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THE SHAPE INVARIANCE CONDITION

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Abstract

The shape invariance condition is well known in supersymmetric quantum mechanics. This condition is sufficient for the solvability of eigenvalue problem of the Schroedinger equation. A method is described, with which the shape invariance condition can be derived by means of expansion of the SUSY superpotentials in terms of the continued fractions.

The concept of a shape invariant potential within the structure of SUSY QM was introduced by Gendenshtein [2]. A potential is said to be shape invariant if its SUSY partner potential has the same spatial dependence as the original one, but they differ in the parameters. This condition is sufficient for the eigenvalue equation to be analytically soluble and, furthermore, that the eigenvalues and eigenfunctions can be obtained by a simple algebraic procedure [3,4,5]. The purpose of this paper is to show, that the shape invariance condition is related with expansion of the SUSY superpotentials in terms of the continued fractions [1].

Suppose $\Psi_0(x)$ is the solution to equation

$$A\Psi_0(x) = 0, \tag{1}$$

where

$$A = \frac{1}{\sqrt{2}} ip + f_0(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} + f_0(x), \tag{2}$$

$f_0(x)$ is a real function (here we take $\hbar = m = 1$). Thus we have

$$A^+ A\Psi_0 = 0, \qquad f(x) A\Psi_0 = 0, \tag{3}$$

where $f(x)$ is also a real function. Hence

$$[A^+ A + 2f(x)A]\Psi_0 = 0. \tag{4}$$

This equation is obtained by recognizing that once we satisfy (1), then we automatically have (4). Factorizing

$$\Psi_0(x) = h(x)\psi_0(x) \tag{5}$$

means that eq. (4) takes the form

$$[(A^+ - g(x))(A + g(x)) + 2f(x)(A + g(x))]\psi_0 = 0. \tag{6}$$

Substituting

$$g(x) = f(x), \tag{7}$$

we find

$$B^+ B\psi_0(x) = 0, \tag{8}$$

where

$$B = A + f(x), \qquad B^+ = A^+ + f(x), \qquad B\psi_0 = 0. \tag{9}$$

We thus have shown that a known ground state wave function (2) leads us to another one which satisfies (8). Using eqs.$A\Psi_0 = 0, B\psi_0 = 0$ and eliminating function $h(x)$, it is easy to show that there is a relationship between them in the form

$$\psi_0(x) = e^{-\sqrt{2}\int^x f(y)dy}\Psi_0(x). \tag{10}$$

Choosing

$$f(x) = -f_0(x) + W_0(x), \tag{11}$$

we have

$$B = \frac{1}{\sqrt{2}}ip + W_0(x), \tag{12}$$

$$\psi_0(x) = e^{-\sqrt{2}\int^x W_0(y)dy} \tag{13}$$

which is the simpler form of (9) and (10). According to (12) we can show that

$$B^+B\psi_0(x) = (-\tfrac{1}{\sqrt{2}}ip + W_0(x))(\tfrac{1}{\sqrt{2}}ip + W_0(x))\psi_0(x) =$$

$$= (\tfrac{1}{2}p^2 + W_0^2 - \tfrac{1}{\sqrt{2}}W_0')\psi_0(x) = 0.$$

Hence we obtain the Schroedinger equation with the potential

$$V(x) = W_0^2(x) - \frac{1}{\sqrt{2}}W_0'(x), \tag{14}$$

the eigenfunction $\psi_0(x)$ and eigenvalue equal to zero. The quantity $W_0(x)$ is generally referred to as the superpotential in SUSY QM literature.

Now, we can repeat the above procedure. According to (9) and (14) we have

$$[B^+B + 2W_1(x)B]\psi_0(x) = 0. \tag{15}$$

Substituting $\psi_0(x)$ given by

$$\psi_0(x) = h_1(x)\psi_1(x), \tag{16}$$

we find

$$(B^+ + W_1(x))(B + W_1(x))\psi_1(x) = 0, \tag{17}$$

where

$$(B + W_1(x))\psi_1(x) = 0, \tag{18}$$

and

$$h_1(x) = e^{\sqrt{2}\int^x W_1(y)dy}. \tag{19}$$

Thus

$$\psi_1(x) = e^{-\sqrt{2}\int^x W_1(y)dy}\psi_0(x) = e^{-\sqrt{2}\int(W_0+W_1)dx}. \tag{20}$$

By means of (12) the eigenvalue equation (18) can also be written in the form

$$[\frac{1}{2}p^2 + (W_0 + W_1)^2 - \frac{1}{\sqrt{2}}(W_0 + W_1')]\psi_1(x) = 0. \tag{21}$$

We wish (22) to be the Schroedinger equation with the same potential (15), but with the nonzero energy eigenvalue. Thus we must choose $W_1 \neq W_0$ in the form which leave the potential (15) unchanged. Let

$$\widetilde{W_1} = W_0 + W_1, \tag{22}$$

and suppose that $\widetilde{W_1}$ has the form

$$\widetilde{W_1} = W_0(a_1, x) + \alpha\frac{1}{W_0(a_1, x) + W_0(a_2, x)}, \tag{23}$$

where α is a real number and a_2 is a new set of parameters uniquely determined from the old set a_1. Then

$$\widetilde{W_1}^2 - \tfrac{1}{\sqrt{2}}\widetilde{W_1}' =$$

$$= W_0^2(a_1, x) - \tfrac{1}{\sqrt{2}}W_0'(a_1, x) +$$

$$+\frac{\alpha^2 + 2\alpha W_0(a_1,x)[W_0(a_1,x)+W_0(a_2,x)] + \frac{\alpha}{\sqrt{2}}[W_0(a_1,x)+W_0(a_2,x)]'}{[W_0(a_1,x)+W_0(a_2,x)]^2}.$$

This is the potential (15) if

$$\alpha + 2W_0(a_1, x)[W_0(a_1, x) + W_0(a_2, x)] + \tfrac{1}{\sqrt{2}}[W_0(a_1, x) + W_0(a_2, x)]' =$$

$$= [W_0(a_1, x) + W_0(a_2, x)]^2$$

or, after simply calculations

$$W_0^2(a_2,x) - \frac{1}{\sqrt{2}} W_0'(a_2,x) = W_0^2(a_1,x) + \frac{1}{\sqrt{2}} W_0'(a_1,x) + \alpha. \quad (24)$$

The last equation is the well known shape invariance condition.

We should also notice, that by applying B^+ to the ground state, we generate the wavefunction for the first excited state with energy eigenvalue equal to $-\alpha$. Using (13),(21) and (27), we find

$$B^+\psi_0(a_2,x) = \psi_1(a_1,x). \quad (25)$$

If we want to obtain the next excited state, then we should repeat the above procedure. Substituting, with the different parameters, $\widetilde{W_1}$ instead of W_0 in (24) and expressing the result in terms of W_0, we find

$$\widetilde{W_2} = W_0 + \frac{1}{W_0 + \frac{1}{W_0 + W_0}}, \quad (26)$$

where the parameters are dropped. Thus the n-th superpotential which provides us with n-th excited state can be written, without parameters, in the form

$$\widetilde{W_n} = W_0 + \frac{1}{W_0 + \frac{1}{W_0 + \frac{1}{W_0 + \dots}}}. \quad (27)$$

what means that W_n is expressed as a continued fraction. Moreover, we should notice that each W_n leads us to the shape invariance condition, although with the different parameters.

We now consider one application of these results. The simplest example is the one-dimensional oscillator, for which the superpotential W_0 is given by

$$W_0(x) = \frac{x}{\sqrt{2}}. \quad (28)$$

Substituting this in (13) and (15) we obtain the potential

$$V(x) = \frac{x^2}{2} - \frac{1}{2} \quad (29)$$

and eigenfunction

$$\psi_0(x) = e^{-\frac{1}{2}x^2}. \quad (30)$$

According to (24) and (25)

$$\widetilde{W_1} = \frac{x}{\sqrt{2}} + \frac{\sqrt{2}\alpha_1}{x}, \tag{31}$$

hence

$$\alpha_1 = -\frac{1}{2}, \tag{32}$$

then the eigenvalue is $-\alpha = 1$. From (21) we have

$$\psi_1(x) = xe^{-\frac{1}{2}x^2}. \tag{33}$$

To determine the next excited state we have

$$\widetilde{W_2} = \frac{x}{\sqrt{2}} + \frac{\alpha_2}{\frac{x}{\sqrt{2}} + \frac{\sqrt{2}\alpha_3}{x}}. \tag{34}$$

Similar calculations lead us to

$$\psi_2 = (2x^2 - 1)e^{-\frac{1}{2}x^2}, \tag{35}$$

and the eigenvalue

$$-\alpha = 2. \tag{36}$$

This procedure can be repeated for all of the eigenvalues and eigenfunctions. Notice that all of the eigenfunctions presented here are without the normalization constant.

Concluding remarks

By using the first order differential equation, we have shown how to obtain the Schroedinger equation with the potential expressed in terms of the superpotential W_0. The subsequent states can be calculated by means of the expansion of W_0 in terms of the continued fractions. As a consequence we obtain the shape invariance condition. We must emphasize that there is a possibility to use this method to others physically interested second order differential equations. This topic seems to be interesting for future calculations.

References

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Comment on THE SHAPE INVARIANCE CONDITION

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The aim of supersymmetric quantum mechanics, nowadays called SUSY QM, is to find new solutions of the Schroedinger equation.The shape invariant condition is a basic tool which enables us, in the framework of the SUSY QM, to construct potentials for which solutions to the Schroedinger equation may be found explicitly. The author of the commented paper shows that formulating the Schroedinger equation in the Riccati form, *i.e.* in the form for which the potential may be written down as

$$V(x) + \text{const} = W^2(x) - \frac{1}{\sqrt{2}} W'(x), \tag{1}$$

we are able to construct chains of solvable potentials given as continued fractions generated by the solution of the problem we began with. From the opposite side, if two solutions are joined by a continued fraction - type relation then the shape invariance condition appears to be a straighforward consequence of this fact which suggests much deeper meaning of this relation to the solvability of the

Schroedinger equation. The presented method is also enough general to be useful and effective tool for solving other equations of quantum physics.

The author illustrated his considerations only on the simplest example of the harmonic oscillator and it would be very interesting if he adopted his approach also to other examples of 1-dimensional Schroedinger equation and if he compared his results with another methods widely used in the SUSY QM.

I also would like to emphasize that in his considerations the author successfully uses mathematical methods which, although interesting and longtime known for their effectiveness, are now almost completely forgotten. The fact that such methods may be applied in analysis of contemporary and extensively studied physical problems should forced us to pay more attention to the XIX-th century mathematics (which in fact stimulated and determined development of the XX-th century physics) and to think about its comeback to modern physics.