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DUFFIN-KEMMER-PETIAU EQUATION, PROCA EQUATION AND MAXWELLS EQUATION IN 1+1 D

Nuri Unal
Akdeniz University, Physics Department PK 510, Antalya 07058, Turkey
E-mail: nuriunal@akdeniz.edu.tr

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Abstract

We study the relation between the spin-1 sector of the Duffin-Kemmer-Petiau (DKP) Equation and Proca Equation in the 1+1 dimensional Riemann-Cartan (RC) spacetime and show that DKP equation is equivalent to complex Proca Equation. The massless particle limit of the Proca equation gives the Maxwells Equation in 1+1 D RC spacetime.

1 Introduction

The Duffin-Kemmer-Petiau (DKP) equation is a first order relativistic wave equation and represents spin-0 and spin-1 particles [1, 2, 3]. Recently, there is a renewed interest in DKP equation in the context of QCD [4], covariant Hamiltonian dynamics [5], in the causal approach [6], in the context of five-dimensional Galilean invariance [7], in the scattering of K^+ nucleus [8] and in the curved spacetimes [9]. The spin-1 (symmetric) part DKP equation is also derived from the quantization of the Barut's classical model of zitterbewegung proposed by Proca and studied by Barut and Zanghi in 3+1 D spacetimes [10, 11, 12, 13].

Recently R. Casana et al discussed the massless particle limit of the DKP equation in the Riemann-Cartan spacetimes by using a singular matrix and showed that there are massless particle limit and a topological field limits for the spin zero and one sectors of the DKP equation [14].

The aim of this study is to discuss the relation between the spin-1 part of the DKP equation in 1+1 dimensional Riemann-Cartan spacetimes and the complex Proca equation and show that the massless particle limit of it gives the two set of real Maxwell equations for the potential, A_μ and field strength, $F_{\mu\nu}$.

2 DKP equation in 1+1 Dimension

In a Riemannian geometry the spacetime coordinates, $x^\mu = \left(x^{\underline{0}},\ x^{\underline{i}}\right)$ are related to local Minkowski coordinates $x^a = \left(x^0,\ x^i\right)$ by

$$dx^\mu = e^\mu_a(x) dx^a,$$

where $e^\mu_a(x)$ is a 2×2 matrix . We use the Greek (Latin) indices for the global Riemannian (local Minkowski) coordinates. Then the metric tensor is defined as

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}. \tag{1}$$

where η_{ab} is the metric of the flat Minkowski coordinates:

$$\eta_{ab} = (1, -1).$$

The DKP equation in the 1+1 dimensional curved spacetime is

$$(\gamma \otimes 1 + 1 \otimes \gamma)^{\mu} \pi_{\mu} \Psi(x) = 2m\Psi(x), \tag{2}$$

where $\gamma^{\mu}(x)$ is the space depended Dirac matrices:

$$\gamma^{\mu}(x) = e_a^{\mu}(x)\gamma^a.$$

The constant matrices, γ^a are chosen as

$$\gamma^a = \left(\sigma^3, i\sigma^1\right),$$

where γ^0 and γ^i are

$$\gamma^0 = \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^i = i\sigma^1 = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

They are related to η_{ab} by the relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}.$$

The generalized momentum, π_{μ} is written in terms of the covariant derivative as

$$\pi_{\mu} = i\nabla_{\mu} = i\left(e_{\mu}^{a}(x)\frac{\partial}{\partial x^a} - \Gamma_{\mu}\right), \tag{3}$$

where Γ_{μ} is the generalized Fock-Ivanenko two-vectors:

$$\Gamma_{\mu} = \frac{1}{4}\Gamma_{\mu ab}\left(\Sigma^{ab} \otimes 1 + 1 \otimes \Sigma^{ab}\right). \tag{4}$$

In Eq. (4) Σ^{ab} is the two-vector of the Dirac algebra and given by

$$\Sigma^{ab} = \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right).$$

We assume the metric, $g_{\mu\nu}$ as diagonal and evaluate the components of Γ_{μ}. These are

$$\Gamma_0 = -\frac{1}{2}e_{\underline{0}}^{0}\Gamma_{001}\left(\sigma^3 \otimes 1 + 1 \otimes \sigma^3\right), \tag{5}$$

and

$$\Gamma_1 = -\frac{1}{2} e_{\underline{1}}^{1} \Gamma_{101} \left(\sigma^3 \otimes 1 + 1 \otimes \sigma^3 \right), \tag{6}$$

where Γ_{abc} is antisymmetric under the exchange $b \rightleftharpoons c$. Then the explicit expressions of π_μ are

$$\pi_0 = i\nabla_0 = ie_{\underline{0}}^{0}(x) \left[\frac{\partial}{\partial x^0} + \frac{1}{2} \Gamma_{00i} \left(\sigma^3 \otimes 1 + 1 \otimes \sigma^3 \right) \right], \tag{7}$$

$$\pi_i = i\nabla_i = ie_{\underline{i}}^{i}(x) \left[\frac{\partial}{\partial x^i} + \frac{1}{2} \Gamma_{i0i} \left(\sigma^3 \otimes 1 + 1 \otimes \sigma^3 \right) \right]. \tag{8}$$

Then the wave equation becomes

$$\begin{aligned} &\left[\left(e_{\underline{0}}^{0}(x) \right) \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \nabla_0 \right. \\ &\left. - \left(e_{\underline{1}}^{1}(x) \right) \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \nabla_1 \right] \begin{bmatrix} \psi_+ \\ \psi_0 \\ \psi_0 \\ \psi_- \end{bmatrix} \\ &= -2mi \begin{bmatrix} \psi_+ \\ \psi_0 \\ \psi_0 \\ \psi_- \end{bmatrix} \end{aligned} \tag{9}$$

We substitute ∇_0 and ∇_1 in Eq. 9:

$$\left\{ \left[i \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial x^0} \right. \right.$$

$$\left. + 2i\Gamma_{001} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]$$

$$
-\left[i \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \frac{\partial}{\partial x^1} \right.
$$

$$
\left. \left. + i\Gamma_{101} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \right\} \begin{bmatrix} \psi_+ \\ \psi_0 \\ \psi_0 \\ \psi_- \end{bmatrix}
$$

$$
= -2mi \begin{bmatrix} \psi_+ \\ \psi_0 \\ \psi_0 \\ \psi_- \end{bmatrix}
$$

After some matrix algebra we obtain the following three equations:

$$
\left(\frac{\partial}{\partial x^0} + \Gamma_{101} \right) (\psi_+ - \psi_-) - \left(\frac{\partial}{\partial x^1} + \Gamma_{001} \right) i\psi_0 = -im\,(\psi_+ + \psi_-)\,, \tag{10}
$$

$$
\frac{\partial}{\partial x^0} (\psi_+ + \psi_-) = -im\,(\psi_+ - \psi_-)\,, \tag{11}
$$

$$
\frac{\partial}{\partial x^1} (\psi_+ + \psi_-) = 2m\psi_0. \tag{12}
$$

3 Proca Equation

To obtain the Proca equation from Eqs. (10), (11) and (12) we organize them as

$$
\left(\frac{\partial}{\partial x^0} + \Gamma_{101} \right) \left(\frac{\psi_+ - \psi_-}{-im} \right) - \left(\frac{\partial}{\partial x^1} + \Gamma_{001} \right) \left(\frac{2i\psi_0}{-im} \right) = (\psi_+ + \psi_-)\,, \tag{13}
$$

$$
\frac{\partial}{\partial x^0} (\psi_+ + \psi_-) = -m^2 i \left(\frac{\psi_+ - \psi_-}{m} \right), \tag{14}
$$

$$
\frac{\partial}{\partial x^1} (\psi_+ + \psi_-) = -m^2 \left(\frac{2i\psi_0}{-im} \right). \tag{15}
$$

We notice

$$
\Gamma_{101} = -\frac{1}{\det e^{-1}} \frac{\partial \det e^{-1}}{\partial x^0}, \quad \text{and } \Gamma_{001} = -\frac{1}{\det e^{-1}} \frac{\partial \det e^{-1}}{\partial x^1}.
$$

We substitute these expressions into the Eq. (13):

$$\frac{\partial}{\partial x^0}\left[\left(\frac{\psi_+ - \psi_-}{-im\det e}\right)\right] - \frac{\partial}{\partial x^1}\left[\left(\frac{-2\psi_0}{m\det e}\right)\right] = \frac{(\psi_+ + \psi_-)}{\det e}. \tag{16}$$

We organize the Eqs. 14 and 15 as

$$e_0^{\widehat{0}}\frac{\partial}{\partial x^0}\det eF^{\widehat{0}\widehat{1}} = m^2 e_{\widehat{1}}^{1} A^{\widehat{1}},$$

$$e_1^{\widehat{1}}\frac{\partial}{\partial x^1}\left(-\det eF^{\widehat{0}\widehat{1}}\right) = m^2 e_{\widehat{0}}^{0} A^{\widehat{0}},$$

where the local Minkowski coordinates are shown by $\left(\widehat{0},\widehat{1}\right)$. We rewrite them as

$$\frac{\partial}{\partial x^0}\det eF^{\widehat{0}\widehat{1}} = m^2\det eA^{\widehat{1}}, \tag{17}$$

$$\frac{\partial}{\partial x^1}\left(-\det eF^{\widehat{0}\widehat{1}}\right) = m^2\det eA^{\widehat{0}}. \tag{18}$$

Then we identify

$$A^{\widehat{0}} = \left(\frac{-2\psi_0}{m\det e}\right), \quad A^{\widehat{1}} = \left(\frac{\psi_+ - \psi_-}{im\det e}\right),$$

$$A_{\widehat{0}} = \frac{-2\psi_0}{m\det e} = A^{\widehat{0}}, \quad A_{\widehat{1}} = \frac{(\psi_+ - \psi_-)}{-im\det e} = -A^{\widehat{1}},$$

and

$$F_{\widehat{0}\widehat{1}} = -F^{\widehat{0}\widehat{1}} = \frac{\partial A_{\widehat{1}}}{\partial x^0} - \frac{\partial A_{\widehat{0}}}{\partial x^1} = \frac{(\psi_+ + \psi_-)}{\det e},$$

where the complex field strength and potentials are

$$F_{\mu\nu} = F_{01} = e_0^{\widehat{0}} e_1^{\widehat{1}} F_{\widehat{0}\widehat{1}} = \frac{(\psi_+ + \psi_-)}{(\det e)^2},$$

$$A_\mu = \left(e_0^{\widehat{0}} A_{\widehat{0}},\ e_1^{\widehat{1}} A_{\widehat{1}}\right) = \left[e_0^{\widehat{0}}\left(\frac{-2\psi_0}{m\det e}\right),\ e_1^{\widehat{1}}\left(\frac{\psi_+ - \psi_-}{-im\det e}\right)\right].$$

The real and imaginary parts of the potentials and field strengths satisfy the Proca equation separately.

4 Maxwell Equations

To obtain the massless particle limit of DKP equation and Proca equation, we take m^2 as zero in Eqs. (17) and (18).

Then we write the $m^2 \to 0$ limit of the Eqs. (13), (14), and (15) as

$$\frac{\partial}{\partial x^0} A_{\hat{1}} - \frac{\partial}{\partial x^1} A_{\hat{0}} = F_{\widehat{01}}, \tag{19}$$

$$\frac{1}{\det e} \frac{\partial}{\partial x^0} \left(\det e F^{\widehat{01}} \right) = 0, \tag{20}$$

$$\frac{1}{\det e} \frac{\partial}{\partial x^1} \left(- \det e F^{\widehat{01}} \right) = 0. \tag{21}$$

where the field strength tensor F_{01} corresponds to $(\psi_+ + \psi_-) / (\det e)^2$. Since the wave function $\Psi(x)$ is complex in DKP equation, the vector potential, A_μ and field strength tensor, $F_{\mu\nu}$ are complex. The real and imaginary parts of them gives two independent or separate sets of Maxwell equations.

5 Current

To define the current we consider the Hermitian conjugate of the wave equation. The Hermitian conjugate of the wave function, $\Psi(x)$ is defined as

$$\overline{\Psi}(x) = [\Psi^*(x)]^T \gamma^0 \otimes \gamma^0.$$

Then the generalized momentum, π_μ is Hermitian and the Hermitian conjugate of the wave equation is

$$-i \nabla_\mu \overline{\Psi}(x) (\sigma \otimes 1 + 1 \otimes \sigma)^\mu = 2m \overline{\Psi}(x). \tag{22}$$

We define the current as

$$j^\mu = \overline{\Psi}(x) (\sigma \otimes 1 + 1 \otimes \sigma)^\mu \Psi(x). \tag{23}$$

The j^μ is a conserved current:

$$\nabla_\mu j^\mu = 0. \tag{24}$$

To derive the massless particle limit of current we evaluate $\Psi\overline{\Psi}$ in terms of A_μ and $F_{\mu\nu}$:

$$\Psi\overline{\Psi} = \frac{1}{2}F_{01}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} F_{01} + O\left(m\right). \tag{25}$$

Then the scalar product of $\Psi\overline{\Psi}$ with $(\sigma \otimes 1 + 1 \otimes \sigma)^\mu$ gives the current j^μ in Eq (25) and it is zero for $\mu = 0, 1$. Since the scalar product of $\Psi\overline{\Psi}$ with Σ^{01} is also zero, the spin current vanishes for the photon in 1+1 dimension.

The normalization for the massive particle is

$$\int d^2x \det e^{-1}\left[\left|\psi_+\right|^2 + \left|\psi_-\right|^2 - 2\left|\psi_0\right|^2\right] =$$

$$= \frac{1}{2}\int d^2x \det e^{-1}\left[\left|\psi_+ + \psi_-\right|^2 + \left|\psi_+ - \psi_-\right|^2 - \left|2\psi_0\right|^2\right] \tag{26}$$

When m^2 vanishes, the limit of the normalization integral is

$$\frac{1}{2}\int d^2x \det e^{-1}\left(\det e\right)^2\left[\left|\frac{(\psi_+ + \psi_-)}{\det e}\right|\right]^2 =$$

$$= \frac{1}{2}\left\{\int d^2x \det e^{-1}\left(\det e\right)^2\left[\mathrm{re}\left|\frac{(\psi_+ + \psi_-)}{\det e}\right|\right]^2\right. \tag{27}$$

$$\left. + \int d^2x \det e^{-1}\left(\det e\right)^2\left[\mathrm{im}\left|\frac{(\psi_+ + \psi_-)}{\det e}\right|\right]^2\right\} \tag{28}$$

$$= \int d^2x \det e^{-1}\left[\mathrm{re}F_{01}\right]\left[\mathrm{re}F^{01}\right] = \int d^2x \det e^{-1}\left[\mathrm{im}F_{01}\right]\left[\mathrm{im}F^{01}\right] \tag{29}$$

Eq. (29) gives the correct normalization for F_{01} and there is no contribution from the potential A_μ. The same normalization can be derived by taking the scalar product of $\Psi\overline{\Psi}$ in Eq. (25) with unit element of Clifford algebra, 1.

6 Conclusion

In this study we discussed the spin-1 sector of the DKP equation, which is derived from the 1+1 D analogue of the the classical model of the zitterbewegung proposed by Barut in 3+1 D. In 1+1 D the classical holomorphic coordinates are 2 components complex spinors. For this reason Dirac spinor has also 2 components. The spin-1 spinor has three independent components and is represented by 4 component spinors which are written as the symmetric direct product of the two components Dirac spinors.

We derived the Proca equation for these three complex spinors by introducing two complex potential and one complex field strength for the massive charged particle. The $m^2 \to 0$ limit of Proca equation gives two sets of Maxwell equation: One of them corresponds to Maxwell equations and the other set corresponds to the dual of them.

We also show that the probability current can be defined for DKP equation in the similar way to the Dirac equation. The $|F|^2$ is normalized in the Maxwell equations In the Proca equation $|F|^2 + m^2 |A|^2$ is also normalized.

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