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BOGOLIUBOV TRANSFORMATIONS AND ENTANGLEMENT OF TWO FERMIONS

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Abstract

We show that Bogoliubov transformations, widely used in quantum field theory, can be also useful in quantum information theory. Namely, we show that the problem of the choice of tensor product decomposition in the system of two fermions can be analysed with the help of Bogoliubov transformations of creation and annihilation operators.

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1 Introduction

Bogoliubov transformations are the standard tool of the quantum field theory. It is interesting that they can be applied also in quantum information theory (QIT). One of the most important notions of QIT is entanglement—a uniquely quantum phenomenon connected with the question whether the state of a composite quantum system may be determined by the states of its constituent subsystems. Therefore to unambiguously define entanglement requires a partition of the overall system into subsystems. However, the division of a given physical system into subsystems is in general by no means unique. From the theoretical point of view this is closely related to the possible choices of the tensor product decomposition (TPD) of the Hilbert space of the system [13, 14]. As a consequence, the following question arises: how entangled is a given state with respect to a particular TPD?

On the other hand the theory of entanglement can be seen as the general theory of state transformations that can be performed on multipartite systems, with the restriction that only local operations and classical communications (LOCC) can be implemented [3]. For the same reason, it was expected that additional restrictions should lead to new interesting physical effects and applications. It has been shown that such a restriction can be given by a superselection rule (SSR) (see e.g. [2, 8]).

In the present contribution we recall results from our previous papers [4, 5] and indicate further applications. We discuss the problem of the choices of TPD in a system of two fermions, neglecting their spatial degrees of freedom and modifying the tensor product in the rings of operators because of anticommuting canonical variables. We show that TPDs are connected with each other by Bogoliubov transformations of creation and annihilation operators. We also study the behavior of the entanglement of the system under these transformations.

We discuss also the SSR—we restrict the set of physical states of the composite system by the requirement that we prohibit superpositions of fermions and bosons. This leads us to the SSR that is a weaker restriction (i.e., it admits a larger set of states) than the SSR based on the conservation of the number of particles considered in

[2, 8]. Moreover, we find the entanglement of formation taking into account the restriction imposed by our SSR.

2 Two-fermion system and the superselection rule

Let us analyze the Thirring model [7] in $1+0$ dimensional space-time which in such a case describes a fermionic quantum mechanical system. The corresponding Lagrangian is of the form:

$$L = i(\bar{\psi}\partial_t\psi) - m\bar{\psi}\psi - \lambda(\bar{\psi}\psi)^2, \tag{1}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix}. \tag{2}$$

Fields ψ_i together with their canonical momenta $\pi_j = i\psi_j^\dagger$ satisfy the equal-time canonical anticommutation relations $\{\psi_i(t), \pi_j(t)\} = i\delta_{ij}$. The solutions of the equations of motion derived from the Lagrangian (1) are

$$\psi_i(t) = a_i e^{-it(m+\lambda+2\lambda N_j)}, \qquad i \neq j, \tag{3}$$

where the time-independent operators a_i and $a_i^\dagger$ satisfy

$$\{a_i, a_j\} = 0, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \tag{4}$$

and

$$N_i = a_i^\dagger a_i, \qquad (i, j = 1, 2). \tag{5}$$

The Hamiltonian of this system describes two fermionic oscillators with the quartic interaction term

$$H = (m+\lambda)(N_1 + N_2) + 2\lambda N_1 N_2. \tag{6}$$

Creation operators generate all the basis vectors from the "vacuum state" $|0,0\rangle$ via the relations

$$|1,0\rangle = a_1^\dagger |0,0\rangle, \tag{7a}$$

$$|0,1\rangle = a_2^\dagger |0,0\rangle, \tag{7b}$$

$$|1,1\rangle = a_2^\dagger a_1^\dagger |0,0\rangle, \tag{7c}$$

while the vacuum is annihilated by annihilation operators:

$$a_i|0,0\rangle = 0, \qquad (i = 1, 2). \tag{7d}$$

Furthermore, the antiunitary time inversion operator acts in the standard way (see, e.g. [9]):

$$\mathsf{T}a_1\mathsf{T}^{-1} = a_2, \quad \mathsf{T}a_2\mathsf{T}^{-1} = -a_1, \tag{8}$$

$$\mathsf{T}|0,0\rangle = |0,0\rangle. \tag{9}$$

The observables for the system are restricted to combinations of even products of creation and annihilation operators because they should commute with the square of time reflection operator T^2 (see, e.g. [9]). Thus, this operator imposes a superselection rule, which means that superpositions of bosons and fermions are forbidden [1, 11].

3 Decompositions into subsystems

Let us consider the problem of decomposition of our system into two subsystems. Such a decomposition corresponds to the different choices of canonical variables a_1, a_2. This is extremely important because each choice of a_1, a_2 defines in the Hilbert space $\mathbb{C}^4$ the corresponding tensor product structure [4, 5]. Each tensor product decomposition defines a corresponding set of local observables.

On the other hand, different choices of canonical variables a_1, a_2 are connected by Bogoliubov transformations which preserve the canonical anticommutation relations. Therefore Bogoliubov transformations give us all possible decompositions of the two-fermion system into two subsystems (two fermions). Such decompositions of the system correspond to the tensor product decompositions of the space $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ and the corresponding endomorphism spaces, appropriate to the definition of the subsystems. The modified tensor product is defined by the graded (supersymmetric) multiplication rule [4, 5]

$$(A \otimes \mathfrak{b})(\mathfrak{a} \otimes B) = (-1)^{F(\mathfrak{a})F(\mathfrak{b})} A\mathfrak{a} \otimes \mathfrak{b}B, \tag{10}$$

where $\mathfrak{a}$, $\mathfrak{b}$ are monomials in a, $a^\dagger$, i.e. $\mathfrak{a}, \mathfrak{b} \in \{I, a, a^\dagger, aa^\dagger, a^\dagger a\}$, A, B are arbitrary operators acting in $\mathbb{C}^2$ and the "fermion number"

$F(\mathfrak{a})$ is equal to the number of creation operators minus the number of annihilation operators building the monomial $\mathfrak{a}$, i.e. $F(I) = 0$, $F(a^\dagger) = -F(a) = 1$, $F(aa^\dagger) = F(a^\dagger a) = 0$; notice that $(-1)^F = \mathsf{T}^2$.

Therefore, the problem of finding all possible tensor product decompositions consistent with our superselection rule is equivalent to determining all possible Bogoliubov transformations satisfying this rule.

It is easy to show [4, 5] that Bogoliubov transformations satisfying our requirement are composed from the following transformations:

- SU(2) transformations which do not mix creation and annihilation operators

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{11}$$

- SU(2) transformations which mix creation and annihilation operators

$$\begin{pmatrix} a_1' \\ a_2^{\dagger\prime} \end{pmatrix} = \begin{pmatrix} \zeta & \omega \\ -\omega^* & \zeta^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}, \qquad |\zeta|^2 + |\omega|^2 = 1, \tag{12}$$

- nonlinear one-parameter transformations

$$a_1' = a_1 e^{i\chi N_2} = a_1[1 + (e^{i\chi} - 1)N_2], \tag{13a}$$

$$a_2' = a_2 e^{i\chi N_1} = a_2[1 + (e^{i\chi} - 1)N_1]. \tag{13b}$$

In the Hilbert space $\mathbb{C}^4$ Bogoliubov transformations (11)–(13) are represented in the occupation number basis (7) by

$$U_{\mathrm{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^* & -\beta & 0 \\ 0 & \beta^* & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_{\mathrm{II}} = \begin{pmatrix} \zeta & 0 & 0 & -\omega^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & \zeta^* \end{pmatrix},$$

$$U_{\mathrm{III}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\chi} \end{pmatrix}, \tag{14}$$

respectively.

The Bogoliubov transformations which lead to physically distinguishable tensor product decompositions should change the local observables $N_i = a_i^\dagger a_i$. Therefore such transformations have the form: (11) with both $\alpha \neq 0$, $\beta \neq 0$ and/or (12) with both $\zeta \neq 0$, $\omega \neq 0$.

Moreover if for local observers their tensor product decomposition are connected by Bogoliubov transformations then density matrices representing the state are connected by similarity transformations, i.e. $\rho' = U\rho U^\dagger$ where $U = U_{\mathrm{I}}U_{\mathrm{II}}U_{\mathrm{III}}$. In general, such transformations change the entanglement measure $E(\rho)$, i.e. entanglement depends on the choice of tensor product decomposition and hence the local observers. In particular, for any state, there exists a pair of observers for whom this state is separable, since the density matrix (15) can always be diagonalized by means of the transformations (3).

4 Two-fermion states

The superselection rule requires that the density matrix has to commute with T^2, thus the general state of this system is represented by the following density matrix [5]

$$\rho = V \left(\begin{array}{c|c} R_1 I + \vec{x}_1 \cdot \vec{\sigma} & 0 \\ \hline 0 & R_2 I + \vec{x}_2 \cdot \vec{\sigma} \end{array} \right) V^\dagger, \tag{15}$$

where $\vec{x}_i = r_i(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$ and $2(R_1 + R_2) = 1$ and $r_i \leq R_i$, $i = 1, 2$, with

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{16}$$

For a particular case of the Werner state [10, 6]

$$\rho_W = \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & \frac{\gamma}{2} \\ 0 & \frac{1-\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\gamma}{4} & 0 \\ \frac{\gamma}{2} & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix}, \quad \gamma \in [-\tfrac{1}{3}, 1]. \tag{17}$$

The manifold of all states (15) is the set-theoretic sum of cartesian products of two balls of radii R_1 and $R_2 = \frac{1}{2} - R_1$, respectively, i.e.

$$\bigcup_{R_1 \in [0, \frac{1}{2}]} B_{R_1} \times B_{\frac{1}{2} - R_1}. \tag{18}$$

This manifold is shown in the Figure 1.

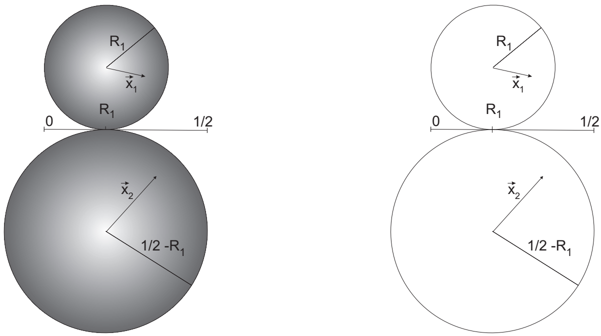

Figure 1: The manifold of states of two fermion system. Each state is represented by the pair of points on two balls. The upper ball has radius R_1 and the lower one R_2, respectively, and both radii vary from 0 to $\frac{1}{2}$ and are connected by relation $R_1 + R_2 = \frac{1}{2}$. If R_1 or R_2 equals 0 then appropriate ball shrinks to a point.

Possible states of subsystems obtained from (15) by partial traces are

$$\rho_1 = \frac{1}{2} I + (r_1 \cos\theta_1 - r_2 \cos\theta_2)\sigma_3, \tag{19a}$$

$$\rho_2 = \frac{1}{2} I + (r_1 \cos\theta_1 + r_2 \cos\theta_2)\sigma_3. \tag{19b}$$

According to definition [10] of separable states $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$, where ρ_A^i and ρ_B^i are admissible states of subsystems and $\sum_i p_i = 1$,

$0 \leq p_i \leq 1$, they have the surprisingly simple diagonal form in our case

$$\rho_{\mathrm{sep}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \tag{20}$$

with $\sum_i \lambda_i = 1$, $\lambda_i \geq 0$. Therefore, nondiagonal density matrices are nonseparable. Thus the standard method of calculating entanglement measures should be taken with care.

The entanglement of formation is defined as [3]:

$$E(\rho) = \min \sum_i p_i S(\rho_A^i), \tag{21}$$

where $S(\rho_A) = -\operatorname{tr}(\rho_A \log_2 \rho_A)$ is the von Neumann entropy and the minimum is taken over all the possible realizations of the state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\rho_A^i = \operatorname{tr}_B(|\psi_i\rangle\langle\psi_i|)$. Taking into account the special form of the density matrix (15) we can find the explicit formula for the entanglement of formation for this state—see [4, 5].

The Wootters concurrence [12] of the Werner state (17) is equal to zero for $\gamma \in [-\frac{1}{3}, \frac{1}{3}]$, therefore for two qubits the Werner state is separable for such values of γ. On the other hand, if this matrix describes two-fermion system, this state *is separable only when* $\gamma = 0$. Thus the Wootters concurrence does not define entanglement measure in our case.

Instead, the calculated entanglement of formation for Werner state is [4, 5]

$$E(\rho_W) = \begin{cases} \frac{1+\gamma}{2} & \gamma \neq 0, \\ 0 & \gamma = 0. \end{cases} \tag{22}$$

Thus, $E(\rho_W) > 0$ for entangled (nondiagonal) states and $E(\rho_W) = 0$ for a separable (diagonal) state. For $\gamma = 1$ we have the maximally entangled Werner state.

We point out that there exists a class of *superseparable* states $\rho_{\mathrm{ss}} = \frac{1}{2}\mathrm{diag}\,(\lambda, 1-\lambda, 1-\lambda, \lambda)$, $\lambda \in [0,1]$, which are separable for every observers. Note that for two qubits only one superseparable state exists, namely the maximally mixed state $\rho_0 = \frac{1}{4}I$.

5 Conclusions

We have discussed the two-fermion system in the presence of the superselection rule forbidding superpositions of bosonic and fermionic states. We have analysed the possible tensor product decompositions of the Hilbert space of this system with the help of Bogoliubov transformations. As a result we get that these states *are not* qubit states. The set of separable states is narrower than in two-qubit case, it consists only of the states represented by diagonal density matrices. We have found that for any state there exist tensor product decomposition of the Hilbert space with respect to which the state is separable. Moreover, we have shown that there exist a class of superseparable states, i.e. the states which are separable with respect to all tensor product decompositions of Hilbert space. Concluding we think that Bogoliubov transformations may be helpful in study of geometry of quantum states as well as entanglement measures.

Acknowledgements

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