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BOGOLIUBOV TRANSFORMATIONS AND QUANTUM DECOHERENCE IN AN ATOMIC BOSE-EINSTEIN CONDENSATE

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Abstract

In this brief review, we discuss quantum decoherence of an atomic Bose-Einstein condensate using Bogoliubov transformation. In particular, condensate phase is introduced and its diffusion is studied. Validity of the Bogoliubov prescription is investigated and restored via the condensate phase. Role of quantum state of the condensate on the phase collapse is examined. Relation between the fluctuations in the condensate particle number and the dynamics of the condensate phase is clarified.

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1 Introduction

After the initial prediction of possibility of macroscopic occupation of the single-particle ground state by many photons [1] or by massive particles [2], it took more than seventy years to develop sufficient experimental ability to realize such a state of matter, so called Bose-Einstein condensate (BEC) in laboratories. The advances in atomic trapping and laser cooling techniques allowed for BEC of dilute alkali atomic gases, in particular for rubidium [3], sodium [4], and lithium [5].

These condensates open path to observe quantum effects on a macroscopic scale. Lifetime of condensates is not too long, which hinders the potential applications of such quantum effects. In the all optical trapped condensates lifetime in the order of 10 s. is achieved [6]. One major source defining the lifetime of a condensate is the phase decoherence.

Following the observation of interference between two Bose condensates [7], recent experimental progress now permits continuous measurement of relative phases between condensates with light scattering techniques [8]. It is believed that in near future it might be possible to monitor phase decoherence as well.

Phase decoherence is closely connected with the understanding of Bose-Einstein condensation as spontaneous symmetry breaking. The global phase symmetry which is broken in the BEC is a continuous symmetry. In this case, according to the Goldstone theorem, a gapless Goldstone mode (or Nambu-Goldstone mode) must arise [9]. Theoretical investigations based upon Bogoliubov transformations reveal that such a symmetry breaking solution decay after a phase diffusion time due to quantum fluctuations. These fluctuations restore the broken symmetry. In the case of confined bosons with fixed number of particles, having a preferred phase is inconsistent with the global phase symmetry. A consistent theory should meet both the conserving and gapless conditions [10]. Bogoliubov prescription, being gapless, satisfy so called Hugenholtz-Pines theorem [11].

In the present brief review paper, we discuss recent progress in understanding phase diffusion of atomic BECs in terms of Bogoliubov transformations. The paper is organized as follows. We will first give a general description of Bogoliubov transformations in Sect. I. Sect. II reviews Bogoliubov theory of a weakly interacting Bose gas. In

Sect. III, quantum decoherence and quantum fluctuations are discussed in the context of the theory developed in the Sect. III. In Sect. IV a generalized Bogoliubov theory taking into account Goldstone modes is discussed and quantum phase operator is introduced. Finally, we summarize and conclude in Sect. V.

2 The Bogoliubov transformation

A second quantized quantum mechanical theory of a weakly interacting Bose gas was developed by N. N. Bogoliubov and applied to the superfluid Helium [12]. This powerful and simple microscopic superfluidity theory was based upon just a few parameters and mainly a single, weak interaction, assumption. It introduced the so called Bogoliubov transformations that allow for a diagonal form of generic bilinear Hamiltonians. These transformations have found many uses beyond the theory of superfluidity, ranging from quantum optics to black holes. Bogoliubov theory of superfluidity provides satisfactory quantum foundations and shows the existence of noninteracting bosonic elementary excitations of the superfluid near absolute zero temperatures. Furthermore, superfluidity condition, being the same as the thermodynamic stability of the system, was obtained.

Bogoliubov transformation can be defined as a unitary canonical transformation between unitary representations of some canonical commutation (or anticommutation) relation algebra. For a Bose gas, a Weyl-Heisenberg algebra is considered with the defining canonical commutation relations $[a, a^\dagger] = 1, [a, a] = 0 = [a^\dagger, a^\dagger]$. Isomorphism of such a canonical commutation relation algebra induces the condition $|u|^2 - |v|^2 = 1$ if one transforms to a new basis, unitary representation, associated with $\alpha, \alpha^\dagger$ such that

$$\begin{pmatrix} \alpha \\ \alpha^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ u^* & v^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \tag{1}$$

The coefficients u, v are conveniently parameterized as

$$u = \mathrm{e}^{\mathrm{i}\theta} \cosh \mathrm{r}, \quad \mathrm{v} = \mathrm{e}^{\mathrm{i}\theta} \sinh \mathrm{r}. \tag{2}$$

Generalization to multimode situations is straightforward. For example, in the case of two modes, associated with operators a, b we assume the Weyl-Heisenberg algebra for each mode and in addition $[a, b] = [a, b^\dagger] = [b, a^\dagger] = [a^\dagger, b^\dagger] = 0$. We introduce new operators α, β such that $\alpha = ua + vb^\dagger$ and $\beta = ub + va^\dagger$ with the condition $|u|^2 - |v|^2 = 1$ to ensure that the transformation remains canonical, meaning the structure of the commutation relations remain the same in either basis.

In the context of superfluidity, an approximate Hamiltonian of the system in the form

$$H = \sum_k \left[E_k a_k^\dagger a_k + \frac{g_k}{2} \left(a_k a_{-k} + h.c. \right) \right] \tag{3}$$

is reduced to its diagonal form

$$H = \sum_k \left[\omega_k \left(b_k^\dagger b_k + \frac{1}{2} \right) - \frac{1}{2} E_k \right], \tag{4}$$

with $\omega_k = (E_k^2 - g_k^2)^{1/2}$ via the Bogoliubov transformation

$$a_k^\dagger = \cosh(r_k) b_{-k}^\dagger - \sinh(r_k) b_k, \tag{5}$$

$$a_{-k} = \cosh(r_k) b_k^\dagger - \sinh(r_k) b_{-k}^\dagger. \tag{6}$$

Here the parameter $r_k = r_{-k}$ is determined from

$$\tanh(2r_k) = \frac{g_k}{E_k}. \tag{7}$$

Clearly if $E_k = 0$ the transformation is not defined. Low energy excitation modes in the superfluid are found to be proportional to the k exhibiting the dispersion property of sound waves. In this case Bogoliubov transformation is not defined for $k = 0$. This innocent looking remark has deep consequences in the decoherence of a Bose system as we shall discuss below. Very recently, this infrared divergence is identified by $r_0 \sim O(\ln N)$ so that it is divergent actually only in the thermodynamic limit, where any thermodynamic effect of squeezing vanish [14].

Excluding $k = 0$, the unitary operator of transformation $S(r_k)$ can be recognized to be a squeezing operator and is given by

$$S(\vec{r}) = \exp \left[\sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k}^\dagger - h.c. \right], \tag{8}$$

where $\vec{r}$ stands for the set of $\{r_k\}$. Bogoliubov transformation can be equivalently written as $Sa_kS^{-1} = b_k$.

3 Weakly interacting Bose gas

We consider a uniform gaseous system of N weakly interacting bosons of mass m confined in a cubic box of volume L^3. In order to describe the interactions between particles at low energies, we assume an effective contact interaction potential $V(\vec{x}-\vec{x}') = u_0\delta(\vec{x}-\vec{x}')$ between two atoms, where $u_0 = 4\pi\hbar^2 a_s/m$ and $\vec{x}, \vec{x}'$ atomic positions. Here a_s is the s-wave scattering length. Despite its simplicity, it brings some UV divergence problems which we shall not consider at that moment. The system is then described by an effective Hamiltonian in the second quantized form

$$H = \int \mathrm{d}^3\tilde{\mathrm{x}}\psi(\tilde{\mathrm{x}})^\dagger \left(\mathrm{T} + \frac{\mathrm{u}_0}{2}\psi(\tilde{\mathrm{x}})^\dagger\psi(\tilde{\mathrm{x}})\right)\psi(\tilde{\mathrm{x}}), \tag{9}$$

where $T = \vec{p}^2/2m$ with $\vec{p} = i\hbar\vec{\nabla}$. In the momentum representation the Hamiltonian becomes

$$H = \sum_{\vec{k}} E_k a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{u_0}{2L^3}\sum_{\vec{k},\vec{p},\vec{q}} a^\dagger_{\vec{p}+\vec{q}} a^\dagger_{\vec{k}-\vec{q}} a_{\vec{k}} a_{\vec{k}}, \tag{10}$$

where the field operators are expanded in the plane wave basis as we consider a uniform, translationally invariant system. We have already noted that Bogoliubov transformation is not well defined for $k = 0$. We can separate the zero momentum part of the Hamiltonian assuming lowest lying single-particle state is macroscopically occupied so that $\langle N_0\rangle = \langle a^\dagger_0 a_0\rangle \gg 1$ or $N_0/N = O(1)$. In the usual Bogoliubov prescription one normally do the replacement $a_0, a^\dagger_0 \to \sqrt{N_0}$. The interacting part of the Hamiltonian H_I can be expanded as

$$H_I \approx \frac{u_0}{2L^3}\left(N_0^2 - N_0 + 4N_0N_{\vec{k}} + a^\dagger_0 a^\dagger_0 a_{\vec{k}} a_{-\vec{k}} + a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} a_0 a_0\right). \tag{11}$$

We can express the Hamiltonian in a decomposed form

$$H = H_{\mathrm{z}} + H_{\mathrm{e}}, \tag{12}$$

where

$$H_{\rm z} = \frac{u_0}{2L^3}(\hat{N}_0^2 - \hat{N}_0), \tag{13}$$

is the so called zero-mode Hamiltonian and

$$H_{\rm e} = \sum_{\vec{k}} \left[\left(E_k + \frac{2u_0}{L^3} N_0 \right) N_{\vec{k}} + \frac{u_0}{2L^3} \left(a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}}, a_0^2 + h.c. \right) \right], \tag{14}$$

with $N_{\vec{k}} = a^\dagger_{\vec{k}} a_{\vec{k}}$. $H_{\rm e}$ is the Hamiltonian for excitations out of the condensate. Note that the excitations are created pairwise with opposite momenta accompanying the annihilation of two condensate atoms with zero momentum. The particle number in this process is conserved. We note that the quantum operator nature of N_0 is emphasized in $H_{\rm z}$ through the notation $\hat{N}_0$.

Using the identity $N = N_0 + \sum_{\vec{k}} N_{\vec{k}}$, the N_0 in the $H_{\rm e}$ can be replaced by N. Furthermore, in $H_{\rm e}$, the replacement $a_0 \to \sqrt{N}$ can be performed. In other words, we can ignore the quantum nature of a_0 in $H_{\rm e}$ and treat it as an ordinary number. The resulting H_e would not commute with N and yields eigenstates which are not eigenstates of $\hat{N}$. This reflects that Bogoliubov theory is gapless yet not conserving.

The resulting approximate Hamiltonian do not conserve the particle number. Introducing the particle density $n = N/L^3$, we write $H_{\rm e}$ in the form

$$H_{\rm e} = \sum_{\vec{k}\neq 0} \left[\epsilon_k N_{\vec{k}} + \frac{u_0 n}{2} (a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} + h.c.) \right] \tag{15}$$

with $\epsilon_k = E_k + u_0 n$. This Hamiltonian describes two-mode squeezing interaction. It can be diagonalized via a Bogoliubov transformation, which yields

$$H_{\rm e} = \sum_{\vec{k}\neq 0} \left[\omega_k (b^\dagger_{\vec{k}} b_{\vec{k}} + \frac{1}{2}) - \frac{1}{2}\epsilon_k \right], \tag{16}$$

where $\omega_k = [\epsilon_k^2 - (u_0 n)^2]^{1/2}$ is the excitation energy. Low energy excitations (long wavelength limit) have the spectrum $\hbar v_s k$ with

$v_s = \sqrt{u_0 n/m}$ indicating the gapless nature of excitations in agreement with the Hugenholtz-Pines theorem [11]. Diagonal form of the $H_{\rm e}$ describes non-interacting elementary excitations annihilated and created respectively with operators $b_{\vec{k}}$ and $b^\dagger_{\vec{k}}$.

According to the usual Bogoliubov prescription, we would ignore the quantum nature of N_0 in $H_{\rm z}$ as well. Doing so actually is only possible if quantum fluctuations in the particle number is negligible. Indeed the difference between the quantum $H_{\rm z}$ and classical one is proportional to $(\Delta N_0)^2 = \langle \hat{N}_0^2 \rangle - \langle \hat{N}_0 \rangle^2$.

4 Quantum fluctuations and decoherence

The ground state of the noninteracting gas would be a Fock number state $|N_0>= (a_0^\dagger)^{N_0}|vac> /\sqrt{N_0!}$. This state has uniform phase distribution and cannot break the global phase symmetry of the Hamiltonian. Furthermore, it satisfies $\langle N_0|a_0|N_0\rangle = 0$. The symmetry breaking can be described by $\hat{\psi} = \psi + \hat{\delta\psi}$. Comparing this with the plane wave expansion

$$\hat{\psi} = \frac{1}{L^{3/2}}\hat{a}_0 + \frac{1}{L^{3/2}}\sum_{\vec{k}\neq 0} \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\hat{a}_{\vec{k}}, \tag{17}$$

we see that in order to employ Bogoliubov theory we need to have a ground state that would satisfy $\langle \hat{a}_0 \rangle \neq 0$. Replacement $a_0 \to \sqrt{N_0}$ in $H_{\rm e}$ breaks the phase symmetry. Such a state cannot be an eigenstate of the number operator. Thus, it cannot be an eigenstate of the zero-mode Hamiltonian H_z as well. If we denote the symmetry breaking ground state of the noninteracting system by $|\alpha\rangle$ then interacting case would accept a ground state of the form $|\vec{r},\alpha\rangle = S(\vec{r})|\alpha\rangle$. The ground states satisfy the defining conditions $b_{\vec{k}\neq 0}|\vec{r},\alpha\rangle = 0 = a_{\vec{k}\neq 0}|\alpha\rangle$.

A good candidate for symmetry breaking ground state of the non-interacting system can be chosen to be a coherent state with $\alpha \simeq \sqrt{N_0} \sim \sqrt{N}$. It can be expanded in the Fock basis $|n\rangle$, that are eigenfunctions of the $H_{\rm z}$ with energies $E_n = (u_0/2L^3)(n^2 - n)$. We can check for how long we can assume $\langle a_0 \rangle \approx \sqrt{N}$. Short time dynamics yields $\langle a_0(t) \rangle \simeq \exp(-t^2/2\tau_c^2)$, where $\tau_c = \hbar L^3/N^{1/2}u_0$ [13]. This is the so called phase collapse or diffusion time after which symmetry breaking solution can no longer be assumed and the phase coherence is lost. On the other hand it is shown in the long time dynamics that the solution will revive periodically in time with interval $t_r = N^{1/2}\tau_c$ [13].

Alternatively, squeezed coherent zero-mode is also considered in a variational study [14]. Superpositions of squeezed states are also considered [15]. In addition, a generalized coherent state representation of Bose-Einstein condensates is also given recently [16]. Systematic microscopic derivation of squeezed coherent state for the zero-mode is studied under mean-field treatment [20, 21]. Variational states are considered especially to remedy the non-conserving nature of the Bogoliubov theory [22]. Though either coherent or squeezed coherent states are variationally very good approximations neither of them are the eigenstates of the H_z and thus would suffer from phase decoherence during their dynamics. Furthermore, microscopical interpretation of the squeezing in the zero-mode is not obvious despite the clear presence of squeezing in the excitation modes according to the Bogoliubov theory.

5 Goldstone modes and phase operator of BEC

Generalization of Bogoliubov prescription to include the zero-mode systematically was examined by a number of methods [17, 18, 19]. The variety is mainly due to the choice of different vacua.

Following Ref. [17, 13], a simple toy model calculation can be presented to introduce the phase operator of the condensate. Let us write $a_0 = \sqrt{N}_0 + \delta a_0$. Zero-mode Hamiltonian becomes

$$\begin{aligned} H_z &\simeq \frac{u_0}{2L^3}(N_0^2 - N_0) - \mu N_0 + \frac{u_0}{L^3}N_0\left(\frac{\delta a_0^\dagger + \delta a_0}{\sqrt{2}}\right)^2 = \\ &= \ldots + \frac{u_0}{L^3}N_0 P^2, \end{aligned} \tag{18}$$

where μ is a Lagrange multiplier playing the role of chemical potential to fix either N_0 or $\langle\hat{N}_0\rangle = \langle a_0^\dagger a_0\rangle$. Here a quadrature (or momentum) operator is introduced as $P = (1/2^{1/2})(\delta a_0^\dagger + \delta a_0)$. Phase operator is then introduced as its conjugate 'displacement' operator such that

$$Q = -\frac{i\hbar}{\sqrt{2}}(\delta a_0^\dagger - \delta a_0). \tag{19}$$

While P is constant, this operator has time dependence given by

$$Q(t) = Q(0) + 2\frac{u_0}{L^3} N_0 P(0) t. \tag{20}$$

Second term shows the influence fluctuations in the atom number on the phase dynamics. As a result Bogoliubov theory looses its validity. On the other hand this can be remedied by an ansatz which allows for converting the expression

$$a_0(t) = \sqrt{N}_0 + \frac{1}{\sqrt{2}} P(0) - i\frac{1}{\sqrt{2}\hbar} Q(t) \tag{21}$$

into a form given by

$$a_0(t) \simeq \exp\left[-iQ(t)\Big/\sqrt{2N_0}\hbar\right]\left(\sqrt{N}_0 + \frac{1}{\sqrt{2}} P(0)\right). \tag{22}$$

More systematically and mathematically appearance of the zero-mode and the P, Q operators are in fact a general results arising in diagonalizing quadratic forms [23]

$$H = \sum_{i,j} A_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{ij} (B_{ij} a_i^\dagger a_j^\dagger + h.c.), \tag{23}$$

where $A = A^\dagger, B^T = B$ which can be written in the matrix form

$$H = \frac{1}{2} \alpha^\dagger M \alpha - \frac{1}{2} \mathrm{tr} A, \tag{24}$$

with $\alpha = (a \;\; a^\dagger)^T$ is a row vector and

$$M = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}. \tag{25}$$

A canonical transformation is then employed such that $\beta = T\alpha$. Canonical condition can be summarized using σ_z Pauli matrix and given by $T\sigma_z T^\dagger \sigma_z = 1$ or $T^* = \gamma T \gamma$, where

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

Hamiltonian becomes diagonal in the form

$$H = \frac{1}{2}\beta^\dagger \sigma_z T \sigma_z M T^{-1} \beta - \frac{1}{2}\mathrm{tr}A, \tag{27}$$

as $T\sigma_z MT^{-1}$ is a diagonal matrix. Diagonalization procedure requires distinguishing left and right eigenvectors. P is identified as the eigenvector to a zero eigenvalue and associated with vector Q via $\sigma_z MQ = -iP/m$, where $m > 0$ is an inertial parameter. Without them the closure relation is not complete and the diagonalization is actually a reduction to a Jordan normal form. by introducing their linear combinations $v_0 = (1/2^{1/2})(P + iQ)$ and $w_0 = -(1/2^{1/2})(P + iQ)$ the closure relation is completed and a transformed Hamiltonian is finally obtained in the form

$$H = \sum_{n>0} \omega_n b_n^\dagger b_n + \frac{P^2}{2m} + \frac{1}{2}\sum_{n>0} - \frac{1}{2}\mathrm{tr}A. \tag{28}$$

First term is the usual normal mode Hamiltonian describing independent elementary excitations. Second term, zero-mode Hamiltonian arises due to the collective translational or rotational motion and associated with broken continuous symmetry. In the light of our previous discussions it is connected with the effect of quantum fluctuations introducing microscopic force restoring the broken symmetry in a system having symmetry inertia parameter m. After a characteristic time, the system will recover its symmetry. In the case of atomic BEC this is the phase diffusion time.

6 Conclusion

Summarizing, we have reviewed the phase diffusion problem in weakly interacting atomic Bose-Einstein condensate of a uniform system based upon Bogoliubov prescription. After reviewing Bogoliubov transformations, we have presented their application in the theory weakly interacting Bose gas. The effect of quantum nature of the condensate mode and the quantum fluctuations in the particle number is clarified within the context of Bogoliubov theory. Phase operator of the condensate and its dynamics derived by the quantum fluctuations in the particle number is explained. Recent developments in the experimental techniques in continuous monitoring of relative phases between the condensates may allow for observing such phase diffusion phenomena. Its understanding may lead to its control and enhancing condensate lifetimes for making BECs more practical for their potential applications.

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