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SOLUTION OF DIRAC EQUATION FOR AN ELECTRON MOVING IN A HOMOGENEOUS MAGNETIC FIELD: EFFECT OF MAGNETIC FLUX QUANTIZATION

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Abstract

The flux associated with electron's spin is calculated, based on the relativistic theory of the electron in the presence of a uniform magnetic field. In the relativistic limit, it is found that the flux associated with the electron's spin is exactly equal to $+\Phi_0/2$ for a spin down, and $-\Phi_0/2$ for a spin up, of unity.

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1 Introduction

Recently, a special issue of the journal Old and New Concepts in Physics [1] was dedicated to Professor Asim Orhan Barut, who passed away in 1994. One of the authors of this article (Z.Z.A.) was his Ph.D. student. Another author (M.S.) took a number of courses from Professor Barut.

Among the scientific interests of this outstanding physicist, it was also an investigation of effects caused by the magnetic flux quantization. In this paper, we examine the flux associated with spin of an electron in the relativistic limit. We dedicate the paper to the blessed memory of Professor Barut.

The magnetic flux quantization was first discovered by London and Onsager [2] in the form $\Phi = n\Phi_0$ $(n = 0, 1, 2, \ldots)$, where the flux unit $\Phi_0 = \frac{2\pi\hbar c}{e} = 4.14\ 10^{-7}$gauss cm^2 is called a fluxoid or fluxon [3]. Later this quantization was experimentally confirmed in 1961 [4]. The quantized unit was not $\frac{2\pi\hbar c}{e}$, but $\frac{2\pi\hbar c}{2e}$, which was explained by "correlated states" of pairs of electrons (with a charge of $2e$) [3, 5]. By using Bohr-Sommerfeld's phase integral formula, Onsager, L. and Lifshitz, I.M. showed that the orbit of the electron is quantized in such a way that the magnetic flux through it is $\Phi = (n + \frac{1}{2})\Phi_0$. In their calculation Onsager and Lifshitz did not consider the spin of the electron. The main reason for neglecting the flux associated with the electron's spin was due to the smallness of the size of the electron. In conventional calculations it has been treated like a point charge with a mass M and the spin vector is assumed to be attached to this point charge. Under the above assumptions it is not possible to get a flux associated with the spin. But it is also well-known that the electron has a non-zero radius ($\sim 10^{-15}$ m) and a spin which was measured through Stern-Gerlach experiment. It behaves in many ways as if it were continually rotating around an axis of its own. In an external magnetic field $\mathbf{B}$ it experiences a torque equal to that which would act on a current loop of equivalent dipole moment. Sağlam and Boyacıoğlu [6] were the first who suspected that, because of the spinning motion, the electron would allow an additional magnetic flux through itself. Their model was based on the magnetic top model [7, 8, 9] which can be made equivalent to a circular current loop with the radius R in x-y plane. It was shown that as far as the magnetic flux is concerned the radius R of this loop is a phenomenal concept and it gets eliminated in the end. In that model the electron motion is considered in two parts namely an "external" motion which can be interpreted as the motion of the center of mass (and hence the central of charge) and an "internal" one whose average disappears in the classical limit. The latter is caused by the spin of the electron. The important thing is that although the average of the internal motion disappears, the average of the magnetic flux associated with the internal motion does not. It was shown that to calculate the quantum flux the area integral of the magnetic field must be converted to a time integral over the cyclotron period , $T_c = \frac{2\pi}{\omega_c}$, of the magnetic field. It is important to note that electron's spinning frequency ω_s is very high compared to the cyclotron frequency ω_c. During the cyclotron period T_c, electron will complete only one turn around its cyclotron orbit, but it will spin ($\frac{\omega_s}{\omega_c} \gg 1$) times about itself. Although the radius of the electron (and hence the radius of the loop) is very small, because of the rapid spinning, the total flux during the cyclotron period, T_c would be comparable with the flux quantum, $\Phi = \frac{hc}{e}$. Following a similar way as in the Lenz law, they showed that the total flux coming from the internal motion during the cyclotron period, T_c is exactly equal to $\pm\frac{\Phi_0}{2}$ depending on the spin orientation. When the spin contribution is added to the flux of the orbital motion, the total spin dependent magnetic flux takes the form of $\Phi_n = n\Phi_0$ $(n = 0,1,2,\ldots)$ which is consistent with the results of London and Onsager [2] The aim of the present study is to solve the Dirac equation for an electron moving in a uniform magnetic field and show that a quantized flux of $\pm\frac{\Phi_0}{2}$ is associated with electron spin depending on the spin orientation. The present work is therefore a kind of full quantum mechanical justification of the previous work of Sağlam and Boyacıoğlu [6] who used a semiclassical model to prove their idea. In passing we note that in both studies we are considering the magnetic flux through the electron in the presence of externally applied uniform magnetic field. Therefore the present result has nothing to do with the "Anyon" feature of the particles carrying a flux tube even in the absence of an external magnetic field [10, 11].

We know that the Dirac formulation uses a relativistic equation to describe the electron, including its spin. Using the Dirac equation for a charge moving in an electromagnetic potential $A_\mu =$

$(A_0(\mathbf{x}), -\mathbf{A}(\mathbf{x}))$ we seek a solution. Therefore we make the usual replacement $\pi_\mu = p_\mu - eA_\mu$ in

$$(\gamma^\mu \pi_\mu - M)\psi(x) = 0 \tag{1}$$

where $p_\mu = (E, -\mathbf{p})$ is the momentum four vector, e is the electric charge, M is the rest mass of the electron and γ^μ matrices satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2}$$

where $g^{\mu\nu} = g_{\mu\nu}$ is the Minkowski metric, whose components are chosen to be $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$; and all the rest are zero. Then a particular solution of the wave equation

$$\psi(\mathbf{x}, t) = e^{-iEt}\psi(\mathbf{x}) \tag{3}$$

is said to represent a stationary state of the particle, since $|\psi|^2$ is constant in time. To find solution of this equation [12, 13], let us decompose this four-spinor into two smaller two-spinors:

$$\psi(\mathbf{x}) = \begin{pmatrix} \phi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{pmatrix}. \tag{4}$$

Then the Dirac equation can be decomposed as the sum of two two-spinor equations, and eliminating $\chi(\mathbf{x})$, one then finds:

$$\phi(\mathbf{x}) = \frac{[\boldsymbol{\pi}^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}]}{[E - M - eA_0(\mathbf{x})][E + M - eA_0(\mathbf{x})]}\phi(\mathbf{x}), \tag{5}$$

where we have used the fact [12, 13] that:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \boldsymbol{\pi}^2 - e\boldsymbol{\sigma}\cdot\mathbf{B} \tag{6}$$

where σ^i are the familiar Pauli spin matrices.

2 Relativistic Landau levels and solution of the Dirac equation in a constant uniform magnetic field

Here, we consider the motion of a charged particle in a constant uniform magnetic field $\mathbf{B}$ pointing in the $+z$ direction. Then in the symmetric gauge, one has $\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B}\times\mathbf{r} = \frac{1}{2}\mathrm{B}(-y, x, 0)$. Since it is very hard to solve these equations for the case of $A_0(\mathbf{x}) \neq 0$, it is usual to write $A_0(\mathbf{x})=0$ in order to solve the problem. Under these assumptions:

$$\boldsymbol{\pi}^2 = \mathbf{p}^2 - 2\kappa L_z + \kappa^2\rho^2, \tag{7}$$

where $\mathbf{p} = -i\nabla$, and ∇^2 is the Laplacian operator in cylindrical coordinates (ρ, φ, z), since the problem has cylindrical symmetry.

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} \tag{8}$$

$$L_z = -i\frac{\partial}{\partial\varphi}, \qquad \kappa = \frac{eB}{2}. \tag{9}$$

We can also split $\phi(\mathbf{x})$ as follows:

$$\phi(\mathbf{x}) = \begin{pmatrix} \phi_+(\mathbf{x}) \\ \phi_-(\mathbf{x}) \end{pmatrix}, \tag{10}$$

then the equation for $\phi(\mathbf{x})$ in Eq. (5) becomes:

$$\left(\beta_\lambda + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - 2\kappa i\frac{\partial}{\partial\varphi} - \kappa^2\rho^2\right)\phi_\lambda(\rho, \varphi, z) = 0, \tag{11}$$

where ϕ_λ stands for upper (lower) components of the spinor $\phi(\mathbf{x})$ for $\lambda = +1(-1)$, that is, the spinor can be expanded in terms of the spin-up and spin-down states, respectively. $\beta_\lambda = E^2 - M^2 + 2\lambda\kappa$ is a constant, which similarly takes two different values depending on the value of the λ [12].

After some calculations, one can obtain the solution for the ϕ_λ in terms of the confluent hypergeometric function as follows [5, 14, 15]:

$$\phi_\lambda(t, \varphi, z) = N_\lambda t^{\frac{|m|}{2}} e^{-\frac{t}{2}} e^{im\varphi} e^{ip_z z} F(-n_\lambda, |m|+1, t) \tag{12}$$

where the multiplying constant N_λ is the normalization constant, $t = \eta^2 = \kappa\rho^2$ and with $n_\lambda = 0, 1, 2\ldots$ This implies that the relativistic energy is given by

$${E_\lambda}^2 = {p_z}^2 + M^2 + 2\kappa\left(2n_\lambda + |m| - m + 1 - \lambda\right). \tag{13}$$

The requirement that the wave function be single-valued, so that the phase factor $e^{im\varphi}$ is unity, implies that $m = 0, \pm1, \pm2, \ldots$ where now all physically meaningful solutions are included if m is allowed to be a positive or negative integer, or zero. The components of the spinor χ can be written as follows

$$\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{1}{E+M} \begin{pmatrix} \pi_z \phi_+ + \pi_- \phi_- \\ \pi_+ \phi_+ - \pi_z \phi_- \end{pmatrix} \tag{14}$$

where π_+, π_- and π_z are the derivative operators defined such as:

$$\pi_\pm = \pi_x \pm i\pi_y = \sqrt{\kappa}\, e^{\pm i\varphi} \left(\pm \frac{1}{\eta} \frac{\partial}{\partial \varphi} - i \frac{\partial}{\partial \eta} \mp i\eta \right) \tag{15}$$

$$\pi_z = -i \frac{\partial}{\partial z} \tag{16}$$

By choosing $\phi_- = 0$ ($\phi_+ = 0$) four four-spinor states $\psi(x)$ can be found such that the spin directions are parallel (antiparallel) to the magnetic field; that is, spin-up states or spin-down states:

$$|+; m > 0\rangle = N f(t, \varphi, z) \begin{pmatrix} F(-n, |m|+1, t) \\ 0 \\ \frac{p_z}{(E_n+M)} F(-n, |m|+1, t) \\ \frac{2ni\sqrt{\kappa}\eta e^{i\varphi}}{(E_n+M)(|m|+1)} F(-n+1, |m|+2, t) \end{pmatrix} \tag{17}$$

$$|+; m < 0\rangle = N f(t, \varphi, z) \begin{pmatrix} F(-n, |m|+1, t) \\ 0 \\ \frac{p_z}{(E_n+M)} F(-n, |m|+1, t) \\ \frac{-2|m|i\sqrt{\kappa}e^{i\varphi}}{(E_n+M)\eta} F(-n, |m|, t) \end{pmatrix} \tag{18}$$

$$|-; m > 0\rangle = N f(t, \varphi, z) \begin{pmatrix} 0 \\ F(-n, |m|+1, t) \\ -\frac{2|m|i\sqrt{\kappa}e^{-i\varphi}}{(E_n+M)\eta} F(-n-1, |m|, t) \\ \frac{-p_z}{(E_n+M)} F(-n, |m|+1, t) \end{pmatrix} \tag{19}$$

$$|-; m<0\rangle = Nf(t,\varphi,z)\begin{pmatrix} 0 \\ F(-n,|m|+1,t) \\ \frac{2i(n+|m|+1)\sqrt{\kappa}\eta e^{-i\varphi}}{(E_n+M)(|m|+1)}F(-n,|m|+2,t) \\ \frac{-p_z}{(E_n+M)}F(-n,|m|+1,t) \end{pmatrix} \tag{20}$$

where $f(t,\varphi,z)$ is defined as follows:

$$f(t,\varphi,z) = t^{\frac{|m|}{2}} e^{-\frac{t}{2}} e^{im\varphi} e^{ip_z z} \tag{21}$$

and the energies $E_n^2 = 4\kappa n + M^2 + p_z^2$, $E_n^2 = 4\kappa(n+|m|) + M^2 + p_z^2$, $E_n^2 = 4\kappa(n+1) + M^2 + p_z^2$ and $E_n^2 = 4\kappa(n+1+|m|) + M^2 + p_z^2$ correspond to these states, respectively. It is seen that there are infinite number of degenerate states for $m \geq 0$, that is, when the orbital angular momentum of the electron is parallel to the magnetic field.

After this point, let us consider the electron to move only on the xy-plane, that is, $p_z = 0$. In that case these energies are called the Landau levels. We require that the eigenstates $|\lambda; m>$ be normalized to unity, so that

$$<\lambda; m|\lambda; m> = 1 \tag{22}$$

which are normalised on the plane. The normalisation condition implies that

$$N = \sqrt{\frac{\kappa}{2\pi}\left(1+\frac{M}{E_n}\right)\frac{(n+|m|)!}{n!(|m|!)^2}} \tag{23}$$

Next, we use the definition of the magnetic flux:

$$\Phi = \oint \mathbf{B}\cdot d\mathbf{s} \tag{24}$$

Performing the $d\mathbf{s}$ integration one finds Φ=B$S = \Phi_0 t$, where $S = \pi\rho^2$ is the area of the circular loop. Since we assume that the electron rotating in a circular orbit with the radius ρ on the plane. Note that here $ds = 2\pi\rho d\rho = \frac{1}{2\kappa} dt d\varphi$ is a differential element of surface area and the integration is to be carried out over the entire plane surface which is infinite. The expectation value of the magnetic flux Φ for the function $|\lambda; m >$ gives us a relation between energy and flux

$$\Phi_\lambda = \Phi_0 < \lambda; m|t|\lambda; m > \tag{25}$$

where the fluxoid or fluxon $\Phi_0 = \frac{2\pi}{e} = 4.14\ 10^{-7}$gauss cm^2 where normally only $\hbar$ and c are eliminated, i.e $\hbar = c = 1$ in natural units. Therefore, the flux for spin up states and spin down states are calculated as follows:

$$\Phi(\uparrow) = \frac{\Phi_0}{2}(1+\zeta)[2n + |m| + 1 + \frac{1-\zeta}{1+\zeta}(2n + |m|)] \tag{26}$$

$$\Phi(\downarrow) = \frac{\Phi_0}{2}(1+\zeta)[2n + |m| + 1 + \frac{1-\zeta}{1+\zeta}(2n + |m| + 2)] \tag{27}$$

where $\zeta = \frac{M}{E}$. For low energies, as $\zeta \longrightarrow 1$, we have the nonrelativistic result, that:

$$\Phi(\uparrow) = \Phi(\downarrow) = \Phi_0(2n + |m| + 1) \tag{28}$$

In the relativistic limit, as $\zeta \longrightarrow 0$, this formula reduces to:

$$\Phi(\uparrow) = \Phi_0\left(2n + |m| + \frac{1}{2}\right) \tag{29}$$

$$\Phi(\downarrow) = \Phi_0\left(2n + |m| + \frac{3}{2}\right) \tag{30}$$

3 Results and Discussion

We have calculated the magnetic flux associated with electron spin. Our calculations are based on the relativistic theory of the electron in the presence of a uniform magnetic field. In the nonrelativistic limit, as we have seen from Eq. (28), the electron's spin does not any contribute to the flux, which confirms earlier results [2, 3, 4, 5]; i.e.,

$$\Phi(\uparrow) = \Phi(\downarrow) = n'\Phi_0 \tag{31}$$

where $n' \equiv 2n+|m|+1$ ($n' = 1, 2, 3, \dots$). However, in the relativistic limit, by comparing Eq. (29) and (30), we have found that

$$\Phi(\uparrow) = (n' - \frac{1}{2})\Phi_0,$$
$$\Phi(\downarrow) = (n' + \frac{1}{2})\Phi_0 \qquad (32)$$

the spin contribution to the flux being $-\frac{\Phi_0}{2}$ for spin-up and $+\frac{\Phi_0}{2}$ for spin-down states. Therefore we find that the magnetic flux associated with electron spin is exactly equal to $\frac{\Phi_0}{2}$. The present result is in agreement with the recent result of Saglam and Boyacıoğlu [6], who used a semiclassical model (current loop model), for electron spin. As was explained in [6], as far as the magnetic flux is concerned, the radius R of this current loop is a phenomenological concept (which disappears in the end), whose detailed calculation is not important. When this spin term is added to that of the orbital motion, the total spin dependent fluxes are found to be quantized in units of $\Phi_0 = \frac{hc}{e}$. The present result is in agreement with those of London and Onsager [2] who found $\Phi_n = n\Phi_0$, using the property of the wave function being single-valued. Because, spin automatically lifts the degeneracy, the wave function becomes non-degenerate, implying being single-valued as well.

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References

[1] Barut Memorial Issue, Old and New Concepts in Physics **2**, No 4, 2005 (under edition of Prof. J.R. Dowling).

[2] F. London, *Superfluids*, John Wiley and Sons, New York, (1950). L. Onsager, *Proceeding of the International Conference on Theoretical Physics*,935, Science Council of Japan, Tokyo, (1954).

[3] C. Kittel, *Introduction to Solid State Physics*, 7th ed. John Wiley and Sons, New York, (1996) p.358.

[4] B. S. Deaver and W. M. Fairbank, Phys. Rev. Lett. **7**, 43 (1961). R. Döll and M. Nabauer, ibid, **7** , 51 (1961).

[5] S. Gasiorowicz, *Quantum Physics*, John Wiley and Sons, New York, (1974) p.221.

[6] M. Saglam and B. Boyacıoğlu, International Journal of Modern Physics B,**16**, 607 (2002).
M. Saglam and B. Boyacıoğlu, Phys. Stat. Sol. B, **230**, 133 (2002).

[7] N. Rosen, Phys. Rev.,**82**, 621 (1951).

[8] L. Schulman, Phys. Rev.,**176**, 1558 (1968).

[9] A. O. Barut, M. Božić and Z. Marić, Annals of Physics,**214**,53 (1992).

[10] A. Lerda, *Anyons*, Springer-Verlag Heidelberg (1992).

[11] R. Jackiw and A. N. Redlich, Phys. Rev.Lett.,**50**, 555 (1983).

[12] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw Hill, New York,(1964).

[13] K. Michio, *Quantum Field Theory*, Oxford University Press, New York, (1993).

[14] B. H. Bransden and C. J. Joachain, *Quantum Mechanics*, 2th ed., Person Education, England, (2000),p.373,571.

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals*, Series and Products, 6th ed., Academic Press, New York,(2000).

Comment by

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Practically all known properties of an electron are consistent with the assumption that its radius is zero, although it seems to be tempting to associate spin of the electron with a certain spinning spatial charge distribution. The classical estimation of the electron radius gives

$$r_{cl} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2} \approx 2.8179 \times 10^{-15}\ m.$$

This quantity is used, for example, as roughly the length scale at which renormalization becomes important in quantum enelctrodynamics [1].

In the paper [2], this value r_{cl} was used to find a magnetic flux through the area, which electron occupies. This flux is caused by spinning of the electron.

As far as we know, so far all attempts to measure the radius of the electron have failed. The known estimation based on experimental data gives $r_e \leq 10^{-18}\ m$. Therefore, the result of Ref. [2] requires further justification.

The paper [3] contains completely quantum consideration based on the analysis of solutions of Dirac equation in presence of a constant uniform magnetic field. It is shown that the electron spin contributes an additional flux

$$\Phi_{spin} = \pm\frac{1}{2}\Phi_0, \quad \Phi_0 = \frac{2\pi\hbar c}{e}$$

into that caused by external field. Here sign $\pm$ depends on the spin orientation with respect to external field. This result confirms the previous one [2] that has been obtained in a semi-classical way.

The fact that the completely quantum picture and semiclassical approach give the same result may be interpreted as an indication that the "dressing" of a point-like bare electron leads to its smearing over the region bounded by r_{cl} (also see Refs. [4]).

An interesting recent approach to investigation of properties of spinning particles is based on the use of Kerr twistorial structures (see Ref. [5] and references therein). In particular, combination of Kerr approach with Dirac equation has lead to a number of interesting results about the space-time structure of electron [6]. In view of results of Ref. [3], it seems to be interesting to investigate relation between the inherent magnetic flux of the electron and its picture within the Kerr-Dirac model of Ref. [6].

References

[1] J. Zinn-Justin, Quantum field theory and critical phenomena (Oxford University Press, New York, 1993).

[2] M. Saglam and B. Boyacioğlu, Int. J. Mod. Phys. B **16**, 607 (2002).

[3] O. Yilmaz, M. Saglam, and Z.Z. Aydin, New and Old Concepts in Physics

[4] W. Israel, Phys. Rev. D **2**, 641 (1970); C.A. López, Phys. Rev. D **30**, 313 (1984).

[5] A. Burinskii, Phys. Rev. D **70**, 086006 (2004).

[6] A. Burinskii, E-print hep-th/0507109.