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ON UNCERTAINTY RELATIONS AND INTERFERENCE IN QUANTUM AND CLASSICAL MECHANICS

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It is demonstrated that uncertainty relations (UR), being the general property of functions, do not prevent particles in quantum mechanics (QM) to have precisely and simultaneously defined position and momentum. Classical interpretation of QM, different from Bohmian mechanics is proposed.

1 Introduction

Nowadays the role of UR in QM is exaggerated. The relation

$$\langle (x-x_0)^2\rangle\langle (k-k_0)^2\rangle \geq 1/4 \tag{1}$$

is valid for any function $f(x)$, if we define

$$x_0=\langle x\rangle=\int x\,|f(x)|^2\,dx/\int|f(x)|^2\,dx,$$

$$k_0=\langle k\rangle=\int k\,|f_F(k)|^2\,dk/\int|f_F(k)|^2\,dk,$$

$$\langle (x-x_0)^2\rangle=\int(x-x_0)^2\,|f(x)|^2\,dx/\int|f(x)|^2\,dx,$$

$$\langle (k-k_0)^2\rangle=\int(k-k_0)^2\,|f_F(k)|^2\,dk/\int|f_F(k)|^2\,dk,$$

where

$$f_F(k)=\int f(x)\exp(-ikx)dx/\sqrt{2\pi}$$

is the Fourier image of the function $f(x)$, and in the next section we remind to readers the proof of it. Thus the eq. (1) is applicable equally well to wave functions in QM and to trajectories $\boldsymbol{r}(t)$ in classical mechanics.

It is usually claimed that UR (1) prohibits for particles in QM to have position and momentum defined precisely simultaneously. We show in section 3 that it is not true. The problem here is in definition of the position and momentum. In particular, the position for an extended object, such as wave function, is the matter of definition, like position of a classical ball.

In the fourth section we consider the widely discussed two slits experiment, and show that interference is not an exclusive property of a wave mechanics, where you don't know, which is the slit the particle goes through, but it also can be observed in classical mechanics, where you know exactly, which is the slit the particle goes through.

After that, in fifth section we propose a system of two nonlinear classical equations for particle trajectory and its field that possibly can replace Schrödinger equation and preserve causality. We show that this approach is completely different from the so called "Bohmian mechanics", because the last one is only a method for solution of the Schrödinger equation. In conclusion we resume our main points.

2 Mathematical proof of UR

Take an arbitrary function $f(x)$ of finite range, and its Fourier image

$$f_F(k) = \int_{-\infty}^{+\infty} f(x) \exp(-ikx) dx/\sqrt{2\pi}. \tag{2}$$

The functions, because of finite range, have the norm

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |f_F(k)|^2 dk \equiv N < \infty. \tag{3}$$

The gravity centers of these functions are

$$x_0 = \frac{1}{N} \int x|f(x)|^2 dx, \qquad k_0 = \frac{1}{N} \int k|f_F(k)|^2 dk. \tag{4}$$

With the function $f(x)$ we can calculate the integral

$$\frac{1}{N} \int |(\alpha(x - x_0) + d/dx - ik_0)f(x)|^2 dx = \alpha^2 A + \alpha B + C, \tag{5}$$

which is nonnegative for arbitrary α. Here we denoted

$$A = \frac{1}{N} \int (x - x_0)^2 |f(x)|^2 dx = \frac{1}{N} \int (x^2 - x_0^2)|f(x)|^2 dx \equiv \langle (\Delta x)^2 \rangle, \tag{6}$$

$$B = \frac{1}{N}\int x\frac{d}{dx}|f(x)|^2 dx = \frac{1}{N}\int \frac{d}{dx}(x|f(x)|^2) - \frac{1}{N}\int |f(x)|^2 = \quad (7)$$

$$-\frac{1}{N}\int |f(x)|^2 = -1,$$

$$C = \frac{1}{N}\int (k-k_0)^2 |f_F(k)|^2 dk = \frac{1}{N}\int (k^2 - k_0^2)|f_F(k)|^2 dk \equiv \langle(\Delta k)^2\rangle. \quad (8)$$

Since eq. (5) is nonnegative for all α, we have

$$\alpha^2\langle(\Delta x)^2\rangle - \alpha + \langle(\Delta k)^2\rangle \geq 0.$$

This inequality can be satisfied only if

$$\langle(\Delta k)^2\rangle\langle(\Delta x)^2\rangle \geq \frac{1}{4}, \quad (9)$$

which is just the UR used in QM, however it is valid for arbitrary function $f(x)$, and is not specific for QM.

Therefore UR-s contain nothing, specific to QM.

3 Simultaneous definition of position and momentum

For every extended object position point is the matter of definition. In classical electrodynamics position of the electron is commonly defined as the singularity point of its field. In classical mechanics position point of, say, a ball can be bound to its center or to a point, where an observer touches it.

In QM, if a free particle is described by a plane wave, all the space points are equivalent for definition of its position. However the free particle can be described by the nonsingular de Broglie's wave-packet [1, 2, 3]:

$$\psi(\boldsymbol{r},t) = j_0(s|\boldsymbol{r}-\boldsymbol{v}t|)\exp(i\boldsymbol{v}\boldsymbol{r} - i\omega t), \quad (10)$$

in which $j_0(x)$ is the spherical Bessel function, s is a parameter related to the width of the function, and

$$\omega = (v^2 + s^2)/2. \tag{11}$$

Here we use unities $\hbar = m = 1$, so velocity v of the particle is the same as its wave-vector k. The function (11) is a solution of the Schrödinger equation:

$$(i\partial_t + \Delta/2)\psi = 0,$$

and it has a distinguished maximum point $|\psi(\boldsymbol{r} = \boldsymbol{v}t, t)|^2$, which we can define as the position. And we immediately see that the position point is precisely defined and has the precisely defined velocity $\boldsymbol{v}$ and momentum $\boldsymbol{p} = m\boldsymbol{v}$.

If one defines as the momentum the wave number $\boldsymbol{k}$ in the Fourier expansion, then his definition is absolutely not related to such a notion as velocity, or displacement per unit time, so it should not use this word to avoid confusion. The QM creates confusions because it uses the commonly defined words for absolutely different values.

4 Interference in classical mechanics

Let's consider the two slits experiment shown in fig.1. It is usually stated that particle goes through both slits in the screen, and transmitted parts of the particle wave function interfere on the observation screen, which is manifested by the interference pattern. However the interference can be found also in classical physics with particle going through one exactly specified slit.

Consider the same experiment with a classical electron, moving through one specified slit in the target screen, as is shown in fig. 2. Because of interaction of the electron's Coulomb field with the screen (take into account boundary condition on the screen and action of the reflected field on the electron), the electron's trajectory changes

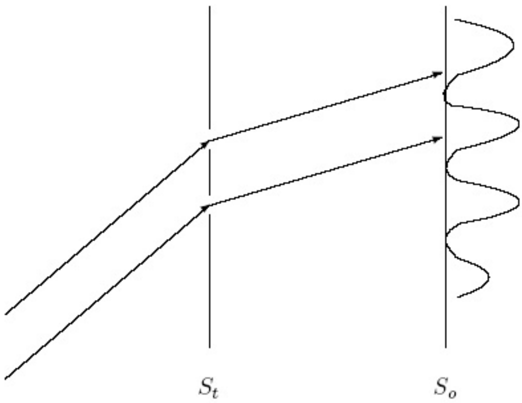

Figure 1: According to standard QM wave function of a particle transmitted through both slits in target screen S_t interferes after S_t and gives a diffraction pattern on observation screen S_o.

after the screen. Interaction of the electron's field with the screen S_t (the boundary condition) depends on the screen's structure. In particularly, it is different, when the screen contains one or two slits. It means that the direction of propagation of the electron after S_t depends on whether the second slit is opened or closed. Thus the second slit interferes the electron motion, even if the electron goes precisely through the same selected upper slit.

It is possible to predict the change of direction of the electron motion after the screen S_t in experiment shown in fig. 2, when the second slit is open or closed with the shutter sh. It is probably impossible with Coulomb field of the electron to predict the interference pattern on the screen S_o, shown in fig. 1, because there are no such a parameter as wave-length, however wavelength can appear, if we take into account relativistic retardation of the interaction of electron with its own field reflected from the screen S_t, or introduce a quan-

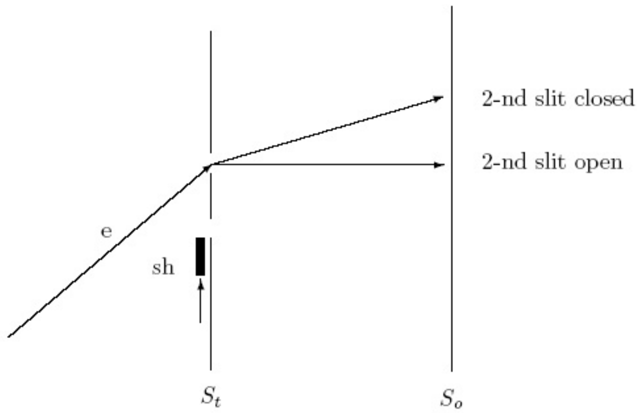

Figure 2: An experiment with classical electron going through the upper slit in the screen S_t. Because of interaction of the electron's field with the S_t its trajectory after the screen depends on whether the other slit is opened or not. It is an illustration of interference of two slits in classical physics.

tum of action. Indeed, we can suppose that the shift of the incident electron by distance l along its trajectory can affect the total field of the electron in presence of the screen S_t, and therefore the motion of electron itself only if $pl = h$. Just at this point the quantization can enter into the classical behavior, and give such a parameter as the wave-length. However, with particle velocity v light velocity c and slit width d we can introduce the wave-length like parameter dv/c, which also can give the interference pattern on the observation screen S_o.

5 Nonlinear classical system of equations instead of QM

All the usual equations in mathematical physics can be sorted into two groups:

1. **Field equations** of the type
$$\hat{L}\psi(\boldsymbol{r}) = j(\boldsymbol{r}), \tag{12}$$
where $\hat{L}$ is an operator, which can be linear or nonlinear in field $\psi(\boldsymbol{r})$, and $j(\boldsymbol{r})$ is a source, which can depend on some particle trajectory $\boldsymbol{r}(t)$, and this trajectory is supposed to be fixed. As an example we can mention Maxwell equations with given currents.

2. **Trajectory equations** of the type
$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}(\boldsymbol{r}(t), t), \tag{13}$$
where the field of force $\boldsymbol{F}(\boldsymbol{r}, t)$ is fixed.

However, in the two slits experiment we have another type of the problem. We have the trajectory equation
$$\frac{d^2\boldsymbol{r}_p}{dt^2} = \boldsymbol{F}(\psi(\boldsymbol{r}_p(t), t)), \tag{14}$$
with the force $\boldsymbol{F}(\psi)$, which depends on the field ψ — a solution of the field equation
$$\hat{L}\psi(\boldsymbol{r}, t) = j(\boldsymbol{r}, \boldsymbol{r}_p(t)) \tag{15}$$
with the source, moving along yet unknown trajectory, defined by the equation (14).

Formally we can exclude $\psi = \hat{L}^{-1}j(\boldsymbol{r}, \boldsymbol{r}_p(t))$ from the equation (14), however then we obtain highly nonlinear equation for trajectory:
$$\frac{d^2\boldsymbol{r}_p}{dt^2} = \boldsymbol{F}(\hat{L}^{-1}j(\boldsymbol{r}_p(t), \boldsymbol{r}_p(t'))). \tag{16}$$

Solution of (16) or of the system (14,15) is a great **challenge for mathematicians**.

QM avoids solution of such a nonlinear system and replaces it by the linear Schrödinger equation, but the payment for that is loss of determinism and introduction of probabilities.

It is great! However it would be very interesting to try to solve such a nonlinear system, which can be easily formulated in classical electrodynamics.

5.1 The problem of classical electrodynamics

We have the Maxwell equation for 4-tensor $\boldsymbol{F}_{\mu\nu}$:

$$\partial_\mu F_{\mu\nu}(\boldsymbol{r},t)=\frac{4\pi}{c}eu_\nu\delta(\boldsymbol{r}-\boldsymbol{r}(t)),\quad \mu,\nu=0\div 3, \tag{17}$$

where u_ν is speed with components $u_0=c$, $\boldsymbol{u}_k=\boldsymbol{v}_k(t)$ for $k=1\div 3$. The functions $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ are not known and are to be determined from the other equation — the trajectory one:

$$m\frac{d\boldsymbol{v}(t)}{dt}=e\boldsymbol{E}(\boldsymbol{r},t)+\frac{e}{c}[\boldsymbol{v}(t)\boldsymbol{H}(\boldsymbol{r},t)],$$

where

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}(t)}{dt},$$

and electric and magnetic fields are the components of the 4-tensor $\boldsymbol{F}_{\mu\nu}$

$$\boldsymbol{E}_k(\boldsymbol{r},t)=F_{0k}(\boldsymbol{r},t),\quad \boldsymbol{H}_k=\epsilon_{ijk}F_{ij}(\boldsymbol{r},t).$$

They are created by field, reflected from the target screen, and the reflection is determined by boundary conditions for the field $F_{\mu\nu}$ on the screen surface. The screen can be an infinitely thin ideal conductor. Position of slits, their width and the distance between them can be arbitrary.

For the beginning it is sufficient to solve the non relativistic, pure Coulomb problem, and later one have to add relativistic retardation.

We want to note that this nonlinear system of equations has nothing to do with Bohmian mechanics. No quantum potential is introduced, and no Schrödinger equation is presupposed. In the next section we briefly review the Bohmian mechanics.

5.2 Bohmian mechanics and hydrodynamical interpretation

In the literature (see, for example, [4] and references there in) one can find interpretation of quantum mechanics in terms of classical trajectories, which is called Bohmian mechanics. However it is not a classical version, which replaces quantum mechanics, but only an alternative way of solving the Schrödinger equation. In Bohmian mechanics one finds solution of the Schrödinger equation, finds with it so called quantum potential, and with this quantum potential finds the trajectory.

The wave function ψ of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t)=\left[-\frac{\hbar^2}{2m}\Delta+V(\boldsymbol{r})\right]\psi, \tag{18}$$

can be represented as $\psi(\boldsymbol{r},t)=R(\boldsymbol{r},t)\exp(iS(\boldsymbol{r})/\hbar)$, where $R(\boldsymbol{r})=|\psi(\boldsymbol{r})|$, and $S(\boldsymbol{r},t)$ is a real function. Substitution of it into (18), and separation of real and imaginary parts of the equation gives two other equations [4]

$$\frac{\partial R^2}{\partial t}+\boldsymbol{\nabla}\left(R^2\frac{\boldsymbol{\nabla}S}{m}\right)=0, \tag{19}$$

$$\frac{\partial S}{\partial t}+\frac{(\boldsymbol{\nabla}S)^2}{2m}+V-\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2R}{R}=0. \tag{20}$$

Solution of these two equations is equivalent to solution of the single (18) equation. When you find R and S, you can calculate

$$Q(\boldsymbol{r},t)=\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2R}{R}, \tag{21}$$

which is called "quantum potential", and

$$\boldsymbol{v}(\boldsymbol{r},t)=\frac{\boldsymbol{\nabla} S(\boldsymbol{r},t)}{m} \qquad (22)$$

which is called speed. If one applies $\boldsymbol{\nabla}$ to Eq. (20) and uses definition (22), one obtains the equation

$$m\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}+m(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=-\boldsymbol{\nabla}(V(\boldsymbol{r})+Q(\boldsymbol{r},t)), \qquad (23)$$

which is equivalent to

$$\frac{d\boldsymbol{v}(\boldsymbol{r},t)}{dt}=-\boldsymbol{\nabla}(V(\boldsymbol{r})+Q(\boldsymbol{r},t)). \qquad (24)$$

However $\boldsymbol{v}$ is not equal to $\dot{\boldsymbol{r}}(t)$, because it is a field, which depends on both $\boldsymbol{r}$ and t.

Now, if you have already solved Eq. (18), you can consider (24) as the Newton equation and find a family of trajectories. However, in this case you arrive at the problem of finding trajectories for given field (13). It has nothing in common with the proposed classical nonlinear system of equations.

6 Conclusion

We think that the wave function ψ in QM represents some kind of a field, and the force of this field can be proportional to $|\psi|^2$. Then it will explain why in QM probability for a particle to be detected is proportional to $|\psi|^2$. The larger is $|\psi|^2$ at the point $\boldsymbol{r}$, the larger is probability that a nucleus in a detector will capture it in the registration process.

If ψ is a field, then the position of the source of the field and velocity of the source can be naturally defined simultaneously, and UR-s do not forbid it. The quantum field of the particle can be represented by a wave packet, and the best candidate for such a packet, in our opinion, is the singular de Broglie wave packet [1, 3]

$$\psi_{dB}(s,\boldsymbol{k},\boldsymbol{r},t)=C\exp(i\boldsymbol{k}\boldsymbol{r}-i\omega(k,s)t)\frac{\exp(-s|\boldsymbol{r}-\boldsymbol{k}t|)}{|\boldsymbol{r}-\boldsymbol{k}t|}, \qquad (25)$$

where $C = \sqrt{s/2\pi}$ is normalization factor, and $\omega(k, s) = (k^2 - s^2)/2$.

Acknowledgement

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Comment on ON UNCERTAINTY RELATIONS AND INTERFERENCE IN QUANTUM AND CLASSICAL MECHANICS

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V. K. Ignatovich claims that uncertainty relations are purely properties of functions and their Fourier transforms and therefore these relations are not specific to quantum physics. He provides an original proof of this statement. By suitable definitions of the position and momentum for quantum objects he shows that these quantities may be simultaneously defined and measured!

I think that the problem of definition of quantum momentum has deeper roots in the passage from classical to quantum physics. Already in classical physics we should distinguish the canonical and mechanical momenta and up to now no discussion of the uncertainty relations took this fundamental fact into account. Moreover, in standard discussion of the notion of position of quantum mechanical objects the notion of position is identified with the operator of multiplication of the Schroedinger wave function by its space argument. V. K. Ignatovich proceeds differently and defines the position of the particle by the maximum point of the wave function. But what to do in the case when the wave function will have more than one maximum?

For me, the most exciting point of the Ignatovich paper is in his statement that "the wave function ψ in QM represents some kind of a field and the force of this field can be proportional to $|\psi|^2$". I think that such point of view naturally follows from the coupled set of Maxwell - Dirac equations for microscopic electrodynamics. Here all quantities implemented as functions defined on spacetime (i.e. the electromagnetic field $F_{\mu\nu}(x)$ and the spinor field $\psi(x)$) should be treated as the same physical entities, namely as fields which jointly describe the microscopic electrodynamical systems. The fact that in macroscopic electrodynamics only the $F_{\mu\nu}$ field survives is the consequence of the short range character of the field ψ. In addition, the ψ field may be viewed as the supersymmetry partner of $F_{\mu\nu}$. In this case, the absence of the ψ field in macroscopic electrodynamics is a manifestation of breaking this symmetry on larger than microscopic scales.

Looking at the Hamiltonian for the Maxwell - Dirac electrodynamical systems we clearly see that both $F_{\mu\nu}$ and ψ contribute to the energy of the system and that the electric charge e is the coupling constant which determines the interaction of both fields. Therefore, both $F_{\mu\nu}$ and ψ are nothing else but quantities which describe the degrees of freedom of the microscopic electrodynamical systems. Omitting ψ (or $F_{\mu\nu}$) is the same as neglecting some of these degrees of freedom and must lead to incomplete description of the systems. In short, microscopic electrodynamics has more degrees of freedom than macroscopic electrodynamics. Looking at charges from the remote distances we see less in comparison with looking at them very closely. But this is a general feature of all physical systems: only close inspections can reveal their structure!