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EVOLUTION OF SPACE-TIME STRUCTURES

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(Received 16 December 2005; accepted 24 December 2005)

Abstract

Reconstructing former and current space-time theories in terms of modern differential geometry discloses a logic inherent in the evolutionary chain of physical space-time theories. By appealing to this logic, it is argued that the next possible step in the evolution of physics should consist in constructing a theory based on the concept of a groupoid over the frame bundle over space-time.

1 Introduction

The aim of this paper is twofold: to present the evolution of space-time theories and to guess the logically next step in this evolution. To do so we reconstruct, in terms of modern differential geometry, space-time theories starting from Aristotelian dynamics to Einstein's theory of relativity, and even further on – to the post-Einstein search for a unification of relativity and quanta. This is, of course, a strong stylization of the real history of science, but it nicely shows a certain logic inherent in the scientific progress. The fact itself that such a reconstruction is possible testifies to a certain logic inherent in the evolution of science. We shall try to read from this logic indications concerning the next step in the progress of our understanding of the world.

The reconstruction of physical space-time theories in terms of modern geometric tools is not new. It has been explored by many authors (see Bibliographical Note at the end of the paper). The second goal of the present study—to foresee the next logical step in the chain of physical theories—refers us back to the program of the present author and his coworkers to elaborate a noncommutative model unifying general relativity with quantum mechanics [2, 3, 4, 5]. Only when the first stages of this model had been worked out, we have realized that it well fits into the logic of the evolution of space-time structures. At the present state of the model elaboration we do not claim that it is the one so intensely looked for by theoreticians; we only show that, from the conceptual point of view, it nicely inscribes itself into the logic of physical progress.

2 Dynamics and Geometry

Dynamical physical theories investigate motions of bodies under the influence of forces. The first thing every dynamical theory has to establish is a criterion allowing one to decide whether any force has appeared. Usually, this is done by defining the so-called free motions, i.e., motions that require no forces. This can be thought of as a "standard of motion". A deviation from a free motion witnesses to the appearance of a force. To establish such a standard of motion is a task of the first law of dynamics. In many physical theories it is (often tacitly) assumed that motions under no influence of forces are determined by the geometric structure of space-time. When this is the case, the first law of dynamics contains information of the space-time structure and, consequently, suitably interpreted geometry becomes a part of physics, and as such it can be subject to empirical verification. It is enough to reconstruct old dynamical theories in modern mathematical terms to see that this machinery did work almost from the very beginning. This is, of course, a strong stylization of the history of science, but it enables us to see a logic of the evolution of physical theories which otherwise would remain hidden from our eyes.

3 Aristotelian Space-Time

In his 7th book of Physics, Aristotle tried to formulate a complete dynamics, he even attempted at doing this in a quantitative manner. His **first law** can be expressed in the following way: *There exists natural rest*, or, in analogy with Newton's first law: *A body, non acted upon by any force, remains in the absolute rest*.

Let us see how this law implies the structure of space-time. To begin with, let us assume that we can combine the Aristotelian space and the Aristotelian time into a 4-dimensional flat space-time manifold; we will denote it by A. This is quite natural assumption. The absolute rest enables us to meaningfully speak about the same place at different times. Indeed, a body in a state of absolute rest always remains at the same place. This fact determines the projection

$$pr_S : A \to S$$

from the 4-dimensional space-time manifold A into a 3-dimensional manifold S representing *absolute space*. The projection pr_S is equivalent to the existence of absolute space.

Two simultaneous events in the Aristotelian space-time (even if they are distant from each other) are simultaneous for every observer. This defines the projection

$$pr_T : A \to T$$

from space-time A into the "axis of absolute time" T. The existence of this projection is equivalent to the existence of the Aristotelian *absolute time*.

The existence of these two projections, pr_S and pr_T, implies that the Aristotelian space-time A is the Cartesian product of absolute time T and absolute space S, $A = T \times S$ (Fig. 1).

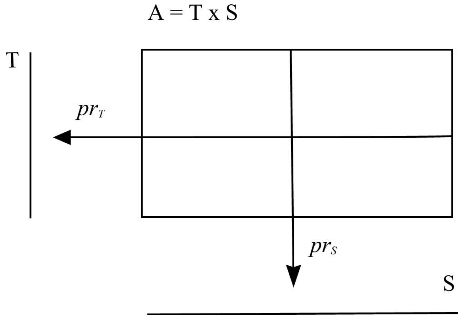

Figure 1: Aristotelian space-time A has a product structure.

We can briefly say that the Aristotelian space-time is a bundle (A, pr_S, S) over space S with the projection pr_S. Since $A = T \times S$, the bundle is trivial.

In Aristotle's view, the appearance of absolute velocity with respect to absolute space, testifies to the action of a force. This force is proportional to the velocity it produces. The above statement can be regarded as the **second law** of the Aristotelian dynamics. In modern notation we would write

$$\mathbf{F} = m \cdot \mathbf{v},$$

where $\mathbf{F}$ is a force, $\mathbf{v}$ – velocity, and m – the coefficient of proportionality which we call mass. We should notice that if $\mathbf{F} = 0$ then $\mathbf{v}$ must be zero as well (since mass cannot disappear), and the body is in the state of absolute rest. This does not mean, however, that the first law of dynamics is superfluous. Without the first law, which establishes the standard of motion, the second law would be meaningless.

Again, we should emphasize that this is a strong stylization of the authentic Aristotle's views. Although from his writings the above equation can be reconstructed, he was lacking the correct concepts of force and mass. Instead of force he used terms such as "moving factor", and instead of mass he spoke about "body". The difference is essential. The correct physical concepts must be defined operationally, i.e., they must correspond to measurable quantities (quantities that can be expressed in numbers). We can measure force and mass, but we cannot measure a "moving factor" or a "body". A long evolution of science was necessary to arrive at the correct operational concepts. Modern physics was born together with them.

4 Space-Time of Classical Mechanics without Gravity

Gravitational field introduces essentially new elements into the structure of space-time. Therefore, we should distinguish space-time of classical mechanics with and without gravity. Newton believed that space and time, as they appear in his mechanical theory, are absolute. In other words, he ascribed to them Aristotelian properties (he tacitly presupposed the existence of projections pr_T and pr_S). He was wrong. The first law of the Newtonian dynamics: *A body, non acted upon by any force, remains in the state of uniform motion* introduces a full democracy among all inertial reference frames. The standard of absolute rest disappears from Newton's physics: all reference frames, moving uniformly with respect to each other, are on equal footing. This introduces the "standard of motion". Being at rest means something different for every inertial observer; every such observer determines the meaning of "being at rest" with respect to his own inertial reference frame. In other words, the universally valid projection pr_S disappears; it is replaced by many "private" projections, each for every inertial observer. The consequence of this is that the Newtonian space-time N cannot be presented as a Cartesian product $T \times S$. However, the absolute simultaneity still exists in the Newtonian mechanics which admits physical signals propagating instantaneously (with infinite velocity). By using such signals we can establish simultaneity of events even if they are arbitrarily distant from each other. This leaves intact the projection $pr_T : N \to T$ from the Newtonian space-time N to the "time axis" T. Therefore, absolute time do exist in classical mechanics. It is interesting to notice that Newton misinterpreted his own first law of dynamics as implying the existence of absolute space.

The second law of the Newtonian dynamics says that a force $\mathbf{F}$ is needed to change the velocity of a body, i.e., to give it an acceleration

a. The measure of resistance against this accelerating force is called mass, denoted by m. This law is expressed by the formula

$$\mathbf{F} = m \cdot \mathbf{a},$$

It is instructive to compare this formula with the one expressing the second law of the Aristotelian dynamics. The difference is seemingly very small: only velocity $\mathbf{v}$ is replaced by acceleration $\mathbf{a}$. In fact, this difference is of outermost importance. Now, it is acceleration that is absolute: the appearance of acceleration testifies to the reality of motion. This allows us to compare velocities at distant points of space-time. If two distant bodies move with no acceleration, their velocities should be represented by two parallel vectors. We express this fact by saying that space-time N of classical mechanics (without gravity) admits *teleparallelism*.

How to express this property geometrically? The natural way to do this is by constructing the *frame bundle* over the Newtonian space-time N. It consists of the following things:

1. the collection $F(N)$ of all local reference frames in N; $F(N)$ is called the *total space* of the frame bundle,

2. the projection $\pi : F(N) \to N$ ascribing to a given reference frame its attachment point in space-time N, and

3. the group G of transformations which allows one to go from one reference frame to another reference frame. In the present case, G is the Galileo group, and the frame bundle is called the *Galileo frame bundle*.

Space-time N admits teleparallelism if $F(N)$ can be represented as the Cartesian product $F(N) = N \times G$. Indeed, the set $N \times \{g\}$, where g is a fixed element of the group G, gives us all reference frames in space-time N parallel to each other.

We thus have the following situation: The Cartesian product structure of the Newtonian space-time N has disappeared (as compared with the Aristotelian space-time A), but it has reappeared in the total space of the Galilean frame bundle over N (Fig. 2).

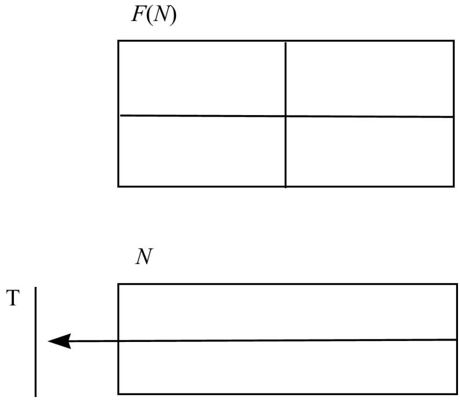

Figure 2: Space-time N of classical mechanics without gravity has no product structure, but admits absolute time. Frame bundle $F(N)$ over N has a product structure.

5 Space-Time of Classical Mechanics with Gravity

The Galileo principle asserts that all bodies fall with the same acceleration in a given gravitational field. This allows us to treat freely falling reference frames as locally non-accelerating, i.e., reference frames in which laws of the Newtonian dynamics are locally valid. The presence of gravity causes focusing of all curves representing histories of test particles. For example, the histories of freely falling bodies in the Earth gravitational field should, in principle, meet at the Earth's center. This liquidates teleparallelism. Now, the parallel transport of a vector depends on the path along which it is made. Such a situation occurs when we try to move paralelly a vector on a curved surface, e.g., on a sphere. This means that the total space $F(N)$ of the Galileo frame bundle over space-time N has curvature, and consequently it cannot be presented as a Cartesian product of space-time N and the Galileo group G, i.e., $F(N) \neq N \times G$. It is a remarkable fact that also Newton's gravity manifests itself as a curvature; but, in contrast to general relativity, it is not the curvature of space-time but rather the curvature of the total space of the frame bundle over space-time that should be identified with the gravitational field. N still admits the projection pr_T (Fig. 3).

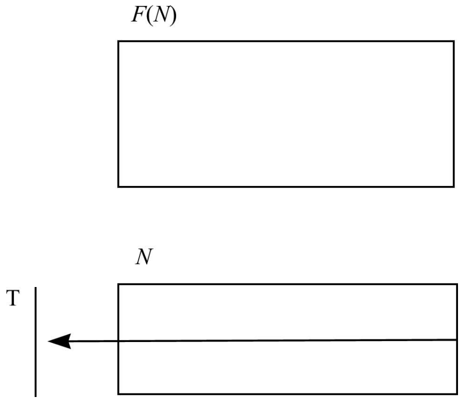

Figure 3: Space-time N of classical mechanics with gravity. Frame bundle $F(N)$ over N has no product structure. Absolute time still exists.

6 Space-Time of Special and General Theories of Relativity

From the elementary course of special relativity it is clear that its space-time M admits neither absolute space (projection pr_S), nor absolute time (projection pr_T). Therefore, space-time M has no Cartesian product structure. However, this space-time being flat admits teleparallelism. This implies that the total space of the Lorentz frame bundle over space-time M has the Cartesian product structure, $F(M) = M \times G$ where G is now the Lorentz group (Fig. 4). Space-time M of general relativity has curvature, and consequently there is no teleparallelism. This in turn implies that the total space of the Lorentz frame bundle over M cannot be presented as the Cartesian product of space-time and the corresponding group. As we have seen above, the same situation occurs in classical mechanics with gravity. However, the essential difference between the classical mechanics with gravity and general relativity consists in the fact that space-time of

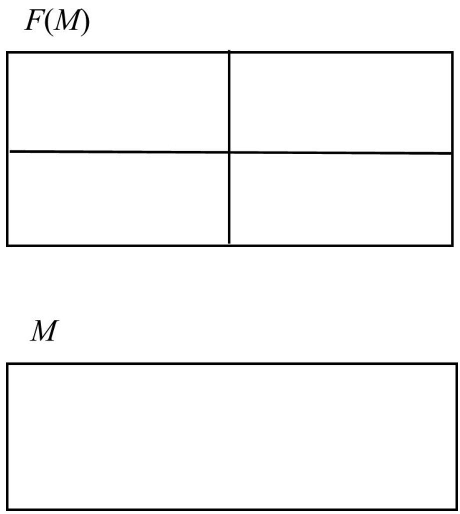

Figure 4: Space-time M of special relativity. No absolute space, no absolute time. Frame bundle $F(M)$ over M has a product structure.

the latter is equipped with the Lorentz metric. This implies that the structural group of the corresponding frame bundle is the Lorentz group.

The new element is that in the total space $F(F(M))$ of the principal fiber bundle over $F(M)$ the Cartesian product structure reappears (i.e., $F(F(M))$ can be presented as a Cartesian product, Fig. 5). This is the consequence of the fact that, in the case of general relativity, the frame bundle over space-time has a linear connection. However, if we take into account the logic of evolution of space-time structures, as displayed in the present essay, we could speculate that this new structure points toward the future evolution of physical theories. It was Trautman [7, 8] who speculated that the future generalization of general relativity could consist in depriving the bundle over $F(M)$ of its Cartesian product structure. Indeed, there exists a model unifying general relativity and quantum mechanics that explores the product structure $F(F(M))$, and introduces a curvature on it. We shall briefly sketch main architectonic features of this model.

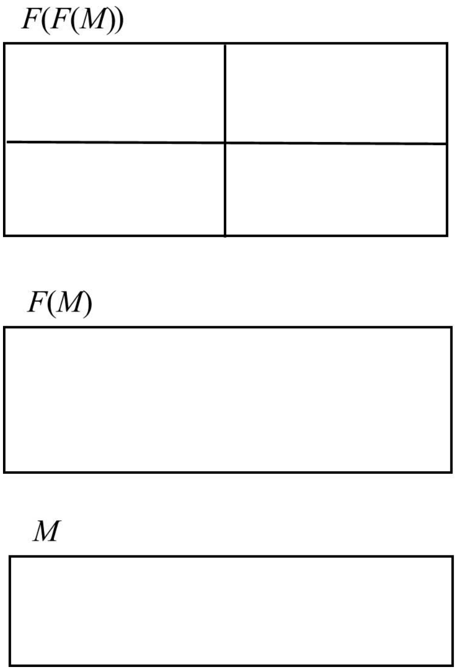

Figure 5: Space-time M of general relativity. Frame bundle $F(M)$ over space-time M has no product structure. Frame bundle $F(F(M))$ over $F(M)$ has a product structure.

7 Groupoid Approach to the Unification of General Relativity and Quantum Mechanics

As it is well known, there is an urgent need to unify general relativity with quantum mechanics, and one of the strategies to reach this goal is to generalize the standard space time geometry so as to make enough "degrees of freedom" to incorporate quantum methods. One of the promising approaches in this direction is the idea of changing the usual space-time geometry into its noncommutative counterpart, and the typical method of doing this leads to the construction of the Cartesian product $\Gamma = F(M) \times G$ where G is the structural group of the frame bundle $\pi_M : F(M) \to M$ (Fig. 6). It turns out that a suit-

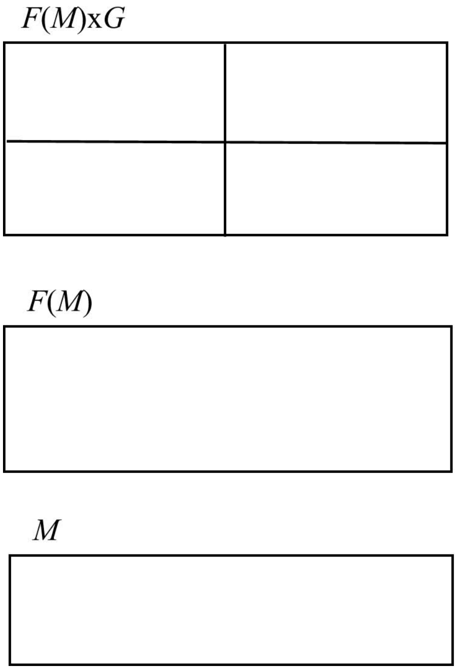

Figure 6: Groupoid $\Gamma = F(M) \times G$ over the frame bundle $F(M)$ over M.

ably defined noncommutative algebra algebra $\mathcal{A}$ associated with Γ (see below) is a noncommutative generalization of the (commutative) algebra $C^\infty(M)$ of smooth functions on space-time M.

Owing to the fact that the group G acts on $F(M)$ (to the right), $F(M) \times G \to F(M)$, the space $\Gamma = F(M) \times G$ has a rich structure known as a groupoid structure. Two elements of Γ, $\gamma_1 = (p_1, g_1)$ and $\gamma_2 = (p_2, g_2)$, can be composed with each other, provided that $p_2 = p_1 g_1$, to give $\gamma_1 \circ \gamma_2 = (p_1, g_1 g_2)$. And the inverse of $\gamma = (p, g)$ is $\gamma^{-1} = (pg, g^{-1})$. In fact, any groupoid can be regarded as a "group with many units" which, in our case, are of the form (p, e) where e is the unit of G. Owing to this fact, not all groupoid elements can be composed with each other (for the precise groupoid definition see, for instance, [6]). As a "generalized group" Γ can be thought of as a symmetry space of the proposed model.

Let $C_c^\infty(\Gamma, \mathbf{C})$ be the algebra of smooth, complex valued, compactly supported functions on Γ with the usual pointwise multiplication. By replacing this multiplication with a convolution

$$(f_1 * f_2)(\gamma) = \int_{\Gamma_p} f_1(\gamma_1) f_2(\gamma_1^{-1}\gamma) d\gamma_1,$$

where $\gamma = (p, g)$ and Γ_p denotes all elements of Γ that begin at p, one obtains the noncommutative algebra $\mathcal{A}$ associated with the groupoid Γ. It is this algebra that can be regarded as a noncommutative generalization of the algebra $C^\infty(M)$ (for details see [5]).

Let us notice that the standard way of making a noncommutative space out of a usual one (e.g. out of a smooth manifold) M is to present M as a quotient space $M = N/R$, where N is a "larger space" and R is an equivalence relation. This procedure naturally leads to the groupoid Γ over $F(M)$ [1, p. 86]. In our case, $N = F(M)$, and two elements, $p, q \in F(M)$ are regarded as R-equivalent if there exists $g \in G$ such that $q = pg$ This is why the right action of G on $F(M)$ is essential to make $\Gamma = F(M) \times G$ a groupoid.

The fact that the algebra $\mathcal{A}$ is noncommutative radically changes the perspective. Geometry based on such an algebra is, in principle, nonlocal, i.e., no local concepts, such as those of a point and its neighborhood, have any meaning in it. Only by analogy with the commutative case one claims that the algebra $\mathcal{A}$ *virtually* describes a certain "noncommutative space" (this is a standard procedure in noncommutative geometry). In this sense, from now on, our groupoid Γ is, in fact, a noncommutative, nonlocal space. However, for simplicity we shall continue to speak of it as of the usual space.

In terms of the algebra $\mathcal{A}$ one develops a geometry on the groupoid Γ, in principle, in the same manner as one usually does on a manifold. In the franework of this geometry, the noncommutative Riemann curvature and the Ricci operator can be defined. With the help of these quantities one can construct a noncommutative version of general relativity (for details see [2, 5].)

And what about quantum mechanics? Happily enough, the algebra $\mathcal{A}$ admits the so-called *regular representation*

$$\pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}^p)$$

in a Hilbert space $\mathcal{H}^p$, for any $p \in F(M)$, where $\mathcal{B}(\mathcal{H}^p)$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}^p = L^2(\Gamma_p)$ with Γ_p denoting the fiber of Γ over p. The representation π_p is defined in the following way. For $f \in \mathcal{A}$ one puts

$$(\pi_p(f))(\psi) = f * \psi$$

where $\psi \in \mathcal{H}^p$. With the help of this representation one constructs the quantum sector of our model [2, 5].

Our model has interesting structural properties. Being noncommutative it exhibits strong nonlocal aspects. Such local concepts as those of point and its neighborhoods are meaningless in it. In consequence, on the Planck level (on which the model is supposed to be valid) the concepts of space and time, as composed of points and point-like instants, have no meaning. The usual space and the usual time emerge only when one goes to lower energy levels. In spite of the fact that on the fundamental level there is no time in the usual sense, there can be defined a generalized dynamics [3]. This dynamics depends on the state (defined as a functional on the algebra $\mathcal{A}$) which also turns out to define a probabilistic measure (in a generalized sense). In this way, two so far distinct concepts, dynamics and probability, are unified below the Planck level [3].

As we can see, the groupoid unification of general relativity and quantum mechanics seems to nicely fit, at least from the conceptual point of view, into the evolutionary chain of space-time theories.

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Comment on EVOLUTION OF SPACE-TIME STRUCTURES

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The article by M. Heller is very interesting for many reasons. In spite of this we should however keep in mind that it is not a prediction but a forecast only. Therefore, the future models of spacetime may be quite different from those described by the Author. I personally do not believe that mathematics can uniquely predict the future shape of physics. It never happened in the past and unlikely it will happen in the future.

A real progress of physics must be based on really new ingredients. Often it happened that for these new ingredients some place must first be opened by removing old concepts which were then replaced by new ones. For instance, for quantum mechanics W. Heisenberg removed the notion of classical trajectories while for relativity A. Einstein removed the absolute time.

I believe that for future and revolutionary new models of space-time we should first remove from our considerations the language of standard mathematical analysis and replace it by some kind of non-standard analysis [1] in which standard numbers will be surrounded by specific "quantum monads". The monads should represent all quantum mechanical quantities needed to implement all higher quantum symmetries seen in Nature. Classical spacetime coordinates are not sufficient to treat quantum mechanical higher symmetries. These symmetries are so important and of primary meaning that on fundamental level they should play the some role as all spacetime symmetries do in classical physics. In standard approaches higher symmetries are realized in fibre bundles over spacetime while in the models I would most welcome all symmetries, both higher and of classical spacetime character, should be implemented on the level of non-standard spacetime with new non-standard notion of elementary events and non-standard spacetime coordinates [2].

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