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BOGOLIUBOV TRANSFORMATIONS FOR FERMI–BOSE SYSTEMS AND SQUEEZED STATES GENERATION IN CAVITIES WITH OSCILLATING WALLS

Victor V. Dodonov
Instituto de Física, Universidade de Brasília
Caixa Postal 04455, 70910-900 Brasília, DF, Brazil
e-mail: vdodonov@fis.unb.br

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Abstract

I discuss applications of the Bogoliubov transformations in the theory of photon creation from vacuum in cavities with moving walls (the nonstationary Casimir effect), their relations to the phenomenon of squeezing and some generalizations of these transformations to the case of coupled Fermi–Bose systems with quadratic Hamiltonians.

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1 Introduction

A famous specific linear canonical transformation of bosonic creation and annihilation operators was introduced by N. N. Bogoliubov [1] in 1947 in connection with the theory of superfluidity [1]. Ten years later, using a specific linear canonical transformation of *fermionic* operators, he constructed a mathematical foundation of the theory of superconductivity [2, 3]. A similar transformation appeared in the paper by Valatin [4], published in the same issue of Nuovo Cimento (although submitted two months later than Bogoliubov's paper). Since that time, due to a great effectiveness of the method proposed in [1, 2, 3], all linear canonical transformations are frequently combined in the literature under the name *Bogoliubov transformations*.

Such transformations turned out to be a very useful tool in many different areas of quantum physics. One of the reasons of this success is that using linear canonical transformations one can diagonalize quadratic multidimensional Hamiltonians exactly. The bosonic case (where canonical transformations are equivalent to *symplectic* ones) was studied in detail, e.g., in Refs. [5, 6, 7, 8, 9, 10, 11, 12, 13]. A diagonalization of fermionic quadratic Hamiltonians was investigated, e.g., in [6, 14, 15]. Canonical transformations of operators in quantum mechanics are performed by means of some unitary operators applied to the initial operators or to the vectors in the related Hilbert spaces. These unitary operators are exponentials of quadratic forms constructed from the annihilation-creation or coordinate-momentum operators, and their kernels in the coordinate, momentum, coherent or Wigner representations are some Gaussian exponentials. The relations between linear canonical transformations and the related unitary operators were extensively studied for bosonic or fermionic operators by Berezin [16, 17] and later by many other authors [18, 19, 20, 21, 22, 23], where explicit expressions for the kernels or matrix elements of these operators in different bases were obtained.

It was also shown by Dodonov, Malkin and Man'ko [24] that using specific forms of linear canonical transformations with time-dependent coefficients, namely, quantum linear operators – integrals

[1]The name of N.N. Bogoliubov in different papers and numerous citations was written differently. I have checked the original publications and put the names in the list of references as they were printed there.

of motion, one can obtain in the most simple way the *propagators* of the time-dependent Schrödinger equation with the most general multidimensional nonstationary *quadratic* Hamiltonians constructed either from Bose or Fermi operators. Detailed reviews with the analysis of numerous examples and important special cases can be found in [25, 26]. One of the goals of this paper is to show (following [25]) how and under which conditions quantum linear integrals of motion (a special case of the Bogoliubov transformation) can be constructed for the most general coupled Fermi–Bose systems with quadratic Hamiltonians. This will be done in Sec. 2.

Another area of applications of the Bogoliubov transformations is the theory of particle creation in quantum systems with time-dependent parameters. Systematic studies in this area, connected mainly with the problem of particles creation in a nonstationary Universe, started at the end of 1960s – beginning of 1970s [27, 28, 29, 30, 31, 32, 33, 34, 35], although some earlier publications can be also found [36]. Among the recent papers we can cite, e.g., [37, 38, 39]. An important special case of the general theory of nonstationary quantum systems is the theory of creation of quanta in systems with moving boundaries [40, 41, 42, 43, 44, 45] or in dielectric media with time-dependent permeability [46, 47, 48, 49, 50]. These phenomena are frequently called as *nonstationary Casimir effect* (NCE, this name was introduced in [51, 52]), *nonadiabatic Casimir effect* [47] or *dynamical Casimir effect* (following [53]). The name *Mirror-Induced Radiation* (MIR) was introduced in [54]. Extensive lists of references can be found in the reviews [55, 56].

Since the second half of 1960s, the Bogoliubov transformations have been systematically used for constructing different kinds of "generalized coherent states". Namely, eigenstates of the operator $\hat{b} = u\hat{a} + v\hat{a}^\dagger$ (for real u and v) were introduced by Miller and Mishkin in 1966 [57] under the name "characteristic states". In order to preserve the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$ one should impose the constraint $|u|^2 - |v|^2 = 1$. Similar states were considered by Stoler [58] ("minimum uncertainty packets"), Lu [59] ("new coherent states") and by I. Bialynicki-Birula [60]. Moreover, Stoler [58] showed that these states can be obtained from the oscillator ground state by means of the unitary operator $\hat{S}(z) = \exp\left[\frac{1}{2}\left(z\hat{a}^2 - z^*\hat{a}^{\dagger 2}\right)\right]$, which was later named the "squeezed operator" [61]. However, it is worth remembering that, as a matter of fact, these states were introduced in the beginning of 1950s by Friedrichs [62], who considered the states $|\psi\rangle = \mathcal{N}\exp\left(-\frac{1}{2}\hat{\mathbf{A}}^{\dagger}G\hat{\mathbf{A}}^{\dagger}\right)|vac\rangle$, where $\hat{\mathbf{A}} = (\hat{a}_1, \hat{a}_2, \ldots)$ is the (infinite-dimensional) vector, whose components are the (Bose or Fermi) "annihilation" operators, and G is a symmetric matrix. ¿From the modern point of view, these states are nothing but *multimode squeezed vacuum states*. Also, in the middle of 1950s, Infeld and Plebański [63] performed a detailed study of the properties of the unitary operator $\exp(i\hat{T})$, where $\hat{T}$ is a generic inhomogeneous quadratic form of the canonical operators $\hat{x}$ and $\hat{p}$ with constant c-number coefficients, giving some classification and analyzing various special cases. Obviously, this operator generates one-mode squeezed coherent states. In 1960, Thouless [64] briefly discussed the importance of states, obtained by the action of the exponential of a general quadratic form of fermionic creation and annihilation operators, for the theory of superconductivity. Nowadays such states are called "squeezed fermion states" [65, 66, 67]. In an explicit form, the connections between "squeezing", Bogoliubov transformations and particle creation in nonstationary quantum systems were discussed, e.g., in Refs. [68, 69, 70, 71, 72, 73]. The Bogoliubov transformations play a significant role in the so called *Thermofield Dynamics*, which can be considered as some kind of multimode squeezed state representation [74, 75, 76]. For more details one can consult the review articles [77, 78, 79, 80]. For recent generalizations see, e.g., [81, 82, 83, 84].

A connection between squeezing and nonstationary Casimir effect was discussed in [42, 43, 51, 85, 86, 87, 88]. A significant progress in this area has been achieved during the past decade [55, 56]. Therefore my second goal (Sec. 3) is to show explicitly how the Bogoliubov transformation works in the theory of NCE, using as example a simple model of a massless scalar field inside a one-dimensional cavity with ideal *oscillating* boundaries and making the emphasis on the squeezing effect. In this connection, the concept of the invariant squeezing will be also discussed.

2 Quadratic Fermi–Bose Hamiltonians and linear integrals of motion

Suppose that the quantum dynamics is described by the equation

$$\partial\psi/\partial t = \hat{L}\psi \tag{1}$$

with a "quadratic" operator $\hat{L}$ of the form

$$\hat{L} = \frac{1}{2}\sum_{\alpha\beta} \hat{Q}^{\alpha} L_{\alpha\beta}(t)\hat{Q}^{\beta}. \tag{2}$$

We confine ourselves here with *homogeneous* quadratic forms. A more general case of inhomogeneous quadratic forms (where linear terms are included) was studied in [25, 26]. It is known since the paper by Lewis and Riesenfeld [89] that Eq. (1) can be easily solved if one knows some complete set of time-dependent operators – integrals of motion $\hat{I}(t)$, satisfying the equation

$$\partial\hat{I}/\partial t = \left[\hat{L}, \hat{I}\right]. \tag{3}$$

In turn, Malkin, Man'ko and Trifonov [90, 91, 92] emphasized the distinguished role played in the theory of quantum systems with quadratic Hamiltonians by *linear integrals of motion* of the form

$$\hat{I}^{\mu}(t) = \sum_{\alpha} \Lambda^{\mu}_{\alpha}(t)\hat{Q}^{\alpha}. \tag{4}$$

Obviously, the right-hand side of (4) is some special case of the Bogoliubov transformation.

Usually, the operators $\hat{Q}^{\alpha}$ in Eq. (2) are assumed to be the operators of canonical coordinates and momenta or bosonic lowering and raising operators. The detailed studies of such multidimensional quadratic systems were performed in reviews [25, 26]. However, in view of a great interest to different "deformations" of the canonical commutation relations (which were introduced on different grounds in [93, 94, 95, 96]), it is worth considering a set of $2N$ operators $\hat{Q}^{\alpha}$ satisfying the relations

$$\hat{Q}^{\alpha}\hat{Q}^{\beta} - \varepsilon_{\alpha\beta}\hat{Q}^{\beta}\hat{Q}^{\alpha} = \Sigma^{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \ldots, 2N, \tag{5}$$

where $\varepsilon_{\alpha\beta}$ can be arbitrary complex numbers, while coefficients $\Sigma^{\alpha\beta}$ are not operators but some other objects (e.g., complex or Grassmann's numbers). The self-consistency conditions of relations (5) are as follows (no summation over repeated indices),

$$\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} = 1, \qquad \Sigma^{\alpha\beta} = -\varepsilon_{\alpha\beta}\Sigma^{\beta\alpha}. \tag{6}$$

It seems to be interesting to know, under which conditions linear integrals of motion of the form (4) exist for systems described by means of equations (1), (2) and (5). One can verify that the commutator $[\hat{L}, \hat{Q}^{\mu}]$ contains no quadratic or cubic terms with respect to operators $\hat{Q}^{\beta}$, provided the following "commutation rules" are imposed on the products between the operators $\hat{Q}^{\mu}$ and the "numbers" $L_{\alpha\beta}$ and $\Sigma^{\mu\nu}$ (no summation):

$$L_{\alpha\beta}\hat{Q}^{\mu} = \hat{Q}^{\mu}L_{\alpha\beta}\varepsilon_{\mu\alpha}\varepsilon_{\mu\beta}, \qquad \Sigma^{\alpha\beta}\hat{Q}^{\mu} = \hat{Q}^{\mu}\Sigma^{\alpha\beta}\varepsilon_{\alpha\mu}\varepsilon_{\beta\mu} \tag{7}$$

(pay attention to different orders of indices in the products of factors $\varepsilon_{\mu\alpha}\varepsilon_{\mu\beta}$ in the right-hand sides of these relations). Analogously, the coefficients Λ^{μ}_{α} must obey the rules

$$\Lambda^{\beta}_{\alpha}\hat{Q}^{\mu} = \hat{Q}^{\mu}\Lambda^{\beta}_{\alpha}\varepsilon_{\mu\alpha}\varepsilon_{\beta\mu}, \tag{8}$$

and the following symmetry conditions can be imposed on coefficients of the quadratic form:

$$L_{\alpha\beta} = L_{\beta\alpha}\varepsilon_{\alpha\beta}\varepsilon_{\alpha\alpha}\varepsilon_{\beta\beta}. \tag{9}$$

If all these conditions are fulfilled, then linear integrals of motion (4) do exist, and the coefficients $\Lambda^{\mu}_{\beta}(t)$ satisfy the set of linear differential equations of the first order (here we *assume summation* over repeated indices)

$$\dot{\Lambda}^{\mu}_{\alpha} = -\Lambda^{\mu}_{\beta}\Sigma^{\beta\gamma}L_{\gamma\alpha}(t), \tag{10}$$

which can be represented also in a compact matrix form

$$\dot{\Lambda} = -\Lambda\Sigma L(t), \qquad \Lambda \left\|\Lambda^{\mu}_{\beta}\right\|, \quad \Sigma = \left\|\Sigma^{\beta\mu}\right\|, \quad L = \left\|L_{\beta\mu}\right\|. \tag{11}$$

An immediate consequence of Eq. (10) is a set of identities

$$\sum_{\beta\gamma}\Lambda^{\alpha}_{\beta}(t)\Sigma^{\beta\gamma}\Lambda^{\mu}_{\gamma}(t)\varepsilon_{\mu\gamma}\varepsilon_{\gamma\gamma} = \Sigma^{\alpha\mu}, \tag{12}$$

which are nothing else that conditions of preservation of commutation relations (5) for the transformed operators (4), if the additional condition $\Lambda^{\alpha}_{\beta}(0) = \delta_{\beta\alpha}$ is imposed. There are two equivalent compact matrix forms of identities (12):

$$\Lambda\Sigma M = \Sigma, \qquad M = \|M_{\gamma\mu}\|, \qquad M_{\gamma\mu} = \Lambda^{\mu}_{\gamma}(t)\varepsilon_{\mu\gamma}\varepsilon_{\gamma\gamma}, \tag{13}$$

$$\Lambda\Sigma' M' = \Sigma', \qquad M'_{\gamma\mu} = \Lambda^{\mu}_{\gamma}(t)\varepsilon_{\mu\gamma}\varepsilon_{\mu\mu}, \qquad \Sigma'_{\alpha\beta} = \Sigma^{\alpha\beta}\varepsilon_{\beta\beta} \tag{14}$$

(there is no summation in these formulas). Matrix M satisfies the equation

$$\dot{M} = L\Sigma M \tag{15}$$

and the consequences of Eqs. (13) and (14) are similar identities for the inverse matrix,

$$\Lambda^{-1} = \Sigma M \Sigma^{-1} = \Sigma' M' (\Sigma')^{-1}, \tag{16}$$

$$\Lambda^{-1}\Sigma M^{-1} = \Sigma, \qquad \Lambda^{-1}\Sigma'(M')^{-1} = \Sigma. \tag{17}$$

A special case of this general construction is a set of bosonic and fermionic operators, when the vector $\hat{\mathbf{Q}} = (\hat{Q}^1, \ldots, \hat{Q}^{2N})$ can be naturally splitted as $\hat{\mathbf{Q}} = (\hat{\mathbf{Q}}^{(+)}, \hat{\mathbf{Q}}^{(-)})$, where the superscript $^{(+)}$ stands for bosonic operators and $^{(-)}$ – for fermionic ones (note that the numbers of bosonic and fermionic operators may be different). Such sets of operators are used in the *supersymmetric quantum mechanics* [97, 98, 99, 100] (the concept of supersymmetry was introduced by Gol'fand and Likhtman [101]). Since all coefficients $\varepsilon_{\alpha\beta}$ can take only values $1, -1, 0$ in this case, there is no need in distinguishing between upper and lower indices. Thus we can put all indices as subscripts. All matrices can be split in the blocks as follows,

$$L = \left\| \begin{array}{cc} L^{(++)} & L^{(+-)} \\ L^{(-+)} & L^{(--)} \end{array} \right\|, \qquad \Sigma = \left\| \begin{array}{cc} \Sigma^{(++)} & \Sigma^{(+-)} \\ \Sigma^{(-+)} & \Sigma^{(--)} \end{array} \right\|,$$

and similarly for matrix Λ. The commutation relations read

$$\hat{Q}^{(+)}_{\alpha}\hat{Q}^{(+)}_{\beta} - \hat{Q}^{(+)}_{\beta}\hat{Q}^{(+)}_{\alpha} = \Sigma^{(+)}_{\alpha\beta} = -\Sigma^{(+)}_{\beta\alpha}, \tag{18}$$

$$\hat{Q}^{(-)}_{a}\hat{Q}^{(-)}_{b} + \hat{Q}^{(-)}_{b}\hat{Q}^{(-)}_{a} = \Sigma^{(-)}_{ab} = \Sigma^{(-)}_{ba}. \tag{19}$$

$$\hat{Q}^{(+)}_{\alpha}\hat{Q}^{(-)}_{n} - \varepsilon\hat{Q}^{(-)}_{n}\hat{Q}^{(+)}_{\alpha} = \Sigma^{(+-)}_{\alpha n} - \varepsilon\Sigma^{(-+)}_{n\alpha}, \tag{20}$$

where now the only coefficient $\varepsilon = \pm 1$ tell us whether bosonic operators commute or anticommute with fermionic ones. The rules of commutation between coefficients $L_{\mu\nu}^{(ab)}$, $\Sigma_{\mu\nu}^{(ab)}$ and $\Lambda_{\mu\nu}^{(ab)}$ (in other words, whether they are complex or Grassmann's numbers; $a, b = +, -$), as well as their symmetry properties, are determined by the relations (7), (8) and (9).

The equation (11) for the matrix $\Lambda(t)$ is untouched, whereas the identities (13) and (14) can be represented as

$$\Lambda\Sigma\left(I_\varepsilon \Lambda I_\varepsilon\right)^T = \Sigma, \qquad I_\varepsilon = \left\| \begin{array}{cc} E_{N_B} & 0 \\ 0 & \varepsilon E_{N_F} \end{array} \right\|, \tag{21}$$

where E_{N_B} and E_{N_F} are the unity matrices in the subspaces of N_B bosonic and N_F fermionic operators. The generalized transposed matrix Λ^T is defined according to the rules of linear algebra in graded spaces [102]

$$\Lambda^T = \left\| \begin{array}{cc} \tilde{\Lambda}^{(++)} & \tilde{\Lambda}^{(-+)} \\ -\tilde{\Lambda}^{(+-)} & \tilde{\Lambda}^{(--)} \end{array} \right\| \tag{22}$$

(the tilde over matrix means the usual matrix transposition).

Let us introduce the *covariance matrix*

$$Q = \left\| \begin{array}{cc} Q^{(++)} & Q^{(+-)} \\ Q^{(-+)} & Q^{(--)} \end{array} \right\|, \tag{23}$$

whose elements are defined as follows,

$$Q_{\alpha n}^{(+-)} = \frac{1}{2}\left\langle \hat{Q}_\alpha^{(+)}\hat{Q}_n^{(-)} + \varepsilon \hat{Q}_n^{(-)}\hat{Q}_\alpha^{(+)} \right\rangle = \varepsilon Q_{n\alpha}^{(-+)},$$
$$Q_{\alpha\beta}^{(++)} = \frac{1}{2}\left\langle \hat{Q}_\alpha^{(+)}\hat{Q}_\beta^{(+)} + \hat{Q}_\beta^{(+)}\hat{Q}_\alpha^{(+)} \right\rangle = Q_{\beta\alpha}^{(++)},$$
$$Q_{mn}^{(--)} = \frac{1}{2}\left\langle \hat{Q}_m^{(-)}\hat{Q}_n^{(-)} - \hat{Q}_n^{(-)}\hat{Q}_m^{(-)} \right\rangle = -Q_{nm}^{(--)}.$$

One can easily verify that the evolution of matrix $Q(t)$ is given by the formula

$$Q(t) = \Lambda^{-1}(t)Q(0)\left[I_\varepsilon \Lambda^{-1}(t) I_\varepsilon\right]^T. \tag{24}$$

But matrix $\Lambda^{-1}(t)$ satisfies the identities (17), which coincide in the case concerned with identity (21). As a consequence, the function

$$\mathcal{D}(\mu) = \text{sdet}\left[Q(t)\Sigma^{-1} - \mu E\right] \equiv \sum_m \mathcal{D}_{2m}\mu^{2m} \tag{25}$$

(where sdet means the "superdeterminant" [102]) *does not depend on time* for any coefficients $L_{\mu\nu}^{(ab)}(t)$ of the "Hamiltonian" $\hat{L}$. This means that the coefficients $\mathcal{D}_{2m}$ of the expansion in the right-hand side of (25), which are certain functions of the covariances $Q_{jk}^{(ab)}$ and elements of the commutator matrix $\Sigma_{jk}^{(ab)}$, are *universal quantum invariants*, i.e., the quantities which are conserved in time independently of a concrete form of the (quadratic) Hamiltonian [26]. Other universal invariants are

$$\mathcal{L}_{2m} = \mathrm{str}\left(\left[Q(t)\Sigma^{-1}\right]^{2m}\right). \tag{26}$$

The universal invariants for pure bosonic or fermionic systems (without interactions between them) were introduced in [103]. Numerous explicit examples can be found in [26, 104]. In the case of *Gaussian states* of bosonic systems, these invariants have close relations with the eigenvalues of matrix $Q(t)\Sigma^{-1}$, which are used under the name "symplectic eigenvalues" in alternative formulations of generalized uncertainty relations [105] and in studies devoted to the problem of entanglement of continuous variable quantum systems [106, 107]. Quantum invariants for time-dependent fermionic systems were considered also in [108].

3 Squeezing of the field quadratures in a 1D cavity with an oscillating boundary

I consider here, following [109], a special case of a 1D ideal cavity, assuming that the left boundary is fixed at the point $x = 0$ (this condition is not significant), whereas the right one performs small oscillations in the (quasi)resonance regime (at $t > 0$), according to the law

$$L(t) = L_0\left(1 + \varepsilon \sin\left[p\omega_1(1+\delta)t\right]\right),$$

with $p = 1, 2, \ldots$, $\omega_1 = \pi/L_0$ (we assume $c = \hbar = 1$) and $|\varepsilon|, |\delta| \ll 1$. The field operator in the Heisenberg picture $\hat{A}(x,t)$ must satisfy the wave equation

$$\hat{A}_{tt} - \hat{A}_{xx} = 0$$

and the boundary conditions

$$\hat{A}(0,t) = \hat{A}(L(t),t) = 0.$$

Remembering the standard decomposition of the field operator for $t < 0$ (when the wall was at rest),

$$\hat{A}(x,t) = \sum_{n=1}^{\infty} \left(2/\sqrt{n}\right) \left[\hat{b}_n \sin\left(n\pi x/L_0\right) \exp\left(-in\omega_1 t\right) \, + \, \text{h.c.}\right],$$

$$\left[\hat{b}_n \, , \hat{b}_m^{\dagger}\right] = \delta_{nm},$$

we write, for $t \geq 0$,

$$\hat{A}(x,t) = \sum_{n=1}^{\infty} \left(2/\sqrt{n}\right) \left[\hat{b}_n \psi^{(n)}(x,t) \, + \, \text{h.c.}\right],$$

and expand each function $\psi^{(n)}(x,t)$ in a series with respect to the *instantaneous basis*

$$\psi^{(n)}(x,t) = \sqrt{L_0/L(t)} \sum_{k=1}^{\infty} Q_k^{(n)}(t) \sin\left[\pi k x/L(t)\right], \qquad (27)$$

with the initial conditions

$$Q_k^{(n)}(0) = \delta_{kn}, \quad \dot{Q}_k^{(n)}(0) = -i\omega_n \delta_{kn}, \quad k,n = 1,2,\ldots$$

This way we satisfy automatically the boundary (and initial) conditions. Putting expression (27) into the wave equation, one can arrive at an infinite set of coupled differential equations [45, 88, 109] $(k,n = 1,2,\ldots)$

$$\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2\sum_{j=1}^{\infty} g_{kj}(t) \dot{Q}_j^{(n)} + \sum_{j=1}^{\infty} \dot{g}_{kj}(t) Q_j^{(n)} + \mathcal{O}\left(g_{kj}^2\right), \qquad (28)$$

where

$$\omega_k(t) = k\pi/L(t)$$

and the time dependent antisymmetric coefficients $g_{kj}(t)$ read $(j \neq k)$

$$g_{kj} = -g_{jk} = \frac{2kj(-1)^{k-j}\dot{L}}{(j^2 - k^2)\, L(t)}. \qquad (29)$$

Writing

$$Q_k^{(n)}(t)\rho_k^{(n)}(t)e^{-ik\omega_1(1+\delta)t} - \rho_{-k}^{(n)}(t)e^{ik\omega_1(1+\delta)t} \tag{30}$$

and using the method of slowly varying amplitudes [110], one can derive an infinite system of coupled ordinary differential equations for the coefficients $\rho_k^{(n)}(t)$ [109]:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_k^{(n)}(-1)^p\left[(k+p)\rho_{k+p}^{(n)} - (k-p)\rho_{k-p}^{(n)}\right] + 2i\gamma k\rho_k^{(n)}, \tag{31}$$

$$\tau = \varepsilon\omega_1 t/2, \qquad \gamma = \delta/\varepsilon.$$

The system (31) was solved in [109], where it was shown that nonzero coefficients $\rho_k^{(n)}$ form p independent subsets $(j = 0, 1, \ldots, p-1)$

$$\rho_{j+mp}^{(j+np)}(\tau) = -\frac{\Gamma(-m-j/p)\,\Gamma(1+n+j/p)\sin[\pi(m+j/p)]}{\pi\Gamma(1+n-m)(\sigma\kappa)^{m-n}} \times\lambda^{m+n+2j/p}F\left(n+j/p, -m-j/p; 1+n-m; \kappa^2\right), \tag{32}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function, $\sigma \equiv (-1)^p$,

$$\kappa = \frac{\sinh(ap\tau)}{\sqrt{a^2 + \sinh^2(ap\tau)}}, \quad a = \sqrt{1-\gamma^2}, \quad \lambda = \sqrt{1-\gamma^2\kappa^2} + i\gamma\kappa.$$

If the wall comes back to its initial position L_0 after some time T, the coefficients $\rho_{\pm k}^{(n)}$ become time independent at $t > T$, but the initial operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ cease to be 'physical', due to the contribution of the terms $\rho_{-k}^{(n)}\exp(ik\omega_1 t)$ with 'incorrect signs' in the exponentials. The 'physical' annihilation operator $\hat{a}_m$, which can be introduced according to the 'standard decomposition'

$$\hat{A} = \sum_{m=1}^{\infty}\left(2/\sqrt{m}\right)\sin\left(\pi m x/L_0\right)\left[\hat{a}_m e^{-im\omega_1(t+\delta T)} + \text{h.c.}\right]$$

at $t \geq T$, is related to the operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ by means of the Bogoliubov transformation

$$\hat{a}_m = \sum_{n=1}^{\infty}\sqrt{m/n}\left[\hat{b}_n\rho_m^{(n)}(\tau_T) - \hat{b}_n^\dagger\rho_{-m}^{(n)*}(\tau_T)\right]. \tag{33}$$

It is remarkable that conditions of unitarity of transformation (33),

$$\sum_{m=-\infty}^{\infty} m\rho_m^{(n)*}\rho_m^{(k)} = n\delta_{nk}\,, \quad n,k=1,2,\ldots \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{m}{n}\left[\rho_m^{(n)*}\rho_j^{(n)} - \rho_{-m}^{(n)*}\rho_{-j}^{(n)}\right] = \delta_{mj}\,, \quad m,j=1,2,\ldots \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\left[\rho_m^{(n)*}\rho_{-j}^{(n)} - \rho_j^{(n)*}\rho_{-m}^{(n)}\right] = 0\,, \quad m,j=1,2,\ldots, \tag{36}$$

are fulfilled *exactly*, as a consequence of the set of equations (31).

To study squeezing in the cavity modes we define the quadrature operators and their (co)variances as (hereafter $\omega_1 \equiv 1$)

$$\hat{q}_m = \left(\hat{a}_m + \hat{a}_m^\dagger\right)/\sqrt{2}, \qquad \hat{p}_m = \left(\hat{a}_m - \hat{a}_m^\dagger\right)/(i\sqrt{2}),$$

$$U_m = \langle \hat{q}_m^2\rangle - \langle \hat{q}_m\rangle^2, \qquad V_m = \langle \hat{p}_m^2\rangle - \langle \hat{p}_m\rangle^2,$$

$$Y_m = \frac{1}{2}\langle \hat{p}_m\hat{q}_m + \hat{q}_m\hat{p}_m\rangle - \langle \hat{p}_m\rangle\langle \hat{q}_m\rangle.$$

For the vacuum initial state, $\hat{b}_n|0\rangle = 0$, we have

$$\left.\begin{matrix} U_m \\ V_m \end{matrix}\right\} = \frac{m}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left|\rho_m^{(n)}\left(\tau_T\right) \mp \rho_{-m}^{(n)}\left(\tau_T\right)\right|^2, \tag{37}$$

$$Y_m = \sum_{n=1}^{\infty}\frac{m}{n}\mathrm{Im}\left[\rho_m^{(n)*}\left(\tau_T\right)\rho_{-m}^{(n)}\left(\tau_T\right)\right]. \tag{38}$$

Calculating the derivatives of the variances with respect to the ‘slow final time’ $\tau_T \equiv \tau$, we find [111]

$$\mathrm{d}U_m/\mathrm{d}\tau = \mathrm{d}V_m/\mathrm{d}\tau 2\sigma m \mathrm{Re}\left(\rho_m^{(j)}\rho_{-m}^{(p-j)}\right), \qquad \mathrm{d}Y_m/\mathrm{d}\tau = 0,$$

for all numbers m, excepting the ‘principal’ modes with the numbers $m = \mu \equiv p(k+1/2)$, $k=0,1,2,\ldots$ (provided p is even). There is no squeezing in the ‘nonprincipal’ modes, since $Y_m = 0$, and the quadrature variances $U_m = V_m = \mathcal{N}_m + 1/2$ (where $\mathcal{N}_m$ is the mean photon number) monotonously increase in time for $\gamma \le 1$, with asymptotical linear dependence

$$d\mathcal{N}_{j+pq}/d\tau \approx 2ap^2 \sin^2(\pi j/p)/\left[\pi^2(j+pq)\right]$$

at $ap\tau \gg 1$. If $\gamma > 1$, then $-1/\gamma \le \kappa \le 1/\gamma$, and the variances oscillate in time with amplitudes inversly proportional to $\gamma^2 - 1$, being always greater than (or equal to) 1/2.

Squeezing can be achieved only in the 'principal' μ-modes:

$$\left.\begin{array}{l} \mathrm{d}U_\mu/\mathrm{d}\tau \\ \mathrm{d}V_\mu/\mathrm{d}\tau \end{array}\right\} \mp \mu \mathrm{Re}\left(\left[\rho_\mu^{(p/2)} \mp \rho_{-\mu}^{(p/2)}\right]^2\right), \tag{39}$$

$$\mathrm{d}Y_\mu/\mathrm{d}\tau = \mu \mathrm{Im}\left(\left[\rho_\mu^{(p/2)*}\right]^2 + \left[\rho_{-\mu}^{(p/2)}\right]^2\right).$$

In the resonance case $\gamma = 0$, all the coefficients $\rho_m^{(n)}$ are real, so $Y_\mu = 0$ and $dU_\mu/d\tau \le 0$. In the special case $p = 2$ we have for $\tau \to \infty$:

$$\left.\begin{array}{l} U_{2m+1} \\ V_{2m+1} \\ Y_{2m+1} \end{array}\right|_{\tau\to\infty} \approx \frac{8a\tau}{\pi^2(2m+1)} \times \left\{\begin{array}{l} 2\sin^2\left[(m+1/2)\,\phi\right] \\ 2\cos^2\left[(m+1/2)\,\phi\right] \\ -\sin[(2m+1)\phi] \end{array}\right.,$$

$$\phi = \arcsin\gamma, \qquad \gamma < 1$$

If $\gamma \ne 0$, then the field goes to the *correlated squeezed* state [112] with $Y_m \ne 0$. For $t > T$ we have a free evolution of the variances,

$$\begin{aligned} U_m(t) &= U_m(T)\cos^2[\omega_m(t-T)] + V_m(T)\sin^2[\omega_m(t-T)] \\ &\quad + Y_m(T)\sin[2\omega_m(t-T)]. \end{aligned} \tag{40}$$

Since the variances $U_m(t)$, $V_m(t)$ and $Y_m(t)$ rapidly oscillate with the frequency $2\omega_m$, one needs some *invariant* characteristics of squeezing, which does not depend on these oscillations (or, equivalently, on the choice of the concrete quadrature component or the initial phase of the annihilation operator describing the selected field mode). For example, it can happen in a generic case, that both variances $U_m(T)$ and $V_m(T)$ are big, but nonetheless the quantum state is highly squeezed due to the existence of big nonzero covariance $Y_m(T)$. It seems that the best characteristics of squeezing are the *minimal* and *maximal* values, which the variance of any quadrature component can attain during the period of free oscillations, because they do not depend on the "fast time" t. Seeking for the extremal values of the function $U(t)$ (40), one can obtain the following expressions [113, 114, 115],

$$\left.\begin{array}{c} u_m \\ v_m \end{array}\right\} = \frac{1}{2}\left[U_m + V_m \mp \sqrt{\left(U_m - V_m\right)^2 + 4Y_m^2}\right]. \qquad (41)$$

An equivalent formula in terms of the mean values of the annihilation and creation operators reads (here the mode indices are suppressed)

$$\sigma_{min}^{max} = 1/2 + \langle \hat{a}^\dagger \hat{a} \rangle \pm |\langle \hat{a}^2 \rangle|. \qquad (42)$$

In the special case $p = 2$ and $\mu = 1$ the coefficients $\rho_{\pm 1}^{(1)}$ can be written in terms of the complete elliptic integrals [109],

$$\rho_1^{(1)} = 2\lambda(\kappa)\mathbf{E}(\kappa)/\pi, \qquad \rho_{-1}^{(1)} = 2\left[\tilde{\kappa}^2\mathbf{K}(\kappa) - \mathbf{E}(\kappa)\right]/(\pi\kappa),$$

where $\tilde{\kappa} \equiv \sqrt{1-\kappa^2}$, and equations (39) can be integrated exactly [111], resulting in the following minimal and maximal variances (41):

$$\left.\begin{array}{c} u_1 \\ v_1 \end{array}\right\} = \frac{2}{\pi^2\kappa}\left[2(\kappa \mp 1)\mathbf{K}(\kappa)\mathbf{E}(\kappa) \pm \tilde{\kappa}^2(1 \mp \kappa)\mathbf{K}^2(\kappa) \pm \mathbf{E}^2(\kappa)\right]. \qquad (43)$$

The minimal variance tends to the unique limit $u_1(\infty) = 2/\pi^2$ for any $\gamma \le 1$ (only the rate of the evolution depends on γ). Moreover, this asymptotical value does not depend on the initial (nonvacuum) state of the field, provided the initial density matrix was diagonal in the Fock basis (in particular, for the thermal of Fock states) [111].

The field appears in a mixed quantum state, and the quantum purity of each mode,

$$\mathrm{Tr}\hat{\rho}_m^2 = \left[4\left(U_m V_m - Y_m^2\right)\right]^{-1/2}$$

(where $\hat{\rho}_m$ is the statistical operator of the mth mode), goes to zero as $\kappa \to 1$, due to strong intermode interactions caused by the Doppler effect on the moving boundary. The total energy (normalized by $\hbar\omega_1$) $\mathcal{E}(\tau) \equiv \sum_m m\mathcal{N}_m(\tau)$ of the initially vacuum state equals [109]

$$\mathcal{E}^{(vac)}(\tau) = \left(p^2 - 1\right)\sinh^2(pa\tau)/\left(12a^2\right).$$

The influence of nonvacuum (especially thermal) initial states on the squeezing and other statistical properties of the field was studied in [44, 111] (for other references see [55]).

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