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WHAT IS WRONG WITH SCHWARZSCHILD'S COORDINATES?

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Abstract

A strict derivation of the Schwarzschild metric, based solely on Newton's law of free fall and the equivalence principle, is presented. In the light of it, regarding Schwarzschild's coordinates as representing the point of view of a distant observer resting relative to the source of a centrally symmetric gravitational field, proves illegitimate. Such point of view is better represented by the Painlevé-Gullstrand system of coordinates, which agrees with Schwarzschild's system with respect to its spatial coordinates and time scale, but disagrees with respect to the relation of simultaneity. A duality of the Schwarzschild solution and its time-irreversibility is suggested. The physical meaning of the coordinate singularity at the Schwarzschild radius is clarified.

1 DERIVATION OF THE SCHWARZSCHILD METRIC

Let us assume that in a centrally symmetric gravitational field outside its source:
(i) The relative space of an "infinitely" distant observer resting relative to the source is Euclidean. This means that in this space ordinary spherical coordinates (r, θ, ϕ) may be introduced, with the origin coinciding with the center of the source.
(ii) The equivalence principle [Rindler(1977)] holds. This implies, among others, that any pointlike clock freely falling radially with the initial value of velocity zero "at infinity" may be so adjusted initially that it reads always, during its fall, the relative time in the reference frame of the above-mentioned observer. Let us denote such time by τ.

Let ρ be the spatial coordinate in the radial direction in the local inertial frame **A** connected with such a clock. As follows from (ii), the standards of simultaneity, time unit and length unit in every direction in **A** agree with the ones in the frame of the "infinitely" distant observer. Thus:

$$d\rho = dr - v d\tau \ . \tag{1}$$

On the other hand, the spacetime metric is locally expressed in **A** by the formula:

$$ds^2 = -d\tau^2 + d\rho^2 + r^2 d\Omega^2 \ , \tag{2}$$

where:

$$d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2 \ .$$

By appropriate substitution we get:

$$ds^2 = -\left(1 - v^2\right) d\tau^2 - 2v d\tau dr + dr^2 + r^2 d\Omega^2 \tag{3}$$

(cf. Refs. [Trautman(1966), Nurowski et al.(1999), Volovik(2001)]). Note that, thanks to the elimination of the local coordinate ρ, although the expression (2) holds only locally, this restriction is no more valid with respect to the formula (3).
(iii) According to the point of view of the "infinitely" distant observer, a test particle freely falling radially in the above way obeys

the Newtonian law of free fall, i.e. its velocity is always equal to the escape velocity for the given value of the radial coordinate:

$$v = -\sqrt{2M/r} \ . \tag{4}$$

where M denotes the mass of the source and geometrized units are adopted, so that $G = c = 1$. Substituting the expression (4) into (3), we obtain:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tau^2 + 2\sqrt{\frac{2M}{r}} d\tau dr + dr^2 + r^2 d\Omega^2 \ , \tag{5}$$

where $r_g = 2M$ is the gravitational radius (Schwarschild radius) of the source.

At first sight, the formula (5) does not resemble the Schwarzschild metric. However, in fact this is its Painlevé-Gullstrand form [Painlevé(1921)][Gullstrand(1922), Israel(1987)]. Let us consider the local inertial frame **B** that at the moment spatially coincides with **A** and rests relative to the center of gravitation. Let τ' be the time coordinate in **B**. Now, in consequence of (ii), it is related to the coordinates in **A** by the formula of the appropriate Lorentz transformation:

$$d\tau' = \gamma \left(d\tau + v d\rho\right) \ , \tag{6}$$

where:

$$\gamma = 1 \Big/ \sqrt{1 - v^2} \ . \tag{7}$$

By regarding (1), we get:

$$d\tau' = \frac{1}{\gamma} d\tau + \gamma v dr \ . \tag{8}$$

The time coordinate τ' disagrees with τ with respect to the unit and to simultaneity. Now, let us introduce a new time coordinate t that agrees with τ with respect to the unit and with τ' with respect to simultaneity:

$$dt = \gamma d\tau' \tag{9}$$

(cf. Refs. [Schiff(1960), Wild(1996)]) or:

$$dt = d\tau + \gamma^2 v dr \ , \tag{10}$$

or equivalently, by regarding (4) and (7):

$$d\tau = dt + \left(1 - \frac{2M}{r}\right)^{-1} \sqrt{\frac{2M}{r}} dr \ . \tag{11}$$

By substituting (11) into (5), we obtain:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ , \tag{12}$$

which is the Schwarzschild's formula. This completes a derivation of it in the way which is commonly regarded as impossible (cf. for example Ref. [Rindler(1968)]).

2 ALTERNATIVE DERIVATION

Let us replace the assumption (iii) by a new assumption:
(iii′) According to the point of view of the "infinitely" distant observer, a test particle freely escaping radially with the final value of velocity zero "at infinity" obeys the Newtonian law of free fall, i.e. its velocity is always equal to the escape velocity for the given value of the radial coordinate. This means that (4) must be replaced by:

$$v = \sqrt{2M/r} \tag{4′}$$

and instead of (5), we get:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tau^2 - 2\sqrt{\frac{2M}{r}} d\tau dr + dr^2 + r^2 d\Omega^2 \ . \tag{5′}$$

Nevertheless, the final result (12) remains the same. Instead of (11), we obtain:

$$d\tau = dt - \left(1 - \frac{2M}{r}\right)^{-1} \sqrt{\frac{2M}{r}} dr \ . \tag{11′}$$

It is easy to see that substituting (11′) into (5′) results in (12). This is no surprise, since (5′) is just another branch of the Painlevé-Gullstrand form of the Schwarzschild metric, discovered by P. Painlevé [Painlevé(1921)] and rediscovered, for example, by G. Lemaître [Lemaître(1933), Lemaître(1997)].

3 TWO VERSIONS OF THE SCHWARZSCHILD FIELD?

In the framework of the Newtonian kinematics, the assumptions (iii) and (iii′) were compatible. However, in the framework of the relativistic kinematics they contradict each other. This is because, in the light of (ii), each of them would distinguish other local inertial frame that would have to agree with the frame of the "infinitely" distant observer, resting relative to the center of gravitation, with respect to the standards of length unit, time unit and simultaneity. Unfortunately, the frames distinguished in the above sense by the alternative assumptions would move relative to each other and thus disagree at least with respect to simultaneity. Consequently, each of the assumptions (iii) and (iii′) is satisfied in a different physical situation. By comparison of (5) and (5′), it is easy to see that the corresponding metrical fields are time reflections of each other.

On the other hand, in each of them it is possible to introduce a new time coordinate in a way that results in transforming the metric into the time-reflexible form (12). One might be tempted, therefore, to interpret (5) and (5′) as two descriptions of the same physical reality in different systems of coordinates. The reaction to such temptation depends on the question which of the descriptions is more fundamental.

To some extent, the situation resembles the one that resulted from analogous duality of the Eddington-Finkelstein forms of the Schwarzschild field [Anderson(1967)]. Finkelstein even interpreted his result as implying past-future asymmetry of such field [Finkelstein(1958)]. Unfortunately, this interpretation has not been taken seriously, partly because he introduced his time coordinate in a purely formal way, whereas Schwarzschild's time coordinate has a clear physical meaning. This suggests regarding the latter as more fundamental and the former as a mere auxiliary variable. Such a suggestion is strengthened by the interpretation of the Finkelstein extension as an intermediate step toward the Kruskal extension [Anderson(1967), Misner et al.(1973)]. However, the matter is otherwise in the case of our time coordinate. Let us, now, go into details.

4 PHYSICAL INTERPRETATION OF THE SPACETIME COORDINATES

Schwarzschild's coordinates are usually regarded as representing the point of view of an "infinitely" distant ("outer") observer (see [Landau and Lifschitz(1981)]). This would mean that they are the result of the most appropriate extrapolation of the local coordinates from a spacetime region where the gravitational field is negligible, onto the regions where it cannot be neglected. Such extrapolation should be based on local reference frames that (a) agree with respect to the standards of length unit, time unit and simultaneity and (b) rest relative to each other.

Unfortunately, the influence of gravitation results in the situation that the frames which satisfy (a) cannot satisfy (b) and *vice versa*. The only way out is constructing the frames that would satisfy both (a) and (b) by transferring somehow the metrical standards from the frames **A** satisfying (a) to the frames **B** satisfying (b) (see Sec. 1). This may be done in two equivalent ways: either by appropriate corrections of measurements performed in resting frames or by transferring to the latter the results of measurements performed in the frames satisfying (a) by a suitable Galilean transformation. It is clear from our derivations that both ways result in defining the coordinates in which the spherically symmetric metrical field acquires the form (5) or (5′).

The first version of the construction consists in correcting the effects of length contraction and time dilation resulting from the motion of frames **B** relative to the corresponding frames **A**, and correcting the influence of gravitation on the result of application of the standard synchronization procedure in the frames **B**. Refraining from correcting the simultaneity is irrelevant with respect to spatial coordinates, but it is of crucial importance when the time coordinate is concerned. It results in defining a coordinate which does not represent the point of view of any observer. Since it will agree with the time in **A**, and

thus with the relative time of the "infinitely" distant observer, with respect to the unit, and with the time in **B** with respect to simultaneity, it is clear
(cf. Ref. [Misner et al.(1973)]) that the time coordinate defined this way can be identified with Schwarzschild's time coordinate t.

There is an important asymmetry between the two versions of the above-mentioned construction. Whereas the second one is, in principle, applicable for any finite value of the radial coordinate for which the tidal forces are not too big, the first one is restricted to the region where its value is greater than the gravitational radius [Rindler(1977)]. This means that the second version is preferable [Gautreau and Hoffmann(1978)], for which the choice of our time coordinate τ, rather than t, is natural.

It follows from the above considerations that our coordinates represent the point of view of an "infinitely" distant observer better than Schwarzschild's coordinates. The latter may be regarded as the result of a compromise between such a point of view and the points of view of observers permanently resting in the Schwarzschild field, since they get the standards of length and time units from the frame resting "at infinity", and the standard of simultaneity from the frame resting in the given spacetime region.

Needless to say, this is true only about the regions with the radial coordinate greater than the gravitational radius. For smaller values of that coordinate, Schwarzschild's time coordinate has no independent physical meaning and it is completely derivative with respect to our. This means that this is rather the former than the latter time coordinate that should be treated as a mere auxiliary variable. For further advantages and applications of the Painlevé-Gullstrand coordinates, see [Volovik(2001), Kraus and Wilczek(1994), Visser(1998), Doran(2000), Parikh and Wilczek(2000), Martel and Poisson(2001)] [Schützhold(2001)].

Although the manifold covered by the Painlevé-Gullstrand coordinates, identical in any case with the appropriate Finkelstein extension, is neither geodesically complete nor maximal [Anderson(1967)], it has another interesting property which may be called *spatial maximality*: a spacetime is spatially maximal iff a *global time* function [Wald(1984)] exists in it, the hypersurfaces of constancy of which are maximal in the sense of the geometry induced on them.

5 THE MEANING OF THE COORDINATE SINGULARITY AT THE GRAVITATIONAL RADIUS

It is clear from the formulae (4) and (4′) that at the gravitational radius the velocity v acquires the absolute value equal to the constant c (in our units $c = 1$), i.e. to the light velocity in vacuum, which is the limit of velocity of matter in local inertial frames. The directions of the worldlines of light signals propagating radially is defined by the condition of the metric and its non-radial components acquiring the value zero. For the field with the metric (5), this condition reduces to the equation:

$$-\left(1-\frac{2M}{r}\right)d\tau^2+2\sqrt{\frac{2M}{r}}d\tau dr+dr^2=0\ . \tag{13}$$

For the metric with the field (5′), the equation 13 must be replaced by:

$$-\left(1-\frac{2M}{r}\right)d\tau^2-2\sqrt{\frac{2M}{r}}d\tau dr+dr^2=0\ . \tag{13'}$$

The solutions of 13 and 13’ are:

$$dr=\left(-1-\sqrt{2M/r}\right)d\tau\ , \tag{14}$$

$$dr=\left(1-\sqrt{2M/r}\right)d\tau \tag{15}$$

and:

$$dr=\left(-1+\sqrt{2M/r}\right)d\tau\ , \tag{14'}$$

$$dr=\left(1+\sqrt{2M/r}\right)d\tau\ , \tag{15'}$$

respectively. Let us have a look at their respective special cases at the gravitational radius:

$$dr=-2d\tau\ , \tag{16}$$

$$dr=0 \tag{17}$$

and:

$$dr = 0 \ , \tag{16'}$$

$$dr = 2d\tau \ . \tag{17'}$$

They mean that, for both fields, one of the signals rests. The other in the field (5) moves toward the center of gravitation and in the field (5′) in the opposite direction. Thus, in the first field rests the outgoing and in the second the ingoing signal. Consequently, in both cases the Schwarzschild sphere works as an unidirectional membrane, but in the field (5) directed inward and in the field (5′) outward.

The above result is no surprise [Anderson(1967)]. What is more interesting is the fact that the equations (14),(15) and (14′),(15′) may be derived from simple equations:

$$dr = (-c + v)d\tau \ , \tag{18}$$

$$dr = (c + v)d\tau \ , \tag{19}$$

which are consequences of the Galilean velocity composition formula, applied to the velocities of ingoing and outgoing light signals, respectively, relative to the appropriate frame **A**, and of **A** relative to the center of gravitation. This means that the influence of gravitation on physical phenomena reduces to some "dragging" toward the source in the field (5), or in the opposite direction in the field (5′), the stronger the closer to the source. At the gravitational radius nothing special takes place, but only the "dragging" velocity reaches the value equal to the light velocity.

For the values of the radial coordinate smaller than the gravitational radius, the "dragging" velocity becomes greater than the light velocity. Thus, in such spacetime regions the resting frames would have to move relative to local inertial frames with extraluminal velocities. This is why they are physically impossible. Moreover, if the standard notion of simultaneity as orthogonality of the spacetime interval to the worldline of a resting object is extrapolated from local inertial frames to such non-physical frames, then two events simultaneous in such a frame may be separated by a timelike interval. No wonder that inside the Schwarzschild sphere the proper time along any worldline of ordinary matter flows backward in Schwarzschild's coordinate time t [Misner et al.(1973)]. No such effect appears with respect to our time τ.

6 DISCUSSION

If our arguments for the suggested duality of the Schwarzschild solution are sound, then a question arises. It seems that only one of the metrical fields (5) , (5′) can represent the gravitational field around a spherically symmetric body. There are strong arguments that it is rather (5) than (5′) [Wald(1984)]. What is, then, if any, the physical meaning of the field (5′)? If gravitation always expresses itself as "dragging" inward (see Sec. 5 above), then how can the field of opposite nature be produced? The lack of answer to this question would suggest that the field (5′) as such is non-physical, being a mere formal result of time reflection of the field (5). However, such a conclusion would imply that some cosmological models (cf. Refs. [Lemaître(1933), Gal-Or(1981)]) are non-physical.

On the other hand, our derivations are based on a very strong reading of the equivalence principle. According to it, at any spacetime point there exists a local inertial frame (LIF) which not only is equivalent to a given LIF in the usual sense in which all LIF-s are, but, in addition, agrees with it with respect to the length unit in all directions, time unit and simultaneity. It is clear, for example, that at the same spacetime point no two LIF-s which are moving relative to one another are *strongly equivalent* in this sense. Moreover, no two LIF-s at spacetime points with different values of the radial coordinate in the counterpart of the Schwarzschild solution in Nordström's second theory [Nordström(1913)] are so equivalent. Thus, that counterpart does not satisfy the equivalence principle in our interpretation. This observation seems to meet R. Sexl's objection cited in Ref. [Ehlers and Rindler(1997)].

Assuming the existence of the relation of strong equivalence is tantamount to assuming the existence of the "ether" vector field in the sense of Ref. [Trautman(1966)]. Now, one may object that such an assumption, although quite natural in the Newtonian framework, is illegitimate from a purely relativistic point of view. This objection does not invalidate our derivations of Secs. 1 and 2, but it calls the above-mentioned duality into question, since the difference between assumptions (iii) and (iii′) may be interpreted as resulting from arbitrary choices of time coordinates to represent the point of view of the "outer" observer. Nevertheless, even if such an objection is raised, our considerations still seem to have shed new light on the problem of the physical meaning of the Schwarzschild metric.

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Comment on WHAT IS WRONG WITH SCHWARZSCHILD'S COORDINATES?

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E. Cartan [1], K. Friedrichs [2], and then A. Trautman [3] have shown that Newton's theory of gravitation can be given an elegant geometric formulation in which Newtonian space-time appears to be a 4-dimensional differentiable manifold endowed with a symmetric connection which leads to a non vanishing curvature. However, the respective geometry is nonriemannian as there does not exist any natural metric on the Newtonian space-time. This last fact is easily understood since in Newtonian mechanics there does not exist any universal velocity $c < \infty$, and we are not able to define any invariant space-time interval (a space-time metric). Consequently, the Einstein theory of gravity is not contained in the Newtonian theory. This is a well known and trivial fact, but we should remember it when we are looking for some deep relations between these two theories. First, **we cannot use the Newtonian theory to prove or justify the results of general relativity**. Of course, one can perform some heuristic considerations founded on the Newtonian theory, but these are only heuristic considerations and not the proofs in the strict sense.

Moreover, **any interpretation of the results obtained must be done within the general relativity formalism only.** In particular this concerns the interpretation of coordinates and observables, what is almost always an involved matter [4,5,6].

To give an analogy of that situation in quantum mechanics we can cite the case of Ehrenfest equations [7]. They resemble very much the respective equations from Newtonian dynamics, but to prove the Ehrenfest equations we employ the formalism of quantum and not of classical physics. The same can be said about Bohr's assumptions on the hydrogen atom. Nowadays, they play, so to say, educative role only and the atom is considered within the formalism of quantum mechanics.

If all that is true I cannot agree with the words written in the abstract of the paper by J. Czerniawski: " A strict derivation of the Schwrzschild metric, based solely on Newton's law of free fall and the equivalence principle is presented. In the light of it, regarding Schwarzschild's coordinates as representing the point of view of a distant observer resting relative to the source of a centrally symmetric gravitational field, proves illegitimate".

The "strict derivation" in my opinion is simply a heuristic derivation which is a mixture of Newton's laws (see Eqs. (1) and (4)) and Einstein's laws (see (2)). What is the meaning of the coordinates: r, ρ and τ? It can be explained only with the use of Schwarzschid's metric and by no means one should refer to Newtonian laws.

The assumptions (1) and (4) seem to be *ad hoc* because if one would find, for example, the Kerr solution one can expect that another assumptions should be made.

Therefore, I am not convinced that anything is wrong with Schwarzschild's coordinates.

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