot{\varepsilon}\varepsilon^*)}\right]\,;\\
w_4 &= \exp\left[-2\,|\,\alpha\,|^2-\frac{\{X-\delta-2i\sqrt{2}\mathrm{Im}\,(\alpha[\varepsilon^*\mu+\nu\dot{\varepsilon}^*])\}^2}{|\,\dot{\varepsilon}\,|^2\,\nu^2+|\,\varepsilon\,|^2\,\mu^2+2\mu\nu\mathrm{Re}\,(\dot{\varepsilon}\varepsilon^*)}\right]\,.
\end{aligned}
$$

The photon-number tomogram of even coherent states is

$$
\begin{aligned}
\omega_+ &= \frac{1}{2\,n\cosh\left(|\gamma|^2\right)}\left[\exp\left\{-|\gamma|^2-2\,\mathrm{Re}(\alpha^*\gamma)\right\}|\alpha+\gamma|^{2n}+\right.\\
&\quad+\exp\left\{-|\gamma|^2+2\,\mathrm{Re}(\alpha^*\gamma)\right\}|-\alpha+\gamma|^{2n}+\\
&\quad\left.+2\exp\left\{-|\gamma|^2\right\}\mathrm{Re}\,\left((\alpha+\gamma)^n(-\alpha^*+\gamma^*)^n\right)\right].
\end{aligned}
$$

The photon-number tomogram of odd coherent states is

$$
\begin{aligned}
\omega_- &= \frac{1}{2\,n!\sinh\left(|\gamma|^2\right)}\left[\exp\left\{-|\gamma|^2-2\,\mathrm{Re}(\alpha^*\gamma)\right\}|\alpha+\gamma|^{2n}+\right.\\
&\quad+\exp\left\{-|\gamma|^2+2\,\mathrm{Re}(\alpha^*\gamma)\right\}|-\alpha+\gamma|^{2n}-\\
&\quad\left.-2\exp\left\{-|\gamma|^2\right\}\mathrm{Re}\,\left((\alpha+\gamma)^n(-\alpha^*+\gamma^*)^n\right)\right].
\end{aligned}
$$

Photon-number tomograms of two-mode odd/even coherent states were considered in [37].

7 Conclusion

We review the results of the papers in which discussed the tomography of an ion in Paul trap. One can conclude that the state of an ion in a Paul trap can be described by three types of different tomograms(symplectic and its particular case optical tomogram, photon-number tomogram) which can be measured experimentally. Measurement of any of the three distributions gives the reconstruction of density operator of the quantum state of the trapped ion. The explicit expressions for the marginal distributions yield the theoretical predictions for measuring experimentally the nonlinear coherent states of an ion in a Paul trap with the help of the three different methods, namely, optical tomography, symplectic tomography, and photon number tomography. All the discussed approaches use as necessary ingredient the canonical Bogoliubov transform.

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