The old and new Concepts of Physics
An open-dialogue journal
www.uni.lodz.pl/concepts

PURE SPINORS AND THEIR POSSIBLE ROLE IN PHYSICS

Paolo Budinich
International School for Advanced Studies Trieste, Italy
E-mail: fit@ictp.it

(Received 20 April 2005)

Abstract

The É. Cartan's equations defining "simple" spinors (renamed "pure" by C. Chevalley) are interpreted as equations of motion for fermions and for fermion's multiplets in momentum spaces which, after the adoption of the Cartan's conjecture on the non elementary nature of euclidean geometry, appear lorentzian and isomorphic to invariant mass spheres.

The equations found are most of those traditionally adopted ad hoc by theoretical physics of elementary particles. It is shown how, the known internal symmetry groups, in particular those of the standard model, might derive from the 3 complex division algebras correlated with the associated Clifford algebras, while the real one might be at the origin of black matter. One of the results is that some of the internal symmetry groups (isospin), might represent reflection groups (conformal) rather than coverings of rotation groups.

It is shown how in the frame of Cartan's conjecture in (quantum) mechanics of fermions, it is necessary and natural to substitute the euclidean concept of point event with that

of string as an integral of null vectors bilinear in pure spinor; fundamentally non local.

1 Introduction

Élie Cartan, in his "Lecons sur la Theorie des Spineurs" [1], explicitly underlined how "simple" spinors (renamed "pure" by Chevalley [2]) may be conceived as the constituents of euclidean geometry in so far euclidean null vectors may be bilinearly expressed in terms of simple spinors, and sums of null vectors generally give ordinary euclidean vectors. Consequently one may formulate the conjecture that simple spinor geometry – rather than euclidean geometry – might represent the fundamental elementary geometry of nature. We will call it the "Cartan's conjecture".

There is a striking analogy with theoretical physics, where fermions; the quanta of spinor fields, have been discovered to be the elementary constituents of macroscopic matter, in so far bosons; the quanta of euclidean tensor fields, may always be represented as bilinears of fermions (even if they are not bound states of those fermions, as in the case of photons).

We will adopt Cartan's conjecture and try to draw from it the natural, straightforward, consequences. To start with, the Cartan's equations defining pure spinors will be interpreted as equations of motion for fermions (or fermion multiplets) in momentum spaces whose vectors, bilinear in pure spinors, will be null and the spaces will naturally result lorentzian and will be equivalent to compact manifolds.

The equations naturally found in this approach are most of those historically defined ad hoc (including Maxwell's) in theoretical physics to represent the observed phenomenology of elementary particles, thus explaining also the possible geometrical origin of some of their properties, among these internal symmetry groups (including those of the standard model), as due to the 3 complex division algebras correlated with the associated Clifford algebras. Precisely $U(1)$ derives from complex numbers and is at the geometrical origin of the electric and strong charges of fermions which are consequently foreseen to steadily appear in charged-neutral doublets (of fermions or fermion doublets) as in fact they appear in nature; $SU(2)$ derives from quaternions and is at the origin of isospin, while $SU(2)_L$ is at the origin of the electroweak model. Quaternions also appear at the origin of the 3 lepton-hadron families and of the 3 colors, in number equal to the 3 imaginary units of quaternions; $SU(3)$ derives from octonions and are at the origin of both flavour and color symmetry.

In some cases (isospin and flavour) these internal symmetries appear as reflection symmetries rather than coverings of rotation groups.

The fact that the first natural consequences of Cartan's conjecture seem to reproduce rather well, and to possibly explain geometrically some of the observed features of elementary particle phenomenology, is an encouraging sign. However, if true, that conjecture might have a far deeper meaning, in so far it would allow, through the consequent identification of (spinor) geometry with elementary particle physics, the explanation of several as yet obscure facts regarding the elementary structure of matter, an explanation also rich of possible epistemological contents. Some of those possible deeper meanings already appear at this preliminary stage and will be outlined in the paper.

2 Hints on spinor geometry

Here we will merely recall some concepts, and define notations, on spinor geometry necessary for the following. For more on the subject see refs.[1,2,3,4].

Given $V = \mathbb{C}^{2n}$ and the corresponding Clifford algebra $\mathbb{C}\ell(2n)$, generated by γ_a, obeying $[\gamma_a, \gamma_b]_+ = 2\delta_{ab}$, with $a, b = 1, 2, \ldots, 2n$; a spinor ψ is a 2^n-dimensional vector of the endomorphism space S of $\mathbb{C}\ell(2n) : \mathbb{C}\ell(2n) = \text{End } S$ and $\psi \in S$.

The Cartan's equation defining ψ is

$$z_a \gamma^a \psi = 0, \qquad a = 1, 2, \ldots 2n, \tag{2.1}$$

where $z \in V$. Given $\psi \neq 0$ (implying $z_a z^a = 0$), it defines the d-dimensional totally null, projective plane $T_d(\psi)$. For $d = n$ (maximal), ψ was named simple (by Cartan [1]) or pure (by Chevalley [2]), a name now prevailing in the literature. A pure spinor is isomorphic (up to a sign) to $T_n(\psi)$.

Given $\mathbb{C}\ell(2n) = \text{End } S$ and $\psi, \phi \in S$, we have [5]

$$\psi \otimes B\phi = \sum_{j=0}^{n} F_j \tag{2.2}$$

where B is the main antiautomorphism [3] of $\mathbb{C}\ell(2n)$ and of S. It is defined by: $B\gamma_a = \gamma_a^t B$; $B\phi = \phi^t B$, where γ_a^t and ϕ^t mean γ_a and ϕ

transposed. The Clifford algebra elements F_j may be written in the form

$$F_j = [\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_j}]T^{a_1a_2\ldots a_j}, \tag{2.3}$$

in which the γ matrices are antisymmetrized and the antisymmetric tensor T is given by

$$T_{a_1a_2\ldots a_j} = \frac{1}{2^n}\langle B\phi, [\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_j}]\psi\rangle \ . \tag{2.4}$$

Proposition 1 Take $\phi = \psi$ in (2.2), then ψ is pure iff:

$$F_0 = F_1 = \ldots F_{n-1} = 0; \quad F_n \neq 0. \tag{2.5}$$

Then eq.(2.2) becomes

$$\psi \otimes B\psi = F_n \tag{2.6}$$

representing the maximal totally null plane $T_n(\psi)$ to which ψ is isomorphic (up to a sign) [5].

For $\mathbb{C}\ell(2n)$ with generators $\gamma_1, \gamma_2 \ldots \gamma_{2n}$ and with the volume element (normalized to one): $\gamma_{2n+1} := \gamma_1\gamma_2\ldots\gamma_{2n}$, the spinors: $\psi_\pm = \frac{1}{2}(1\pm\gamma_{2n+1})\psi$ are named Weyl spinors, and they are 2^{n-1}-dimensional and are associated to the even $\mathbb{C}\ell(2n)$ subalgebra $\mathbb{C}\ell_0(2n)$. Weyl spinors may be pure, and they are such [5] for $n \leq 3$. For $n > 3$ the constraint equations given by eqs.(2.5) are in numbers $1, 10, 66, 364$ for $n = 4, 5, 6, 7$, respectively.

If we multiply eq.(2.2) from the left by γ_a and from the right by $\gamma_a\psi$ and set it to zero after summing over a, we obtain

$$\gamma_a\psi \otimes B\phi\ \gamma^a\psi = z_a\gamma^a\psi = 0, \quad a = 1, 2, \ldots 2n \tag{2.7}$$

that is Cartan's eq.(2.1), where now, because of eq.(2.4), the vector components z_a are bilinear in the spinors ψ and ϕ

$$z_a = \frac{1}{2^n}\langle B\phi, \gamma_a\psi\rangle. \tag{2.8}$$

We have [5]

Proposition 2: For arbitrary ϕ in $z_a = \langle B\phi, \gamma_a\psi\rangle$, $z_az^a = 0$ if and only if ψ is pure.

In this way Cartan's eq.(2.1) or (2.7) admit non null solutions if the involved spinors are pure and the vectors $z \in V$ are of the form (2.8) which, together with eqs.(2.2) and (2.4) represent the formal realization of the Cartan's conjecture.

There are the isomorphisms of Clifford algebras [3]

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \tag{2.9}$$

both central simple, and

$$\mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \tag{2.10}$$

both non simple, from which we have the isomorphisms and subsequent embeddings of Clifford algebras

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \hookrightarrow \mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \hookrightarrow \mathbb{C}\ell(2n+2) \tag{2.11}$$

and the corresponding ones for spinors

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W \oplus \psi_W \simeq \Psi_D \simeq \psi_D \oplus \psi_D \tag{2.12}$$

where D, P, W stand for Dirac, Pauli, Weyl, and (2.12) implies that a Dirac or Pauli spinor is isomorphic to a doublet of Dirac, Pauli or Weyl spinors.

These isomorphisms may be explicitly represented. In fact let γ_a be the generators of $\mathbb{C}\ell(2n)$ with associated Dirac spinors ψ and Γ_A those of $\mathbb{C}\ell(2n+2)$ with associated spinors Ψ. Then we have

$$\begin{aligned} &\text{for} \quad \Gamma_a^{(0)} = 1_2 \otimes \gamma_a : \quad \Psi^{(0)} = \begin{pmatrix} \psi_1^{(0)} \\ \psi_2^{(0)} \end{pmatrix} \\ &\text{and for} \quad \Gamma_a^{(j)} = \sigma_j \otimes \gamma_a : \quad \Psi^{(j)} = \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \end{pmatrix} \end{aligned} \tag{2.13}$$

where $j = 1, 2, 3$. Then $\Psi^{(0)}$ is a doublet of Dirac spinors while $\Psi^{(j)}$ a doublet of Weyl ($j = 1, 2$) or Pauli ($j = 3$) spinors. Because of (2.12) they are isomorphic.

Now define

$$L := \frac{1}{2}(1 + \gamma_{2n+1}); \quad R := \frac{1}{2}(1 - \gamma_{2n+1}), \tag{2.14}$$

there are the unitary transformations U_j

$$U_j = 1 \otimes L + \sigma_j \otimes R = U_j^{-1} \quad j = 1, 2, 3, . \tag{2.15}$$

We have [4]:

Proposition 3: Dirac, Pauli, Weyl spinor doublets are isomorphic since

$$U_j \Gamma_A^{(0)} U_j^{-1} = \Gamma_A^{(j)}; \quad U_j \Psi^{(0)} = \Psi^{(j)}; \quad A = 1, 2 \ldots 2n+2; \; j = 1, 2, 3 \; , \tag{2.16}$$

as easily verified.

3 Spinor and momentum spaces. The signature

We will now, following Cartan [1], start with the simplest, non trivial, two component spinors which may be conceived as Dirac spinors $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ associated with the Clifford algebra $\mathbb{C}\ell(2)$ or equivalently as Pauli spinors associated with the isomorphic Clifford algebra $\mathbb{C}\ell_0(3)$, generated by the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Let $\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$ represent another such spinor and let us insert them in eq.(2.2). We obtain, taking into account that $B = -i\sigma_2 := \epsilon$, the following equation and identities

$$\begin{pmatrix} \varphi_0\psi_1 & -\varphi_0\psi_0 \\ \varphi_1\psi_1 & -\varphi_1\psi_0 \end{pmatrix} \equiv \varphi \otimes B\psi = z_0 + z_j\sigma^j \equiv \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}, \tag{3.1}$$

from which we easily get both the z-vector components bilinear in the spinor ψ and φ : $z_\mu = \frac{1}{2}\psi^t \epsilon \sigma_\mu \varphi$ where $\mu = 0, 1, 2, 3$ and $\sigma_0 = 1$ (compare the matrices) and the nullness of the vector z : $z_\mu z^\mu = z_0^2 - z_1^2 - z_2^2 - z_3^2 \equiv 0$ (compute the determinants of the matrices) in agreement with Proposition 2.

In order to restrict to the real, of interest for physics, z_0 and z_j, we need to introduce the conjugation operator C defined by: $C\gamma_a = \bar{\gamma}_a C, C\varphi = \bar{\varphi}C$ where $\bar{\gamma}_a$ and $\bar{\varphi}$ mean γ_a and φ complex conjugate. Then eq.(3.1) may be expressed, uniquely, [4] in the form

$$\begin{pmatrix} \varphi_0\bar{\varphi}_0 & \varphi_0\bar{\varphi}_1 \\ \varphi_1\bar{\varphi}_0 & \varphi_1\bar{\varphi}_1 \end{pmatrix} = p_0 + p_j\sigma^j = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \tag{3.1'}$$

and now

$$p_\mu = \varphi^\dagger \sigma_\mu \varphi, \quad \mu = 0, 1, 2, 3, \tag{3.2}$$

where $\varphi^\dagger$ means φ hermitian conjugate. Then we have, again identically

$$p_\mu p^\mu = p_0^2 - p_1^2 - p_2^2 - p_3^2 \equiv 0, \tag{3.3}$$

which shows how p_μ are the components of a null or optical vector of a momentum space with Minkowski signature. This is a particular case of Proposition 2. In fact embed $\mathbb{C}\ell_0(3)$ in the non simple $\mathbb{C}\ell(3)$ isomorphic to $\mathbb{C}\ell_0(1,3)$ with generators $\gamma_\mu = \{\sigma_1 \otimes 1, \; -i\sigma_2 \otimes \sigma_j\}$ and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1$. Then we may identify the above Pauli spinor with one of the two Weyl spinors defined by

$$\varphi_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right)\psi \tag{3.4}$$

where ψ is a Dirac spinor associated with $\mathbb{C}\ell(1,3)$. Then eq.(3.2) identifies with one of the two

$$p_\mu^{(\pm)} = \tilde{\psi}\gamma_\mu(1 \pm \gamma_5)\psi; \quad \mu = 0, 1, 2, 3 \tag{3.5}$$

where $\tilde{\psi} = \psi^\dagger \gamma_0$.

Now the vectors $p^\pm$ are null or optical because of Proposition 2, since the Weyl spinors $\varphi_\pm$ are pure. The corresponding Cartan's equations will be

$$p_\mu \gamma^\mu (1 \pm \gamma_5)\psi = 0, \tag{3.6}$$

which may be expressed in Minkowski space-time if p_μ are interpreted as generators of Poincaré translations: $p_\mu \to i\frac{\partial}{\partial x_\mu}$. They identify with the known wave equation of motion for massless neutrinos.

Observe that in this unique derivation, obtained by merely imposing the reality of the p_μ components, Minkowski signature derives from quaternions, as may be seen already from eqs.(3.1) and (3.1′) and from their correlation with Clifford algebras, in fact notoriously $\mathbb{C}\ell(1,3) = H(2)$ where H stands for quaternions. One might then affirm that Minkowski signature is the image in nature of quaternions.

It is interesting to observe that if we define the electromagnetic tensors F (so named already by Cartan [1]) with components

$$F_{\mu\nu}^{(\pm)} = \tilde{\psi}[\gamma_\mu, \gamma_\nu](1 \pm \gamma_5)\psi \tag{3.7}$$

we obtain from Cartan's eq.(3.6) the Maxwell's equations in empty space [6]

$$p_\mu F_+^{\mu\nu} = 0; \quad \epsilon_{\lambda\rho\mu\nu} p^\rho F_-^{\mu\nu} = 0. \tag{3.8}$$

Also the inhomogeneous Maxwell's equations in the presence of external electromagnetic sources may be obtained from spinor geometry [7].

We may now start from eq.(3.6), which is the simplest Cartan's equation, in order to find the other ones for higher dimensional spinors and corresponding real components of momentum spaces. Consider the four-component Dirac spinors ψ of $\mathbb{C}\ell(1,3)$. Exploiting the isomorphisms of Proposition 3, it is easily found that a doublet of them is isomorphic to a doublet of the two Weyl spinors ψ_+, ψ_- of $\mathbb{C}\ell_0(1,5)$ building up the eight component Dirac spinor Ψ of $\mathbb{C}\ell(1,5)$ with generators Γ_a and volume element Γ_7, and then ψ_+, ψ_- are defined by

$$\psi_\pm = \frac{1}{2}(1 \pm \Gamma_7)\Psi \ . \tag{3.9}$$

If we now construct

$$P_a^{(\pm)} = \tilde{\Psi}\Gamma_a(1 \pm \Gamma_7)\Psi, \tag{3.10}$$

where $\tilde{\Psi} = \Psi^\dagger \Gamma_0$, the P_a are the six real components of a vector $P \in \mathbb{R}^{1,5}$ which is null because of Proposition 2, since $\psi_\pm$ are pure.

The Cartan's equations will be

$$P_a \Gamma^a (1 \pm \Gamma_7)\Psi = 0. \tag{3.11}$$

Observe that P_a may also be written in terms of the Dirac spinor ψ of $\mathbb{C}\ell(1,3)$

$$P_\mu = \tilde{\psi}\gamma_\mu\psi; \quad P_5 = i\tilde{\psi}\gamma_5\psi; \quad P_6 = \tilde{\psi}\psi \tag{3.10$'$}$$

and P_μ is the sum of the two null vectors $p_\mu^{(\pm)}$ previously found and given in eqs.(3.5)

$$P_\mu = p_\mu^{(+)} + p_\mu^{(-)}, \tag{3.12}$$

which is obviously non null, in general, however, it is the projection in $\mathbb{R}^{1,3}$ of the null vector in $\mathbb{R}^{1,5}$ with components P_a given by eq.(3.10$'$). In this framework the sum of eq.(3.12) may be considered as a direct sum since it brings from a null vector in $\mathbb{R}^{1,3}$ to one in $\mathbb{R}^{1,5}$ momentum space

$$p_\mu^+ \oplus p_\mu^- \hookrightarrow \{P_\mu, P_5, P_6\}, \tag{3.13}$$

which, together with

$$\psi_+ \oplus \psi_- = \Psi, \tag{3.14}$$

implied by eq.(3.9), is well representing the prescription of Cartan's conjecture. In fact ordinary euclidean vectors result as sums of null vectors constructed bilinearly with pure spinors and bringing to null vectors of higher dimensional spaces. Correspondingly, the direct sums of the simple spinors brings to a double dimensional spinor; we will adopt it as a general rule, naturally following from Cartan's conjecture.

4 From n to $n+1$: the general rule

Given a 2^n-component Dirac spinor ψ of $\mathbb{C}\ell(1, 2n-1)$ generated by γ_a, the vectors of $\mathbb{R}^{1,2n-1}$ with components

$$p_a^{(\pm)} = \tilde{\psi}\gamma_a(1 \pm \gamma_{2n+1})\psi, \quad a = 1, 2, \dots 2n, \tag{4.1}$$

are null because of Proposition 2 provided the 2^{n-1}-component Weyl spinors $(1 \pm \gamma_{2n+1})\psi$ are pure. The vector with real components

$$P_a = p_a^{(+)} + p_a^{(-)} = \tilde{\psi}\gamma_a\psi, \quad P_{2n+1} = i\tilde{\psi}\gamma_{2n+1}\psi, \quad P_{2n+2} = \tilde{\psi}\psi \tag{4.2}$$

is a null vector of $\mathbb{R}^{1,2n+1}$ provided the 2^n-component Weyl spinors

$$\psi^\pm = \frac{1}{2}(1 \pm \Gamma_{2n+3})\Psi, \tag{4.3}$$

are pure; where Ψ is a 2^{n+1} component spinor of $\mathbb{C}\ell(1, 2n+1)$ with generators Γ_A and the volume element Γ_{2n+3}. In fact the above components P_A may be written in the form

$$P_A^{(\pm)} = \tilde{\Psi}\Gamma_A(1 \pm \Gamma_{2n+3})\Psi, \quad A = a, 2n+1, 2n+2. \tag{4.4}$$

The corresponding Cartan's equation in momentum space will be

$$P_A\Gamma^A(1 \pm \Gamma_{2n+3})\Psi = 0\ . \tag{4.5}$$

Above we have exploited the isomorphisms discussed in Chapter 2. In particular, because of those isomorphisms, the spinor Ψ may be considered as a doublet of Dirac, Weyl or Pauli spinors. To set it in evidence we may then substitute (4.5) with the four equations

$$(P^a \gamma_a^{(m)} + iP_{2n+1}\gamma_{2n+1}^{(m)} \pm P_{2n+2})\psi^{(m)} = 0, \quad m = 0,1,2,3, \quad (4.6)$$

where $\psi^{(m)}$ is a 2^n component member of a doublet of Dirac, Weyl, Pauli spinors for $m = 0; m = 1,2; m = 3$, respectively.

We could adopt, as well, the opposite lorentzian signature: $(2n-1,1)$ instead of $(1,2n-1)$. The only formal difference is that in eqs.(4.2) and (4.6) the imaginary unit i appears as a factor in P_{2n+2} rather than in P_{2n+1} [4]; in particular (4.6) becomes

$$(P^a \gamma_a^{(m)} + P_{2n+1}\gamma_{2n+1}^{(m)} \pm iP_{2n+2})\psi^{(m)} = 0. \quad (4.6')$$

The geometrical equations (4.6) or (4.6′) which have been uniquely derived from Cartan's Conjecture will have to be conceived as equations of motion and compared with those adopted by theoretical physics to represent the observed fermion's phenomena. To do this there are two possible ways. The simplest one is to interpret, in the first four terms $P^\mu \gamma_\mu^{(m)} \psi^{(m)}$, contained in all equations, P^μ as generators of Poincaré translations

$$P^\mu := i\frac{\partial}{\partial x_\mu}, \quad (4.7)$$

by which Minkowski space is then generated as a homogeneous space. Consequently, the spinor ψ and P_j with $j \geq 5$ will have to be considered as taking values in such a space and eqs.(4.6) or (4.6′) will be interpreted as wave equations of motion (in first quantisation). The second equivalent way is to start from the traditional approach in a higher dimensional space and set to zero the extra dimensional coordinates. This is described in Chapter 6.

It is remarkable that the equations found are most of those traditionally defined ad hoc, which means that all of them may derive from Cartan's conjecture.

Remark 1. The condition of reality on the vector components give, in the above constructions, steadily momentum space with lorentzian signature.

Remark 2. The above construction is the natural one, in the spirit of Cartan's conjecture, on deriving euclidean geometry by summing null vectors. However, there are other interesting signatures; in particular those of $\mathbb{R}^{(3+j),(1+j)}$ with $j = 1, 2, \ldots$: the conformal extensions of Minkowski ones. They give real vectors for even j. As an example for $j = 1$ the Weyl spinors of $\mathbb{C}\ell(4, 2)$ are twistors, however they give rise only to complexified Minkowski moment spaces or space-times [8]. From Weyl spinors of $\mathbb{C}\ell(5, 3)$ we may instead obtain vectors of real spaces. From this property one may obtain a derivation of the electroweak model [6] which is different from the derivation presented in the next Chapter.

We have now the instrument for constructing the further possible Cartan's equations deriving from Cartan's conjecture.

5 Cartan's equations from $n = 2$ to $n = 5$, the role of division algebras

We will now list the Cartan's equations which derive from the Cartan's conjecture following the rules of Chapter 4. We will underline the role of division algebras, specially of the complex ones, at the origin of internal symmetry groups, in particular those of the standard model: $SU(3) \otimes SU(2)_L \otimes U(1)$.

n = 2: Majorana equation

We obtain for the signature $(3, 1)$ from $(4.6')$, with $n = 2$ and $m = 0$, and for the real spinor of $\mathbb{C}\ell(3, 1) = \mathbb{R}(4)$ [4], [6] the equation

$$(p_\mu \gamma^\mu + p_5 \gamma_5)\psi = 0, \tag{5.1}$$

where γ_μ are the generators of $\mathbb{C}\ell(3, 1)$ and γ_5 its volume element, which may represent Majorana equation. In our frame it is the simplest after eq.(3.6) for massless neutrinos and it derives from the division algebra of real numbers.

n = 3: a) The pion-nucleon equation; isospin SU(2)

We easily obtain from (4.6′) $n = 3$ and $m = 0$ [4] the pion-nucleon equation

$$(p_\mu 1 \otimes \gamma^\mu + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M)N = 0, \tag{5.2}$$

where N is a double of $\mathbb{C}\ell(3,1)$ Dirac spinors representing the proton and neutrons. The second term derives from quaternions and formally presents the internal symmetry $SU(2)$ of isospin where the pion π is represented by

$$\vec{\pi} = \frac{1}{8}\tilde{N}\ \vec{\sigma} \otimes \gamma_5 N, \tag{5.3}$$

whose pseudoscalar nature emerges from the fact that being N a doublet of Dirac spinor the $\Gamma_5, \Gamma_6, \Gamma_7$ generators of $\mathbb{C}\ell(5,1)$ must contain γ_5 in order to anticommute with $\Gamma_\mu = 1 \otimes \gamma_\mu$.

As we will see this $SU(2)$ is not to be thought of as the covering of $SO(3)$; it appears here as generated by $\Gamma_5, \Gamma_6, \Gamma_7$: operators of reflections in spinor space; identical to the corresponding ones of the conformal group, whose Clifford algebra is $\mathbb{C}\ell(4,2)$. In fact if Γ_6, the other time-like generator besides Γ_0, the corresponding reflection operator will be $i\Gamma_6$, since the square of a reflection must be the identity. The fact that isospin originates from conformal reflections may have further interesting consequences, illustrated elsewhere [4], [6].

b) The electric charge; generated by U(1)

From eq.(5.2) the origin of electric charge may also appear. In fact let us write it explicitly for the Dirac spinors ψ_1 and ψ_2 of the doublet N

$$\begin{aligned}(p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,\psi_1 + \gamma_5\,(p_5 - ip_6)\,\psi_2 &= 0\,,\\ (p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\,(p_5 + ip_6)\,\psi_1 &= 0\,.\end{aligned} \tag{5.4}$$

All p_a are real therefore defining

$$p_5 \pm ip_6 = \rho e^{\pm i\frac{\omega}{2}}$$

and multiplying the first of eqs.(5.4) by $e^{i\frac{\omega}{2}}$ we obtain

$$\begin{aligned}(p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,e^{i\frac{\omega}{2}}\psi_1 + \gamma_5\rho\psi_2 &= 0,\\ (p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\rho e^{i\frac{\omega}{2}}\psi_1 &= 0,\end{aligned} \tag{5.4′}$$

where we see that ψ_1 appears with a phase factor $e^{i\frac{\omega}{2}}$ corresponding to a rotation ω in the circle

$$p_5^2 + p_6^2 = \rho^2. \tag{5.5}$$

The corresponding transformation is generated in spinor space by $J_{56} = \frac{1}{2}[\Gamma_5, \Gamma_6]$ which is the complexification of the dilatation generator of $SU(2,2)$ covering of the conformal group.

The corresponding space-time equations are obtained substituting p_μ with $i\frac{\partial}{\partial x_\mu}$ and since, as we will see in Chapter 6, p_j with $j \geq 5$ are x dependent, the consequent local dependence of the phase factor $e^{i\frac{\omega}{2}}$ will impose a covariant derivative. Therefore eq.(5.4′) will become

$$\left\{\gamma_\mu \left[i\frac{\partial}{\partial x_\mu} + \frac{e}{2}\left(1 - i\Gamma_5\Gamma_6\right) A_\mu\right] + \vec{\pi}\cdot\vec{\sigma}\otimes\gamma_5 + M\right\}\begin{pmatrix} p \\ n \end{pmatrix} = 0, \tag{5.6}$$

where $\psi_1 = p$ represents the proton and $\psi_2 = n$ the neutron, well representing the equation for the proton-neutron doublet interacting with the pion and with the electromagnetic potential A_μ. We will see that this appearance of charged-neutral fermion doublets is a general feature of our construction.

$\mathbf{n = 4}$: The baryon-lepton quadruplet; the U(1) of the strong charge

Let Θ represent a 16-component $\mathbb{C}\ell(7,1)$-Dirac spinor

$$\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \tag{5.7}$$

where N_1 and N_2 are $\mathbb{C}\ell(5,1)$-Dirac spinors, then equation (4.6′) for $n = 4$ and $m = 0$ will be

$$(P_a G^a + P_7 G_7 + P_8 G_8 + P_9 G_9 + iP_{10})\Theta = 0, \tag{5.8}$$

where $a = 1, 2, \ldots 6$. If written explicitly in terms of N_1 and N_2 it is easy to see that N_1 (or N_2) presents an $U(1)$ covariance represented by a phase factor $e^{i\frac{\tau}{2}}$ where τ is an angle of rotation in the circle

$$P_7^2 + P_8^2 = \rho^2, \tag{5.9}$$

which, in spinor space is generated by $[G_7, G_8]$ and could be interpreted at the origin of a strong charge or baryon number for N_1, not presented by N_2, and then N_1 may represent a baryon doublet while N_2 a lepton one. In turn, in each of them $[\Gamma_5, \Gamma_6]$ generates $U(1)$ at the origin of the electric charge as seen above. We could then interpret the quadruplet Θ as representing proton-neutron; in N_1 and electron-neutrino; in N_2 and then the electric charges Q_e might be represented by the eigenvalues of the matrix

$$Q_e = \frac{e}{2}\left(-iG_7G_8 - i\Gamma_5\Gamma_6 \otimes 1\right), \tag{5.10}$$

which then results $+e$ for the proton and $-e$ for the electron [11].

Remark: The free Dirac equation for a Dirac spinor may be obtained here from (5.6) only as an approximate equation, when strong and electric interactions may be ignored.

$\mathbf{n = 5}$: The U(1) of baryon number or strong charge

Let Φ represent a 32-component $\mathbb{C}\ell(9,1)$-Dirac spinor

$$\Phi = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \tag{5.11}$$

where Θ_1 and Θ_2 are $\mathbb{C}\ell(7,1)$-Dirac spinors, then eq.(4.6′) for $n = 5, m = 0$ will be

$$\left(P_A\mathcal{G}^A + P_9\mathcal{G}_9 + P_{10}\mathcal{G}_{10} + P_{11}\mathcal{G}_{11} + iP_{12}\right)\Phi = 0, \quad A = 1, 2, \ldots 8, \tag{5.12}$$

where $\mathcal{G}_\alpha(\alpha = 1, 2\ldots 10)$ and $\mathcal{G}_{11}$ are the generators and volume element of $\mathbb{C}\ell(9,1)$. We may now impose Φ to be simple or pure, as Weyl of $\mathbb{C}\ell_0(11,1)$ (then subject to 66 constraint equations) and then P_α, P_{11}, P_{12}, bilinear in Φ [4], will define, because of Proposition 2, a null vector in $\mathbb{R}^{11,1}$

$$P_\alpha P^\alpha + P_{11}^2 + P_{12}^2 = 0, \quad \alpha = 1, 2\ldots 10, \tag{5.13}$$

which is anyway a necessary condition for eq.(5.12) to admit non null solutions for Φ.

As before Θ_1 presents an $U(1)$ covariance due to a phase factor generated by $[\mathcal{G}_9, \mathcal{G}_{10}]$. It could be interpreted at the origin of baryon number or strong charge for Θ_1 from which Θ_2 should be free and then Θ_1 could represent baryons, while Θ_2 leptons.

With Φ simple or pure we may end our analysis since, after $n = 5$, because of the Bott periodicity theorem, stating that: $\mathbb{C}\ell(n+8, m) = \mathbb{C}\ell(n, m+8) = \mathbb{C}\ell(n, m) \otimes \mathbb{R}(16)$, the geometrical structures will repeat themselves [4].

In our approach dimensional reduction will simply consist in reversing the steps which brought us, in this chapter from n to $n+1$. That is to use the projection operator like $\frac{1}{2}(1+\gamma_{2n+1})$ to reduce to one half the dimension of spinor space and consequently decreasing by 2 the dimension of momentum space, and thus eliminating two terms from the equations of motion, which means determining their decoupling. We will see, that this procedure might explain the origin of the 3 families of leptons (and baryons).

6 Correlation with the traditional approach

The traditional approach, in order to explain the origin of internal symmetry groups, is to admit the existence of a higher dimensional space-time; say: $X \in M = \mathbb{R}^{9,1}$ and then operate the restriction to ordinary 4-dimensional space-time: $x \in \mathrm{M} = \mathbb{R}^{3,1}$ by supposing that the extra dimension X_j with $j > 4$ characterize compact manifolds of very small, unobservable size (Kaluza-Klein method). Let us adopt this approach and suppose that the 16-component spinor Θ is a field taking values in M. Its spinor field equation will be

$$\left(\partial_\mu G^\mu + \sum_{j=5}^{9} \partial_j G_j + P_{10}\right) \Theta(x, X) = 0. \tag{6.1}$$

Let us now assume $X_j = 0$ (null-size of the compact manifolds) and let us define

$$i\frac{\partial}{\partial X_j}\Theta(x, X)\Big|_{X_j=0} = P_j(x)\Theta(x), \quad j = 5, 6 \ldots 9, \tag{6.2}$$

and we obtain

$$\left(i\frac{\partial}{\partial x_\mu} G^\mu + \sum_{j=5}^{9} P_j(x) G_j + iP_{10}\right) \Theta(x) = 0, \tag{6.3}$$

which is eq.(5.8) in ordinary space-time, where P_j have to be considered as functions of space-time coordinates x_μ. This means that our equations may be derived from the traditional approach for the null-size of the compact extra manifolds, once the larger dimensional space-times have been postulated with their signatures. In this frame it is now obvious that some of the internal symmetry groups like isospin $SU(2)$ (and flavour) cannot be considered any more as covering of rotation groups: they represent instead reflections in spinor space, at difference with the gauge groups like $SU(2)_L$ (and color). Therefore the former will have little role in dynamics which the latter instead have.

7 The baryon multiplet

We have seen that Θ_1 in (5.11) may represent a quadruplet of $\mathbb{C}\ell(3,1)$-Dirac spinor representing baryons. It obeys eq.(5.8) where the ten-dimensional vector P is null. It defines an invariant mass $\mathcal{M}$ through the equation

$$-P_\mu P^\mu = P_5^2 + P_6^2 + P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = \mathcal{M}^2. \qquad (7.1)$$

This equation defines a sphere S_5 presenting a symmetry $SO(6)$ orthogonal to the Poincaré group. Therefore a maximal internal symmetry for the quadruplet might be $SU(4)$ covering of $SO(6)$.

Observe that for $n = 2$, that is for $\mathbb{C}\ell(3,1)$, the Weyl pure spinors are $\varphi_\pm$ of eq.(3.4) and the corresponding light cone is given by eq.(3.3) defining the sphere S_2 given by $p_0^2 = p_1^2 + p_2^2 + p_3^2$. The corresponding one for $n = 5$ will be

$$P_0^2 = P_1^2 + P_2^2 + P_3^2 + P_5^2 + P_6^2 + P_7^2 + P_8^2 + P_9^2 + P_{10}^2 \qquad (7.1')$$

defining S_8. If we now impose the 32-component spinor Φ obeying eq.(5.12) to be simple or pure, the corresponding sphere derived from (5.13) will be S_{10}.

Observe that $\mathbb{C}\ell(1,9) = \mathbb{R}(32) = \mathbb{C}\ell(9,1)$ therefore it admits Majorana-Weyl spinors. Furthermore [12] Spin$(9,1)$ = Spin$(1,9) \simeq S(2,\mathbf{o})$, where $\mathbf{o}$ stands for octonions and the last isomorphism refers only to the infinitesimal group or, to the Lie algebra. However it is enough to set in evidence the internal symmetry $SU(3)$ generated by octonions. In fact octonions have a group of automorphism G_2 which restricts to $SU(3)$ when one of its seven imaginary units $e_1, e_2 \dots e_7$ is fixed. In our approach eq.(5.8) may be obtained from

$$P_\alpha \mathcal{G}^\alpha (1 + \mathcal{G}_{11}) \Phi = 0, \qquad \alpha = 1, 2, \dots 10, \tag{7.2}$$

and the volume element G_{11} may be identified with e_7 [4]. Now defining

$$\begin{aligned} U_\pm = \tfrac{1}{2}(1 \pm \mathcal{G}_{11}), \quad V_\pm^{(n)} &= \tfrac{1}{2}\mathcal{G}_{6+n}(1 \pm \mathcal{G}_{11}) \\ \text{and} \qquad V_{\mu\pm}^{(n)} &= -\tfrac{1}{2}\mathcal{G}_\mu^{(n)}(1 \pm \mathcal{G}_{11}), \end{aligned} \tag{7.3}$$

where $n = 1, 2, 3$, it is known [13] that $U_\pm$ transforms as an $SU(3)$ singlet and both $V_\pm^{(n)}$ and $V_{\mu\pm}^{(n)}$ as (3) and $(\bar{3})$ representations of $SU(3)$. Now eq.(5.8) may be set in the form [4]

$$\left[i \left(\frac{\partial}{\partial x_\mu} - igA^\mu_{(n)} \right) V_{\mu^+}^{(n)} + P_5 \mathcal{G}_5 + P_6 \mathcal{G}_6 + \sum_{n=1}^{3} P_{6+n} V_+^{(n)} + P_{10} \right] U_+ \Phi = 0 \tag{7.2$'$}$$

and, if we set

$$P_{6+n} = \tilde{\Phi} V_-^{(n)} \Phi \quad \text{and} \quad A_\mu^{(n)} = \Phi^\dagger \mathcal{G}_0 V_{\mu-}^{(n)} \Phi,$$

eq.(7.2′) is $SU(3)$ covariant both in the term containing $V_+^{(n)}$ which could represent $SU(3)$ flavour and in the gauge term representing $SU(3)$ color where $A_\mu^{(n)}$ represent colored gluons whose index n derives from quaternion imaginary units (or Pauli matrices). Eq.(7.2′) may be considered as a preliminary step for physical interpretation which could be only obtained by expressing baryons trilinearly in terms of the spinors $V_+^{(n)}\Phi$ and $V(n)_{\mu^+}\Phi$, say, then acting on them with the 3×3 representation through the pseudo octonion algebra [14].

As seen above the model foresees the possibility of an internal symmetry $SU(4)$ correlated with a fourth quark; which could be discovered at higher energies.

8 The lepton multiplet

If in eq.(5.11) Θ_1 represents baryons then Θ_2 could represent leptons, possessing neither baryon number nor strong charge; this means that the lepton multiplet will need at least one step of dimensional reduction, bringing from $n = 4$ to $n = 3$ of Chapter 5 and, because of which, leptons in nature should only appear in doublets (charged-neutral), as in fact they do.

Let then Θ_2 be of the form

$$\Theta_2 := \Theta_{\mathcal{L}} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} \ell_{11} \\ \ell_{12} \\ \ell_{21} \\ \ell_{22} \end{pmatrix} , \tag{8.1}$$

where $\Theta_{\mathcal{L}}$ is a $\mathbb{C}\ell(1,7)$-Dirac spinor.

8.1 The families

The obvious dimensional reduction will be

$$1) \qquad \Theta_{\mathcal{L}} \rightarrow \frac{1}{2}(1 + G_9)\Theta_{\mathcal{L}} \tag{8.2}$$

bringing it to a Weyl spinor and because of which

$$P_9 = \Theta_{\mathcal{L}}^{\dagger} G_0 G_9 (1 + G_9)\Theta_{\mathcal{L}} \equiv 0 \equiv \Theta_{\mathcal{L}}^{\dagger} G_0 (1 + G_9)\Theta_{\mathcal{L}} = P_{10}. \tag{8.3}$$

The equation of motion for L_1 will reduce to

$$(p_a \Gamma^a + i p_7 \Gamma_7 + p_8) L_1 = 0, \quad a = 1, 2 \ldots 6 \tag{8.4}$$

and the corresponding invariant mass will reduce to

$$1') \qquad p_\mu p^\mu = p_5^2 + p_6^2 + p_7^2 + p_8^2 = m_1^2 \tag{8.5$'$}$$

which will be smaller that the $\mathcal{M}^2$ of eq.(7.1) obtained for baryons. That is, leptons should be lighter than baryons as in fact it happens.

However there are more possibilities; in fact due to the isomorphism of Proposition 3 one may think $\Theta_{\mathcal{L}}$ as a doublet of Dirac, Weyl or Pauli spinors and therefore there are two more projectors

2) $\Theta_{\mathcal{L}} \to \frac{1}{2}(1 + iG_8G_9)\Theta_{\mathcal{L}}$ implying $P_8 \equiv 0 \equiv P_9$,

and

3) $\Theta_{\mathcal{L}} \to \frac{1}{2}(1 + iG_7G_8)\Theta_{\mathcal{L}}$ implying $P_7 \equiv 0 \equiv P_8$.

The corresponding invariant masses will be:

$$2')\qquad p_\mu p^\mu = p_5^2 + p_6^2 + p_7^2 + p_{10}^2 = m_2^2, \qquad (8.5'')$$

$$3')\qquad p_\mu p^\mu = p_5^2 + p_6^2 + p_9^2 + p_{10}^2 = m_3^2. \qquad (8.5''')$$

All 3 lepton doublets may represent pairs of charged-neutral leptons; however with different masses. Reminding that the p_A components represent external field containing multiplicative coupling constants one could expect $m_1 < m_2 < m_3$ (one could also, reminding that p_5, p_6, p_7 give rise to the pion field times its coupling constant, set it to zero and then $m_1 = p_8; m_2 = p_{10}$ and $m_3^2 = p_9^2 + p_{10}^2$). The 3 families could represent the ones found in nature: $(e, \nu), (\mu, \nu_\mu)$ and (τ, ν_τ), respectively.

We may affirm that the origin of families is due to quaternions as may be seen in the isomorphisms discussed in Chapter 2. In fact both the unitary transformations U_j are labelled by quaternion indices and the isomorphic spinor doublets $\Psi^{(j)}$ constitute a quaternion representation. An independent derivation of the 3 lepton-families from quaternions was given in ref.[15].

8.2 The electroweak model

As we have seen in Chapter 5 for $n = 4$, a doublet of Dirac $\mathbb{C}\ell(1,5)$ spinors like $\Theta_{\mathcal{L}}$ in (8.1) presents a $U(1)$ covariance for L_1 represented by a phase factor $e^{i\frac{\tau}{2}}$ where τ represents an angle of rotation in the circle (5.9) generated by $[G_7, G_8]$ in spinor space, from which L_2 is free. It may represent the origin of the electroweak model.

In fact the angle τ should be local, in space-time and, as seen before, it should give rise to a gauge term in the equation of motion for the lepton doublet $L_1 := \Psi$. Therefore, due to the mentioned isomorphisms the 4 equations for $\Psi^{(m)}$, represented in eq.(4.6′) will take the form

$$\left[\left(p^\mu - A^\mu_{(m)}\right)\Gamma^{(m)}_\mu + \sum_{j=5}^{7} p_j \Gamma^{(m)}_j + M\right]\Psi^{(m)} = 0, \qquad (8.6)$$

for $m = 0, 1, 2, 3$.

Observe now that according to Proposition 3

$$U_j \Psi^{(0)} = \Psi^{(j)}, \quad j = 1, 2, 3, \qquad (8.7)$$

and $U_j = 1 \otimes L + \sigma_j \otimes R$. Consequently we have: $1 \otimes L\ \Psi^{(0)} = \Psi^{(0)}_L = \Psi^{(j)}_L$. Therefore the gauge term may be written in the form $A^\mu_{(m)}\Gamma^{(m)}_\mu \Psi^{(m)} = A^\mu_{(m)}\Gamma^{(m)}_\mu(\Psi^{(0)}_L + \Psi^{(m)}_R)$ which, summed over m gives (remember that $\Gamma^{(j)}_\mu = \sigma_j \otimes \gamma_\mu$ for $j = 1, 2, 3$)

$$\sum_{m=0}^{3} A^\mu_{(m)}\Gamma^{(m)}_\mu \Psi^{(m)} = A^\mu_{(0)}\Gamma^{(0)}_\mu \Psi^{(0)} + \sum_{j=1}^{3} A^\mu_{(j)}\Gamma^{(j)}_\mu \Psi^{(j)}_R + \vec{A}^\mu \cdot \vec{\sigma} \otimes \gamma_\mu \Psi^{(0)}_L, \qquad (8.8)$$

presenting on $SU(2)_L$ internal symmetry for $\Psi^{(0)}_L$. It is easily seen that for $\Psi^{(0)} = \begin{pmatrix} e \\ \nu_L \end{pmatrix}$ where e represents the electron and ν_L the left-handed neutrino, one obtains the equation

$$(p_\mu \gamma^\mu + M)\begin{pmatrix} e \\ \nu_L \end{pmatrix} - \vec{A}_\mu \cdot \vec{\sigma} \otimes \gamma^\mu \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} + (B_\mu \gamma^\mu + \tau) e_R = 0, \qquad (8.9)$$

where $\vec{A}_\mu = \tilde{\Psi}^{(0)} \vec{\sigma} \otimes \gamma_\mu \Psi^{(0)}$, B_μ and τ are vector and scalar fields, which is the starting point of the electroweak model.

8.3 The chargeless leptons

Now going back to eq.(8.1) the doublet L_2 should be free from electroweak charge; therefore a further dimensional reduction is needed which will bring from $n = 3$ to $n = 2$ of Chapter 5, and we will then obtain the equations representing Majorana fermions and/or neutrinos presenting neither electric charge, nor the electrically charged weak one transmitted by W_μ. Therefore they should build up invisible matter, subject only to the gravitational interaction (and possibly neutral weak transmitted by Z^0_μ). They obviously candidate themselves as a source of black matter. They could be originated, in a computable fraction, at the Big Bang and then, since apparently they have no way to decay, they could have accumulated during the whole life of the universe through high energy gravitational phenomena like supernovas. If this could be enough to explain their abundance is an open, but perhaps not unsolvable, problem. An obvious consequence of this explanation for the origin of black matter in galaxies is that it should increase with their age.

In the framework of our interpretation of some of the elementary particle phenomenology in terms of division algebras, if, from the existence of L_2 in eq.(8.1), one could derive the explanation of the existence of black matter one could then affirm that it derives from the role in nature of the real division algebra.

9 Further consequences of Cartan's conjecture on the role of pure spinors

Up to now we have simply derived the straightforward consequences of Cartan's conjecture on the fundamental nature of pure spinor geometry and we found them compatible with the main features of our knowledge of elementary particle phenomenology.

However we exploited rather little of that geometry, precisely: Cartan's equations, Proposition 2 by which pure spinors imply the existence of compact momentum spaces and Proposition 3 on the isomorphisms of Clifford algebras and of spinors. But pure spinor geometry is very rich and, should Cartan's conjecture be right, as it would appear from this analysis, it is then to be expected that its relevance for the description of elementary natural phenomena should be much deeper and wider.

We will try here to list concisely some of the arguments from which this deepening and extension could be attained.

9.1 Fermion's dynamics: masses and charges

We have seen that the spaces which deal with fermion's dynamics are projective null-quadrics in lorentzian momentum spaces defining invariant masses. They also define spheres: from the celestial one S_2, for $n = 1$ of Chapter 3, where to deal with massless neutrinos (and photons) up to the possibly maximal one S_8 or S_{10} for $n = 5$ of Chapter 7. Such invariant masses which may be expected to be correlated, in a way to be determined, with those of the concerned fermions, are steadily increasing with n; that is with the dimensions of the fermion multiplets: a general feature in agreement with what is observed in nature where baryons appearing in triplets (or quadruplets) are heavier than leptons appearing in charged-neutral doublets.

At this point one could hope to be able compute the values of those fermion masses. The way might be long but perhaps not impossible.

But one could also hope to obtain more information on masses from the further elaboration of equations like (7.1) deriving from spinor's simplicity. However one has to remember that in our Cartan's equations the terms P_j with $j > 5$ represent external fields which must also contain multiplicative constants representing charges, like in eq.(5.6), whose values are usually inserted by hand. But in our case also charges seem to be correlated with the mentioned spheres. In fact they originate from a $U(1)$ phase covariance of the corresponding fermions arising from rotations in the circles S_1 which are intersections of those spheres with planes spanned by P_5P_6, for the electric charges and, by P_7P_8 or P_9P_{10} for the strong one. One could then expect that from the geometry in those spheres one could derive dynamical information of the corresponding quantum systems including also those on charges.

The confirmation of this possibility derives, somehow surprisingly, from the study of one particularly simple dynamical system: that of the Hydrogen atom. In fact for that system we got from our spinor geometry whatever we needed. First eq.(5.6) gives us the equation of the proton interacting with the electromagnetic potential A_μ whose equation of motion may be obtained from Maxwell's ones (3.8) (and corresponding inhomogeneous ones [7]). Then, going to the non relativistic limit one may obtain the equation of motion proton-electron (contained in Θ of eq.(5.7)) on a sphere which will be the minimal after S_2 (devoted to massless systems); that is S_3 (obtained from (7.1′) after setting to zero all P_j with $j > 5$). The equation one easily gets for the electron spinor ψ (reduced to one component) is:

$$\psi(\mathbf{u}) = \frac{\alpha}{V(S_3)}\frac{mc}{p_0}\int_{S_3}\frac{\psi(\mathbf{u}')}{(\mathbf{u}-\mathbf{u}')^2}d^3\mathbf{u}' \tag{9.1}$$

where $V(S_3) = 2\pi^2$ is the volume of the unit sphere S_3, $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, p_0 is a unit of momentum, m the electron's mass, and $\mathbf{u}$ is a vector indicating a point on the unit sphere S_3. This equation is the one adopted in 1935 by V. Fock [16] (set here in adimensional form) for the description of the H-atom in the one point compactification S_3 of ordinary 3-dimensional momentum space. Fock showed how this equation solves the problem of H-atom stationary states, after a harmonic analysis from the ball B_3 to S_3 which, for $\psi \to \psi_n$: spherical harmonics on S_3, gives for $E = -p_0^2/2m$ the known eigenvalues E_n of the H-atom stationary states. This solution also sets in evidence the $SO(4)$ symmetry of the H-atom system. We see then that in the spheres in momentum space we may solve purely geometrically the dynamical problem of at least a simple system as that of the H-atom.

Now since charges are correlated with that geometry one could try to obtain geometrically their values by using again harmonic analysis in the spherical domains we have found.

Let us start from eq.(9.1) where the electric charge appears in the adimensional fine structure constant α. Now we have seen that the generating $U(1)$ of electric charge corresponds to rotations in the circle (5.5) therefore we have to start from the sphere $S_4 : p_0^2 = p_1^2 + p_2^2 + p_3^2 + p_5^2 + p_6^2$.

Therefore one may hope to obtain the value of the adimensional fine structure constant α in eq.(9.1), through harmony analysis, like Fock did it for the factor mc/p_0 from B_3, but this time from S_4 and the correlated classical symmetric domains D_5 with boundary Q_5. Now it happens that 3 authors [17] have independently computed it in terms of precisely these domains finding

$$\frac{e^2}{\hbar c} = \frac{8\pi[V(D_5)]^{1/4}}{V(S_4)V(Q_5)} = \frac{1}{137,03608} \tag{9.2}$$

which differs less than $1/10^6$ from the experimental value. Therefore eq.(9.1) could be written in the form:

$$\psi(\mathbf{u}) = \frac{8\pi[V(D_5)]^{1/4}}{V(S_3)V(S_4)V(D_5)}\frac{mc}{p_0}\int_{S_3}\frac{\chi(\mathbf{u}')}{(\mathbf{u}-\mathbf{u}')^2}d^3\mathbf{u}' \qquad (9.3)$$

indicating the possibility of a purely geometric solution of the H-atom problem including the computation of α. This possibility is under study (with P. Nurowski) and will be discussed elsewhere. It is obvious that should one be able to compute the value of the fine structure α in eq.(9.1) one could, by reversing the steps which were performed by Fock, arrive at the Schrödinger equation in space-time or, by relativistic generalization, to eq.(5.6) where both the electric charge and Planck's constant would appear with their values defined, upto a multiplicative factor. At difference with the traditional approach in which they have to be inserted by hand, thus opening the door to the possibility of a full geometrization of quantum mechanics.

9.2 Further aspects and conclusion

The possible central role in nature of pure spinor geometry which follows from Cartan's conjecture presents some epistemogical aspects which we will try to concisely illustrate here.

Let us then adopt the hypothesis that, when dealing with fermions, the appropriate geometry is that of pure spinors whose equations of motion have to be represented in projective momentum spaces whose null vectors are bilinearly expresses in terms of those spinors. Then ordinary euclidean vectors may result, according to Cartan's conjecture, only as sums or integrals - conceived as continuous sums - of those null vectors. Now those integrals are correlated with fundamental objects of geometry and mathematics; that is minimal surfaces, as discovered more than one century ago by Enneper and Weierstrass [20] when they parametrized minimal surfaces through integrals of null lines and, in so doing, opened new chapters of geometry and mathematics. It has been shown [4], [21], that for our lorentzian spaces those integrals of null vectors, are precisely strings. In the frame of Cartan's conjecture their introduction for the study of quantum physics then appears quite natural and necessary. In fact they have been successfully adopted, since several years, somehow arbitrarily, however without correlating them with pure spinors.

A striking parallelism then emerges, between mechanics and the geometry appropriate to describe its equations of motion: while euclidean geometry is perfect for classical mechanics of macroscopic bodies - think of celestial mechanics - neither macroscopic bodies nor euclidean geometry are elementary; the constituents of the former are fermions, while those of the latter are pure spinor. Then for dealing with the former in space-time the appropriate elementary concept is the one of the euclidean point-event while, for the latter, is that of string in space-time which renders the corresponding (quantum) mechanics fundamentally non local, both in space-time and momentum space. In the frame of Cartan's conjecture then, to tray to transplant the elementary concept of point-event from euclidean geometry to the quantum mechanics of fermions, thinking of them as moving euclidean-point particles, would appear incoherent and could create difficulties; as in fact it does. And this both in the first quantization, as stated by the uncertainty principle, and in the second one, where presumably the non locality of strings might play a role.

But the main epistemological aspect which emerges from the adoption of Cartan's conjecture (specially after the $U(1)$ geometrical generation of charges) is the possibility of the full geometrization of quantum mechanics; thus continuing and fulfilling the enterprise already started by V. Fock [16] in 1935 when he partially geometrised, in compactified momentum space, the problem of the hydrogen atom. In this way both great revolutions of last century: relativistic and quantum mechanics would have a purely geometric genesis.

Another aspect of interest for us is that some of the internal symmetry groups (isospin and flavour) appear as generated by reflections rather than as covering of rotation groups. In particular the reflections generating isospin appear to identify with those of the conformal group. The important role of this group for physics was widely studied in last decades, specially by A.O. Barut [22]. It results that conformal reflections, in particular Weyl reflections, have the property of mapping space-time to momentum space and viceversa, which could then be conceived as conformally dual and that both have to be conceived a compactified (Robertson Walker compactification). In the frame of Cartan's conjecture this result may furnish a further motivation why the euclidean concept of point-event may not be defined in quantum field theory. But there are further fascinating consequences which might derive from these results [23].

9.3 Final remark

I wish to dedicate this paper to the memory of my dear and old friend Asim Barut. We met in Trieste in the early '60s when the International Centre for Theoretical Physics was created and of which he became a steady visitor and collaborator. With him it was easy to establish links of scientific collaborations as well as of personal friendship. His departure was a great irreparable loss for all of us.

References

[1] É. Cartan, *Lecons sur la Theorie des Spineurs*, (Hermann, Paris, 1937).

[2] C. Chevalley, *The Algebraic Theory of Spinors*, (Columbia U.P., New York, 1954).

[3] P. Budinich and A. Trautman, *The Spinorial Chessboard*, (Springer, New York, 1989).

[4] P. Budinich, *Found. Phys.* **32**, 1347 (2002).

[5] P. Budinich and A. Trautman, *J. Math. Phys.* **30**, 2125 (1989).

[6] P. Budinich, "On Fermions in Compact Momentum Spaces Bilinearly Constructed with Pure Spinors", `hep-th/0102049`, 9 February (2001).

[7] D. Hestenes, *Space-Time Algebra*, (Gordon Breach, New York, 1987).

[8] R. Penrose and W. Rindler, *Spinors and Space-Time*, (Cambridge University Press, Cambridge, 1984).

[9] G.M. Dixon, *Division Algebras: Octonions Quaternions and Complex Numbers and the Algebraic Design of Physics*, (Kluwer, 1994).

[10] G. Trayling and W.E. Baylis, *J. Phys. A. Math. Gen* **34**, 3309 (2001).

[11] P. Budinich, in course of publication in World Scientific.

[12] A. Sudbery, *Journ. Phys.* **A17**, 939 (1987).

[13] F. Gürsey and C.H. Tze, *On the Role of Division, Jordan, and Related Algebras in Particle Physics*, (World Scientific, Singapore, 1996).

[14] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, (Cambridge University Press, Cambridge, 1995).

[15] T. Dray and C.A. Manogue, *Mod. Phys. Lett.* **A14**, 93 (1999).

[16] V. Fock, *Zeitsch. F. Phys.* **98**, 145 (1935).

[17] M.A. Wyler, *C.R. Acad. Sc.* S.A., 743 (Paris);
G. González-Martin, SB/F/277-00, Universitad Simon Bolivar, Caracas;
F.D.T. Smith, `hep-th/9302030`.

[18] D.B. Fairlie and C.A. Manogue, *Phys. Rev.* **D26**, 475 (1987).

[19] N. Berkovits, "Cohomology in the pure spinor formalism for the superstring", `hep-th 0006003`, 1 June (2000);
N. Berkovits, "Pure spinors are high-dimensional twistors", `hep-th/0404243`, 24 September (2004);
M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, *Nucl. Phys.* **B639**, 182 (2002).

[20] K. Weierstrass, *Monatberichte der Berlines Academic* **612** (1988);
Z. Enneper, *Z. Math. Phys.* **9**, 96 (1894).

[21] P. Budinich and M. Rigoli, *Nuovo Cimento B* **102**, 609 (1988).

[22] A.O. Barut and G. Burnzin, *J. Math. Phys.* **12**, 841 (1971);
A.O. Barut and R. Wilson, *Phys. Rev.* **A40**, 1340 (1989).

[23] A.O. Barut, P. Budinich, J. Niederle and R. Raçzka, *Found. Phys.* **24**, 1461 (1994);
P. Budinich, *Acta Phys. Polonica B* **29**, (1998).