The old and new Concepts of Physics
An open-dialogue journal

www.uni.lodz.pl/concepts

TO WHAT EXTENT THE SUPERFLUID ^{4}He IS SQUEEZED?

Nikolay N. Bogolubov, Jr.
V.A. Steklov Mathematical Institute of Russian Academy of Sciences
Moscow, Russia
Alexander S. Shumovsky
Faculty of Science, Bilkent University
Bilkent, Ankara, 06800 Turkey
e-mail: shumo@fen.bilkent.edu.tr

(Received 20 November 2007; accepted 1 March 2007)

Abstract

We show that the Bogoliubov microscopic theory of superfluidity of liquid ^4He allows quantum fluctuations of both condensate and excitations. Comparison of those fluctuations leads to an equation determining the mean number of atoms with zero momentum in a self-consistent way. Obtained results are in good agreement with the experiments on Bose-Einstein condensation in atomic beams. To explain experimental results on low amount of atoms in condensate in the superfluid phase of liquid ^4He, we propose to consider the ground state of the condensate as a squeezed number state and discuss some corollaries coming from this conjecture.

DOI: 10.2478/v10005-007-0016-9

In 1946, in a report to the General Meeting of Section of Physics and Mathematics of the Academy of Sciences of USSR, Professor Nikolay Nikolaevich Bogoliubov formulated his famous microscopic theory of superfluidity in liquid ^4He [1]. This important quantum phenomenon was discovered by Pyotr Leonidovich Kapitsa, John F. Allen, and Don Misener in 1937 [2, 3]. A macroscopic phenomenological theory of superfluidity in ^4He was created by Lev Davidovich Landau [4].

At the heart of the Bogoliubov microscopic theory of superfluidity, there are the following ideas. In analogy to the phenomenological theory [4, 5], he assumed that at low temperatures there are two "liquids" in the ^4He. Namely, the most of ^4He atoms have zero momentum and hence form the Bose-Einstein condensate (BEC). The minor part of atoms forms excitations with $k \neq 0$. He also assumed that interaction between condensate atoms and excitations is weak.

Beginning with the Hamiltonian of pair interactions

$$H = \sum_p \frac{p^2}{2m} a_p^+ a_p + \frac{1}{2} \sum_{kpq} V(p) a_{k+p}^+ a_{q-p}^+ a_p a_q, \tag{1}$$

where a_p^+, a_p are creation and annihilation operators, respectively, for bosons with momentum p and $V(p)$ is the Fourier transform of the potential energy $V(\mathbf{r}-\mathbf{r}')$ of a pair of bosons (^4He atoms) located at $\mathbf{r}$ and $\mathbf{r}'$, Bogoliubov introduced an effective Hamiltonian of interaction between BEC bosons and excitations

$$H_{eff} = \sum_{p\neq 0} \frac{k^2}{2m} a_k^+ a_k + \frac{V(0)}{2} a_0^+ a_0^+ a_0 a_0 + V(0) \sum_{p\neq 0} \big(a_p^+ a_p a_0^+ a_0 + a_p^+ a_{-p}^+ a_0 a_0 + a_0^+ a_0^+ a_{-p} a_p \big) . \tag{2}$$

For the ground state with N_0 particles in the BEC, the zero-momentum operators act as follows

$$a_0 |N_0\rangle = \sqrt{N_0}\ |N_0 - 1\rangle, \quad a_0^+ |N_0\rangle = \sqrt{N_0 + 1}\ |N_0 + 1\rangle,$$

According to Bogoliubov [1], the difference between the factors $\sqrt{N_0+1}$ and $\sqrt{N_0}$ is negligible since $N_0 \gg 1$. Thus, it is possible to replace all operators a_0 and a_0^+ by the c-number $\sqrt{N_0}$. After that, the effective Hamiltonian (2) becomes a quadratic form with respect to operators a_p and a_p^+ with $p \neq 0$ that describe the excitation above BEC state. This form can be diagonalized by means of the famous Bogoliubov canonical transformations of the form

$$b_p = u_p a_p + v_p a_{-p}^+, \quad b_p^+ = u_p a_p^+ + v_p a_{-p}. \tag{3}$$

This procedure of diagonalization leads to definition of spectrum of excitations in superfluid ^4He [1, 6].

In connection with the effective Hamiltonian (2) and its reduction to the quadratic form we now note that:

• The reduced quadratic form (Hamiltonian) contains terms, describing simultaneous creation and annihilation of excitations with opposite momenta and hence the corresponding state is a squeezed one [7]. In fact, this is a superposition of two-mode squeezed states generated from the vacuum by means of the squeezing operator of the form [8]

$$\exp\left(\xi_p^* a_p a_{-p} - \xi_p a_p^+ a_{-p}^+\right),$$

where ξ_p is a known function of coefficients in (3).

• The reduced effective Hamiltonian does not conserve the number of particles and therefore should by considered in the grand canonical ensemble.

The reduction of the effective Hamiltonian by means of the substitution of $\sqrt{N_0}$ for both a_P and a_P^+ can also be treated as an averaging of the effective Hamiltonian (2) over a certain state [9, 10].

Taking into account almost classical behavior of the system with $p = 0$ provided by the approximate commutation relation

$$[a_0, a_0^+] \approx 0,$$

it seems to be natural to perform averaging over a coherent state that is known to be maximally close to classical one [11].

Let us now note that both coherent and squeezed states of Bose fields manifest certain quantum fluctuations of the number of particles. This means that the mean number of atoms $\bar{N}_0$ in coherent state associated with $p = 0$ can be measured with a certain precision

$$\bar{N}_0 \pm \frac{1}{2}\sqrt{V(N_0)}, \tag{4}$$

where

$$V(X) \equiv \langle X^2 \rangle - \langle X \rangle^2$$

denotes the variance (uncertainty). Taking into account the properties of coherent state, we get from (4) that $\bar{N}_0$ can be measured to within $\pm\sqrt{\bar{N}_0}/2$.

Taking into account the fact that in modern experiments on BEC the number of atoms can be varied from a huge one to a very few, we cannot *a priori* neglect the above fluctuation of the number of atoms in condensate. Moreover, assuming that the total number of atoms N_{tot} is conserved, we have to assume that the number fluctuation in the state with $p = 0$ means transition of some atoms into the excited state and vice versa. This assumption is justified by the above listed facts that there is no conservation of the number of atoms in excited state and that the excited atoms are in squeezed state. The point is that squeezed states manifest quite strong number fluctuations (e.g., see [12]), at least much stronger than coherent states. The mean number of atoms in excited state with $p \neq 0$ is therefore measured with the precision of

$$\pm\frac{1}{2}\sqrt{V(N_{exc})}, \quad V(N_{exc}) = \sum_{p\neq 0} |u_p|^2 |v_p|^2.$$

In view of conservation of the total number of atoms in the system, we should conclude that the exchange by atoms between the condensate and excitations caused by quantum fluctuations is described by the equation

$$V(N_0) = V(N_{exc}), \tag{5}$$

so that

$$\bar{N}_0 = \sum_{p\neq 0} |u_p|^2 |v_p|^2. \tag{6}$$

The right-hand side here also depends on $\bar{N}_0$ because of known relations [6]

$$|u_p|^2 - |v_p|^2 = 1, \quad \frac{|u_p|}{|v_p|} = -\eta_p + \sqrt{\eta_p^2 - 1},$$
$$\eta_p = \frac{1}{V(0)\bar{N}_0}\left(\frac{p^2}{2m} + 2V(0)\bar{N}_0\right).$$

Thus, this expression (6) can be considered as an equation for definition of the mean number of atoms in condensate $\bar{N}_0$ in a self-consistent way [13].

The numerical analysis of this equation shows that the share of excited atoms rapidly decreases with the increase of the total number of atoms N_{tot} and vanishes at $N_{tot} \to \infty$. This result is in a good coincidence with recent experiments on BEC in atomic beams.

At the same time, in the superfluid ^4He the excitations above condensate can be observed in macroscopic systems at reasonably low temperatures. Moreover, experiments on neutron scattering show quite low amount of atoms with $p = 0$ in the superfluid phase – below a few percents (e.g., see [14, 15, 16] and references therein). The qualitative difference with the case of atomic beams can be caused by the significant difference in densities of liquid system and atomic beams.

This experimental fact contradicts with the idea to associate the BEC state in superfluid ^4He with the coherent state. As a matter of fact, coherent state cannot be represented as an eigenstate of Hermitian operators and hence it does not correspond to any thermodynamic phase.

We have no direct indication for the choice of a state to be used for reduction (averaging) of the effective Hamiltonian (2), so that our further discussion should be considered as a conjecture.

The low number of atoms with $p = 0$ in superfluid ^4He ($\bar{N}_0 \leq 0.02 N_{tot}$ [14]) allows to assume that the real condensate in ^4He may fluctuate stronger than that in coherent state. In particular, we can assume that the state of condensate is a *squeezed number state* rather than coherent one [17]. Thus, both condensate and excitations are in squeezed states.

This assumption can be justified by the form of the effective Hamiltonian (2) that involves terms of the form $a_0 a_0$ and $a_0^+ a_0^+$, describing simultaneous creation and annihilation of pairs of atoms in condensate. Physically the mechanism, creating pair states in condensate, can be associated with exchange between the atoms with $p = 0$ by a pair excitations with opposite $p \neq 0$.

The above conjecture leads to the following corollaries.

Corollary 1. Variance of atoms in the condensate has the form

$$V(N_0) = 2\bar{N}_0(\bar{N}_0 + 1), \tag{7}$$

which is much greater than that calculated with coherent state $V(N_0)_{coh} = \bar{N}_0$. This means that the mean number of atoms in the condensed state can be measured with the following quantum precision

$$\bar{N}_0 \pm \frac{1}{\sqrt{2}}\bar{N}_0.$$

Corollary 2. The ground state of condensate consists of the Fock number states with *even* number of atoms. In other words, it is formed by virtual clusters similar to Cooper pairs in superconductors and consisting of $2, 4, \cdots, 2n, \cdots$ atoms, where n is a positive integer. In particular, the probability to have exactly N_0 atoms in condensate is

$$P(N_0) = \frac{1}{N_0! \cosh^2(\mathrm{arsinh}\sqrt{\bar{N}_0})} \left(\frac{1}{2}\tanh(\mathrm{arsinh}\sqrt{\bar{N}_0})\right)^{N_0} \mathcal{H}_{N_0}(0),$$

where $\mathcal{H}_n$ is the Hermite polynomial of degree n. The quantity $\bar{N}_0$ can be determined here by means of the self-consistent equation (5).

Corollary 3. Experiments on neutron scattering by condensate in superfluid ^4He should be focused on estimation of contribution given by different pair clusters rather than individual atoms. This means that the wave length of incident neutrons should be increased in a proper way. An indirect justification of the above picture comes from the early experiments on neutron scattering with high enough wavelength that have shown about 10% of atoms in the condensate [18].

A detailed analysis of the proposed conjecture requires further investigations both theoretical and experimental.

References

[1] N.N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947).

[2] P. Kapitsa, Nature **141**, 74 (1938).

[3] J.F. Allen and A.D. Misener, Nature **141**, 75 (1938).

[4] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941); Phys. Rev. **60**, 356 (1941); J. Phys. **11**, 91 (1947).

[5] L. Tisza, Comples Rendus **207**, 1035, 1186 (1938); J. Phys. Radium **1**, 164 (1940); Phys. Rev. **75**, 885 (1949).

[6] N.N. Bogoliubov, *Selected works*, Part II, edited by N.N. Bogoliubov, A.M. Kurbatov, and A.S. Shumovsky (Gordon and Breach, New York, 1991).

[7] D. Stoler, Phys. Rev. D **1**, 3217 (1970); ibidem **4**, 1925 (1971).

[8] C.M. Caves and B.L. Schumaker, Phys. Rev. A **31**, 3068 (1985).

[9] R. Haag, Nuovo Cim. **25**, 694 (1962).

[10] N.M. Hugenholtz, Progr. Phys. **28**, 201 (1965).

[11] I.A. Malkin and V.I. Man'ko, *Dynamic symmetries and cohrent states of quantum systems* (Nauka, Moscow, 1979); A.M. Perelomov, *Generalized coherent states and their applications* (Springer-Verlag, Berlin, 1986).

[12] V.V. Dodonov, J. Opt. B **4**, R1 (2002).

[13] A.S. Shumovsky, in *9-th Gen. Conf. Condens. Matter Div. EPS*, editors J. Lahuerte, M. Le Bellac, and F. Raymond, V. 13A, p. 223 (European Phys. Soc., 1989).

[14] I.V. Bogoyavlenskii, L.V. Karnatsevich, J.A. Kozlov, and A.V. Puchkov, Physica B **76**, 151 (1992).

[15] W.G. Stirling and K.H. Andersen, J. Phys.: Cond. Matter **6**, A63 (1994).

[16] D.M. Ceperley, R.O. Simmons, and R.C. Blasdell, Phys. Rev. Lett. **77**, 115 (1996).

[17] A.S. Shumovsky, JINR Commun. E17-92-355 (JINR, Dubna, 1992).

[18] R.A. Cowly and A.D.B. Woods, Canadian J. Phys. **49**, 177 (1971).