

## NOTES ON THE THEORY OF QUANTUM CLOCK

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### Abstract

In this paper we argue that the notion of time, whatever complicated and difficult to define as the philosophical category, from physicist's point of view is nothing else but the sum of 'lapses of time' measured by a proper clock. As far as Quantum Mechanics is concerned, we distinguish a special class of clocks, the 'atomic clocks'. It is because at atomic and subatomic level there is no other possibility to construct a 'unit of time' or 'instant of time' as to use the quantum transitions between chosen quantum states of a quantum object such as atoms, atomic nuclei and so on, to imitate the periodic circular motion of a clock hand over the dial. The clock synchronization problem is also discussed in the paper. We study the problem of defining a time difference observable for two-mode oscillator system. We assume that the time difference observable is the difference of two single-mode covariant

time observables which are represented as a time shift covariant normalized positive operator measures. The difficulty of finding the correct value space and normalization for time difference is discussed.

*And the heaven departed as a scroll when it is rolled together;...  
And sware by him that liveth for ever and ever, who created heaven,  
and the things that therein are, and earth and the things that therein,  
and the sea, and the things which are therein, that THERE SHOULD  
BE TIME NO LONGER.*

*The Revelation of St. John the Divine*

## 1 Introduction

As it was mentioned by the philosopher Kenneth Denbigh [1]: "The concept of time is not one we could easily do without and yet it presents us at almost every turn with tantalizing paradoxes and largely unresolved problems".

It seems impossible to review briefly even the most important treaties devoted to various aspects of temporality. To mention but a few, are the book by K. Denbigh already cited or more recent ones by H. D. Zeh [2], H. Price [3], J. Barbour [4], or the volume of collected papers [5].

It becomes almost a platitude to quote in the beginning of such discourses the St. Augustine's words, which we following the tradition also attach here: "If nobody asks me, I know what time is, but if I am asked then I am at loss what to say". The Socratic maxim that the recognition of our ignorance is the beginning of wisdom has the profound significance for our understanding of what the things are, and yet the fundamental question: 'What time is?' seems still be unanswered. Perhaps, the most radical solution to this problem was proposed by Julian Barbour, an English independent thinker and theoretical physicist. In his fascinating, controversial and provocative book entitled 'The End of Time', he claims that time is merely an illusion, that time does not exist at all. Among those who also claims that at most fundamental level the standard time of macroscopic physics entirely loses its meaning or even totally absent, are such prominent physicists as B.S DeWitt (see [6], one of the most frequently cited work by this author), J. A. Wheeler [7] and C. Rovelli [8](see in this context also the Ref.[9]).

Whatever the truth is, the concept of time, however, 'is not one we could easily do without'. As J. Barbour himself states: "I shall have to use many more words to express everything in a timeless fashion". We also cannot help to agree with this author, that time

is often mistakenly thought of as it were some sort of existent, some sort of palpable thing just like river or flow. Indeed, in the philosophical discussions 'time' has been bedevilled by such use of slipshod or pictorial language ('time flow', etc.) We are convinced however, that 'time' is not 'out there' as a substantial thing like river in flow, it is rather as abstract quantity, a construction. As J. Barbour formulated it, "The ultimate and only truly real things are the instants of time identified with possible instantaneous arrangements of all things in the Universe". One can draw a parallel between this statement and that one formulated by Bertrand Russell [10]: "... matter is nothing but sets of events" and the other one excerpted from another paper [11]: "When the whole class of events can be well ordered and also when methods exist of constructing certain kinds of well-ordered series of events, the existence of instants can be proved... It is generally agreed that instants are *mathematical constructions* not physical entities".

According to J. Barbour, the first step to a proper theory of time was taken by the German mathematician Carl Neumann. C. Neumann asked how one could make sense of Newton's claim expressed in the law of inertia, that body free of all disturbances would continue at rest or in straight uniform motion for ever. He concluded that for a single body itself such statement could have no meaning. How can we say that a body is moving in a straight line? How can we tell that it is not a subject to force? How are we to tell time if we cannot any bodies free of forces? The answers to these questions will tell us, first, the meaning of duration and second, that what time is, is told us by matter - something has to move or change in some other way if we are to speak of time.

So far we were on the slightly 'shifting soil' of philosophical speculations; in order to proceed to physics, one should do the same thing as all physicists used to do starting from Galilei's epoch. Namely, one should ask the following question: 'How time can be measured?' It is simply because for the physicists time is nothing else but the sum of measurable lapses of time, or in other words, for the physicists 'time' is something which is measured by a proper clock. Notice, that it does not contradict neither J. Barbour's and C. Rovelli's views, nor the B. Russel's theory of 'instants' and order in time, or the A. Einstein statement that "Space and time are modes by which we think,

not the conditions under which we live”.

No doubt that the conundrum of time on most fundamental level, that is in quantum mechanics, is in fact intimately related to understanding of quantum mechanics (QM) itself. We are not going to discuss different approaches to its interpretation (see, for instance [12,13]) as well as the different approaches to the problem of time in QM (for the sample of references, see [5]). Our aim is much modest: we are rather going, in the spirit of the ideas of thinkers quoted above, to treat the 'instants' as measurable lapses of time and to attract attention to the question of how these lapses can be measured in atomic and subatomic levels, or in other words, to the questions of what are the quantum clocks and clock synchronization, in particular.

## 2 Time in Special and General Theory of Relativity

To begin with let us outline briefly the role and meaning of time in classical, that means non-quantum, physics concentrating merely on those aspects of the notion of time which will be important to the following discussion of the quantum clock.

The laws of classical physics, which found their exhaustive expression in Newton's work, assign to time the role of empty duration, without beginning or end, flowing externally at a constant rate, regardless of the events which take place in the world. Therefore, from the mathematical point of view, space and time in Newtonian point mass mechanics is the Galilean space consisting of a 3-dimensional Euclidean space and time axis. In some coordinate representation the physical space is  $\mathbf{R} \times \mathbf{R}^3$ , time is absolute and essentially the same at any point of the 3-space. Dynamics of a point mass is generated by a second order differential equation  $\ddot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$ , the solution to which is  $\mathbf{x} : (t_0, t_1) \mapsto \mathbf{R}^3$  (with given initial conditions) where time is a parameter.

In Hamiltonian mechanics, the phase space is  $2n$ -dimensional smooth symplectic manifold  $\Omega$ . For example, if we have  $k$  particles with masses  $m_i$  located in the space  $\mathbf{R}^3$ , that is,  $k$  position vectors  $\mathbf{x}_i \in \mathbf{R}^3$  then the configuration space of the system is an open submanifold  $M_k := \{(\mathbf{x}_1, \dots, \mathbf{x}_k) \mid \mathbf{x}_i \neq \mathbf{x}_j, i \neq j\}$  of  $\mathbf{R}^{3k}$ . Defining a "mass metric"  $g = (g_{\alpha\beta}) := \text{diag}(m_1, m_1, m_1, \dots, m_k, m_k, m_k)$  we can

form canonically conjugated moments by  $(\mathbf{p}_1, \dots, \mathbf{p}_k) := g(\dot{\mathbf{x}}_1, \dots, \dot{\mathbf{x}}_k) = (m_1 \dot{\mathbf{x}}_1, \dots, m_k \dot{\mathbf{x}}_k)$  from the velocity vectors (which lie in the tangent bundle  $TM_k$ ). Hence, we get the cotangent bundle  $TM_k^*$  which is a  $6k$ -dimensional symplectic manifold. Note that a symplectic manifold is not always the cotangent bundle of some manifold.

For any phase space  $\Omega$  we define the state space  $S := \mathbf{R} \times \Omega$  consisting of pure states. Nonpure states can be defined either as probability measures or, more generally, as positive normalized distributions of  $\Omega$ ; pure states are then either Dirac measures or Dirac  $\delta$ -distributions. The (classical) observables are dynamical variables, that is, Borel or smooth functions  $S \rightarrow \mathbf{R}$  depending on the choice of nonpure states. Note that if we choose Borel functions then we must give up the Lie algebra structure (defined by the Poisson bracket) of observables.

If one chooses a (local) canonical chart

$$\Omega \supseteq U \ni s \mapsto (q_1(s), \dots, q_n(s), p_1(s), \dots, p_n(s)) \in \mathbf{R}^{2n}$$

then, by the inverse mapping of the chart, a physical system can (locally) be described by  $n$  pairs of canonical conjugate dynamical variables  $(q_k, p_k)$  (generalized position and momentum variables). They satisfy the following Poisson bracket relations:  $\{q_k, p_l\} = \delta_{kl}$ ,  $\{q_k, q_l\} = \{p_k, p_l\} = 0$ . For example, in the case of  $k$  point masses, one can choose the identity mapping as a canonical chart and then denote  $\mathbf{q} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$  but generally 3+1-dimensional space-time must be sharply distinguished from the  $2n$ -dimensional phase space of the system, and the space-time coordinates  $(\mathbf{x}, t)$  must be sharply distinguished from the dynamical variables  $(\mathbf{q}, \mathbf{p})$  characterizing physical system in space-time. In particular, the position variable  $\mathbf{q}$  of a point particle must be distinguished from the point  $\mathbf{x}$  where the particle occupies, although using the identity mapping as a chart, we have the relations:  $q_1 = x$ ,  $q_2 = y$ ,  $q_3 = z$  [14]. And again, in Hamiltonian mechanics time is considered as absolute and dynamics is given by the solution of the Hamilton equations.

On the other hand, all the equations of relativity theory, the special as well as general, are formulated in such a way that all four coordinates enter them on the same footing, that is symmetrically, and yet there is no absolute symmetry. The measurements make us to speak about space and time separately, after all! This asymmetry

enters the structure of relativity theory for instance, due to signature, which is determined by the signs of the diagonal elements of metric tensor  $g_{\alpha\beta}$ :

$$(g_{\alpha\beta}) := \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence, the physical space-time of relativity theory has the signature  $(+ - - -)$ ; notice that nothing changes, if one choose the  $(- + + +)$  signature.

It is important to stress that according to General Theory of Relativity (GTR), time is not longer the primordial notion as in Newtonian or Hamiltonian mechanics, but is defined in more fundamental level, namely on space-time which in its turn, is defined as the set of elementary events called 'points'. These are the truly prime entities which are introduced axiomatically. Intuitively they are understood as something which has neither extention, nor duration. Upon this set of "points" one can define at first (topological) Hausdorff space, and then introduce a structure of a differentiable manifold. Only now, using this construction one can meaningfully assign to space-time its dimensions. According to GTR, the time intervals, as well as the object's length, are defined for each observer in accordance with his/her state of motion; each observer describes the events in his/her reference frame. Only in this reference frame space and time can be definitely separated from one another. However, to make this separation in general case of observer's arbitrary motion, is by no means a trivial task.

In the Special Theory of Relativity (STR) the special class of reference frames, so called 'inertial reference frames' is considered. The inertial reference frame is determined by the totality of similar observers moving uniformly with respect to each other along the straight lines. Plainly speaking, it is supposed that in each space point there is an 'observer', and their world lines make the congruence of straight lines. In the GTR the congruence of the observers' world lines makes the family of curves. The space-time is distorted, it is impossible in principle to determine in it the straight lines and hence, the inertial reference frames. In this case the 'direction of time' is determined for

each observer as the tangent to his/her own world line:  $\tau^\alpha = dx^\alpha/ds$  where  $dx^\alpha$  is the coordinate displacement along the world line,  $ds$  its length. In the GTR each observer determines the space directions as to be orthogonal to the tangential vector  $\tau^\alpha$ . Note, that one can do it only in his/her own vicinity, that is locally, not globally. In the other points of space-time there are other observers who determine space directions of their own. As a result, in place of a homogeneous endless space of STR, in GTR we have the local space cross-sections, and it is not obvious whether or not one can combine these cross-sections into a global space. In general case one can only speak of a local 3 + 1-splitting of a 4-dimensional semi-Riemannian manifold.

For our purposes the most convenient way to describe the congruence of the world lines is to use the so called monad method [16]. Then, if congruence of the world lines is given, beside the metric tensor  $g_{\alpha\beta}$ , we have also the 4-vector of monad  $\tau^\alpha$ , by means of which one can construct the new tensor  $h_{\alpha\beta}$ :

$$h_{\alpha\beta} := \tau_\alpha \tau_\beta - g_{\alpha\beta} \quad (1)$$

Its physical meaning is being the metric tensor of a local 3-dimensional space cross-section orthogonal to  $\tau$ . By means of  $\tau_\alpha$  and  $h_{\alpha\beta}$  one can, using an arbitrary vector  $B_\alpha$ , construct some scalar  $B = B_\alpha \tau^\alpha$ , which is an observable time component of vector  $B_\alpha$ . Note, that  $B$  does not depend on the choice of coordinates. Then  $ds^2$ -interval can be expressed as follows:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = d\tau^2 - dl^2, \quad (2)$$

where  $d\tau = dx^\alpha \tau_\alpha$  is the lapse of time and  $dl^2 = h_{\alpha\beta} dx^\alpha dx^\beta$  is the displacement square in the chosen coordinates. It is also worthy to mention that now time is inseparably connected to the time-like lines and only for the chosen time-like line one can meaningfully speak about the lapses of time between two events which are on the same time-like lines.

### 3 Time in Quantum Mechanics: Quantum Clock

As we mentioned in the *Introduction*, our aim here is to discuss some aspects of the problem of quantum clock. Despite the fact that



quantum clock was already discussed in the literature (see [5], esp. *Introduction* and *Chapter. 8*), much remains to be done and many issues are still open questions. We are concerned here mainly with the following issues: the synchronization of quantum clocks and some mathematical aspects of time shift covariant observables in Quantum Mechanics (QM).

The starting point of our inquiry of what time in QM is, was experiences the surprise. Reading the book by Rudolf Peierls entitled 'Surprises in Theoretical Physics' [17], we were astonished coming across the next piece of text devoted to the discussion of time-energy uncertainty relation in QM: 'Landau was fond of making this point (*emphasizing that the physical meaning of time-energy uncertainty relation is quite different from that of position-momentum*, Authors) saying: "There is evidently no such limitations - I can measure the energy and look at my watch; then I know both energy and time".

The Landau's analysis of time-energy uncertainty relation was already revised in the literature (see, for example [18,19]) and we are not going to repeat the critics again. We were startled by this piece of text for entirely different reason. Sanctified by Landau's authority, despite of all its witty, this statement misses the aim simply because at the atomic and subatomic level there are nobody to look at the watch and there is no any watch to look at, and yet in QM there is indeed the parameter, which we refer to and attribute as 'time'.

As we already mentioned in the previous section, in classical mechanics the position variable  $\mathbf{q}$  to be distinguished from the space-point  $\mathbf{x}$  the particle occupies. If we shift a space-point  $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$  then, to call a dynamical variable  $\tilde{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t)$  a *position* observable, it should change *covariantly* in the shift:  $\tilde{\mathbf{q}}(\mathbf{x} + \mathbf{a}, \mathbf{p}, t) = \tilde{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t) + \mathbf{a}$ . Similarly for time: if we shift time  $t \mapsto t + \tau$  then the time observable, i.e. a *clock-variable*,  $\theta(\mathbf{x}, \mathbf{p}, t)$  should change in a time-shift in a covariant way,  $\theta(\mathbf{x}, \mathbf{p}, t + \tau) = \theta(\mathbf{x}, \mathbf{p}, t) + \tau$ . But since Hamiltonian  $H$  generates time shifts, it follows that

$$1 = \left. \frac{\partial \theta(\mathbf{x}, \mathbf{p}, t + \tau)}{\partial \tau} \right|_{\tau=0} = \{\theta, H\}(\mathbf{x}, \mathbf{p}, t)$$

for all points  $(\mathbf{x}, \mathbf{p}, t)$ , that is,  $\{\theta, H\} \equiv 1$ . For example, consider a free one-dimensional particle with  $H(p) = p^2/(2m)$ . Then  $mx/p$  satisfies  $\{mx/p, H\} \equiv 1$ . But now we know that the solutions of

Hamilton equations are of the form  $x(t) = p_0 t/m + x_0$  and  $p(t) = p_0$ , so that time observable is  $m x(t)/p(t) = t + m x_0/p_0$ . Moreover, we already know, in GTR also one should sharply distinguish the coordinate system from the reference frame. But what about QM? As we will see, quantum time observables stand in a particularly simple relation to  $t$  and do exist in specific physical systems, namely in quantum clocks.

One of the first definitions of a clock was given by A. Peres [20]: "A clock is a dynamical system which passes a succession of states at constant time intervals" (see our comments [21]). In accordance with this definition, one may conclude that every dynamical variable of the physical system marks the 'passage of time' as well as gives quantitative measure of the length of the time interval between two events. However, the 'passage of time', or in other words, time as a physical magnitude can only then be properly defined, when on one hand, physical system undergoes changes and on the other, there is some periodic process which provides 'natural scale' for these changes. That is why we believe that at atomic and subatomic levels not an every observable can serve as a clock, but there is a distinguished class of clocks making quantum clocks. These are the quantum objects (atoms, atomic nuclei and so on) in which the transitions between quantum states imitate the periodic circular motion of a clock hand over the dial.

From exhaustive analysis of quantum mechanics, there is an increasingly agreement that self-adjoint operators cannot be used to describe all observables in QM. For example, there is no self-adjoint phase operator of the single-mode electromagnetic field in the sense of the spectral theorem (and phase shift covariance). As we know from the spectral theorem, for any self-adjoint operator  $A$  there exists a unique spectral measure  $E_A$  such that  $A = \int_{\mathbf{R}} x dE_A(x)$ . Moreover, we know that if we measure  $A$  in a (vector) state  $\psi$ , the measurement outcome statistics is given by the probability measure  $X \mapsto \langle \psi | E_A(X) \psi \rangle$ . Giving up the idempotence of  $E_A$  this can be generalized: we simply define that an *observable in quantum mechanics* is a *normalized positive operator measure (POM)  $E$* ; then, similarly as in the case of  $E_A$ , the number  $\langle \psi | E(X) \psi \rangle$  is interpreted as the probability that a measurement of  $E$ , when the system is in the state  $\psi$ , leads to a result in the set  $X$ . The phase problem can be easily solved

using POMs which are covariant under the phase shifts generated by the number operator [22,23].

Consistently with the classical mechanics, one could define time (or clock) observables in QM as POMs which are covariant under the time shifts generated by the Hamiltonian (or energy) operator. Hence, time and energy would form a true canonical pair as in classical mechanics. This topic is discussed in the next two sections.

As the model example let us consider the 'atomic clock', the basic idea of which is straightforward. First, one has to choose the type of atoms on which he/she would operate, second, identify a transition between two nondegenerate eigenstates of an atom. In accordance with our assumption, one can introduce the quantum clock by means of the Hamiltonian

$$H = \sum_i \epsilon_i a_i^\dagger a_i, \quad (3)$$

where  $\epsilon_i$  is the  $i$ th energy level,  $a_i^\dagger$  and  $a_i$  are Fermi's creation and annihilation operators. The next step is to produce an ensemble of these atoms - in an atomic beam or other storage device. Next, illuminate them with radiation from a tunable source that operates near the transition frequency  $\omega_i = \epsilon_i/\hbar$ . When maximum of the absorption is achieved, one has to count the cycles of the oscillator: a certain number of elapsed cycles generates a standard interval of time. However, since everyone has its own right to choose the type of atoms as well as a pair of their eigenstates to record the transition, an important question of the synchronization of these clocks arises, even if all observers are in the same reference frame. Only after establishing the synchronization of the clocks everybody will agree about the durations of time intervals. It becomes almost obvious that synchronization of atomic clock for the different observers which are in the same reference frame, simply means that

$$\begin{aligned} \Delta t &= \left( \frac{\Delta\varphi'}{\omega'} \right)_{\text{mod}(2\pi/\omega')} = \left( \frac{\Delta\varphi''}{\omega''} \right)_{\text{mod}(2\pi/\omega'')} = \\ &= \dots = \left( \frac{\Delta\varphi^{(n)}}{\omega^{(n)}} \right)_{\text{mod}(2\pi/\omega^{(n)})} \end{aligned} \quad (4)$$

where  $\omega', \omega'', \omega^{(n)}$  are the frequencies of the atomic transitions chosen by the particular observer,  $\Delta\varphi', \Delta\varphi'', \Delta\varphi^{(n)}$  are the corresponding

phase differences measured by the observers. One even risk to say that it is the way of how *time* as an itself emerges. It should be noted also that clock synchronization is an important issue with many practical and scientific applications and it was discussed already in recent works [24], although in completely different context. The authors of these papers discussed mainly the clock synchronization protocols based on quantum entanglement. Our aim is somewhat different: to stress the physical meaning of the synchronization according to (4).

Now it is clear that the synchronization of atomic clocks of the observers moving with respect to each other can be achieved by means of an operational line-of-sight exchange of the light pulses between two observers who are co-located with their clocks in their own reference frames. Then, if the atomic clock characterized by the self-frequency  $\omega_0$  passes by the observer with relative velocity  $v$ , the frequency of the signal received by the observer, due to Doppler effect, is

$$\omega = \omega_0 \frac{\sqrt{1 - v^2/c^2}}{1 - (v \cos \alpha/c)}, \quad (5)$$

where  $\alpha$  is the angle between velocity vector  $\mathbf{v}$  and the line-of-sight. In particular, at  $\alpha = \pi/2$ ,  $\omega = \omega_0 \sqrt{1 - v^2/c^2}$  and we immediately arrive at the relativistic effect of the slowing down of "passage of time".

In our analysis we supposed the existence of well-defined harmonic phase or more precisely, phase difference. It should be noted that the authors of Ref. [25] have constructed the phase difference observable only for harmonic oscillator, while it is desirable, in the spirit of our approach, to construct such observable for a wider class of atomic Hamiltonians. However, in the next two sections we define the time difference observable of harmonic oscillators and compare it to the case of phase difference presented in [25].

## 4 Covariant observables in periodic systems

In this section, we introduce the mathematical structure of covariant POMs when the shift-generator has an integer-valued spectrum.

Let  $A$  be a self-adjoint operator with the discrete non-degenerate integer-valued spectrum  $\text{Sp}A \subseteq \mathbf{Z}$  acting on the Hilbert space  $\mathcal{H}$  spanned by the eigenvectors  $\{|a\rangle \mid a \in \text{Sp}A\}$ , i.e.,  $A = \sum_{a \in \text{Sp}A} a|a\rangle\langle a|$ .

Let  $x \mapsto U_\lambda(x) := e^{i2\pi\lambda^{-1}xA}$  be a unitary representation of the real line  $\mathbf{R}$  generated by  $A$  where  $\lambda > 0$  is the period of the shifter  $U_\lambda$ . Thus, for any  $x \in \mathbf{R}$ ,  $U_\lambda(x) = \sum_{a \in \text{Sp}A} e^{i2\pi ax/\lambda} |a\rangle\langle a|$  and  $U_\lambda(\lambda) = I$  (the identity operator). Next we define a  $\lambda$ -periodic covariant observable  $E$ .

Let  $c \in \mathbf{R}$ . Suppose that for any Borel [26] subset  $X$  of  $[c, c + \lambda)$  we have a (bounded) positive operator  $E_\lambda(X)$  of  $\mathcal{H}$  such that for any set  $X$  and real number  $x$ ,

$$U_\lambda(x)E_\lambda(X)U_\lambda(x)^* = E_\lambda(X \dot{+} x)$$

where  $\dot{+}$  means addition mod  $\lambda$  (i.e.  $x \dot{+} y := x + y + k\lambda$  where  $k \in \mathbf{Z}$  is such that  $x + y + k\lambda \in [c, c + \lambda)$ ). Moreover, we normalized  $E_\lambda$  such that  $E_\lambda([c, c + \lambda)) = I$  (and assume that a set function  $X \mapsto E_\lambda(X)$  is  $\sigma$ -additive). Thus,  $X \mapsto E_\lambda(X)$  is a normalized positive operator measure and we say that  $E_\lambda$  is a  $\lambda$ -periodic covariant observable (associated to  $A$ ).

**Theorem.**  $E_\lambda$  is  $\lambda$ -periodic covariant observable associated to  $A$  if and only if

$$E_\lambda(X) = \lambda^{-1} \sum_{a,b \in \text{Sp}A} c_{ab} \int_X e^{i2\pi(a-b)x/\lambda} dx |a\rangle\langle b| \quad (6)$$

for any  $X \subseteq [c, c + \lambda)$  where  $(c_{ab})$  is a positive semidefinite complex matrix with  $c_{aa} \equiv 1$ .

For the proof of this theorem see, e.g., [22,23].

Immediately one sees that the choice of  $c$  is irrelevant; we can define a positive operator  $E_\lambda(X)$  for any bounded subset  $X$  of  $\mathbf{R}$  using equation (6), and hence get the periodicity condition

$$U_\lambda(x)E_\lambda(X)U_\lambda(x)^* = E_\lambda(X + x)$$

for any bounded  $X \subset \mathbf{R}$  and  $x \in \mathbf{R}$ . Then, restricting  $E_\lambda$  to any interval  $[a, a + \lambda)$ ,  $a \in \mathbf{R}$ , one gets a *normalized* positive operator measure; note that  $E_\lambda([0, n\lambda)) = nI$ ,  $n \in \mathbf{N}$ , so that  $E_\lambda$  is not

normalized. If we define a (possibly unbounded) sesquilinear form

$$C := \sum_{a,b \in \text{Sp}A} c_{ab} |a\rangle \langle b|,$$

a kernel of  $E_\lambda$ , we may write (see [27])

$$E_\lambda(X) = \lambda^{-1} \int_X U_\lambda(x) C U_\lambda(x)^* dx.$$

Note that any kernel  $C$  defines a whole family  $\{E_\lambda\}_{\lambda>0}$  of  $\lambda$ -periodic covariant observables associated to  $A$ . If  $C = C_{\text{can}} := \sum_{a,b \in \text{Sp}A} |a\rangle \langle b|$  then we say that  $E_\lambda$ , with the kernel  $C_{\text{can}}$ , is the *canonical  $\lambda$ -periodic covariant observable  $E_\lambda^{\text{can}}$  (associated to  $A$ )* [23]. Defining a (generalized) vector

$$|x; \lambda\rangle := \lambda^{-1/2} U_\lambda(x) \sum_{a \in \text{Sp}A} |a\rangle = \lambda^{-1/2} \sum_{a \in \text{Sp}A} e^{i2\pi ax/\lambda} |a\rangle$$

we can write

$$E_\lambda^{\text{can}}(X) := \int_X |x; \lambda\rangle \langle x; \lambda| dx.$$

Note that  $E_\lambda^{\text{can}}$  is a spectral measure (or formally  $\langle x; \lambda | x'; \lambda \rangle = \delta(x - x')$ ) if and only if  $\text{Sp}A = \mathbf{Z}$  [23].

**Examples.** Physically relevant examples are the following cases:

1. A particle in the one-dimensional box  $[c, c + \lambda]$  where  $\text{Sp}A = \mathbf{Z}$ ,  $A$  is the momentum operator of the particle and  $E_\lambda^{\text{can}}$  (restricted to  $[c, c + \lambda]$ ) is the canonical position observable.
2. Spin-angle systems where  $\text{Sp}A = \{-j, -j + 1, \dots, j - 1, j\}$ ,  $A$  is a spin operator,  $\lambda = 2\pi$ , and  $E_{2\pi}^{\text{can}}$  is the canonical angle observable.
3. Number-phase systems of the single-mode harmonic oscillator where  $\text{Sp}A = \mathbf{N}$ ,  $A$  is the number operator,  $\lambda = 2\pi$ , and  $E_{2\pi}^{\text{can}}$  is the canonical phase observable.
4. Periodic energy-time systems which are described in the next section.

## 5 Covariant time observables associated to energy operators

Let  $H$  be a Hamiltonian (or energy) operator acting on the Hilbert space  $\mathcal{H}$ ; we assume  $H$  to be a positive self-adjoint but possibly unbounded operator. Assuming that time is one-dimensional, that is, an element of a one-dimensional connected manifold we get (essentially) two different types of covariant time observables, namely, a time observable with  $\mathbf{R}$  (the non-periodic case) or the circle (the periodic case) as its value-space (see also [28,29]):

**Definition (nonperiodic system).** A positive operator measure  $E_\infty$  of  $\mathcal{H}$ , defined on the set of subsets of  $\mathbf{R}$ , is a *covariant time observable (associated to  $H$ )* if  $E_\infty(\mathbf{R}) = I$  and

$$e^{itH/\hbar} E_\infty(T) e^{-itH/\hbar} = E_\infty(T + t)$$

for all  $T \subseteq \mathbf{R}$  and  $t \in \mathbf{R}$ .

**Definition (periodic system).** Suppose that there is such a  $\lambda > 0$  that  $e^{i\lambda H/\hbar} = I$  and let  $c$  be a real number. A positive operator measure  $E_\lambda$  of  $\mathcal{H}$ , defined on the set of subsets of  $[c, c + \lambda)$  is a  *$\lambda$ -periodic covariant time observable (associated to  $H$ )* if  $E_\lambda([c, c + \lambda)) = I$  and

$$e^{itH/\hbar} E_\lambda(T) e^{-itH/\hbar} = E_\lambda(T + t)$$

for all  $T \subseteq [c, c + \lambda)$  and  $t \in \mathbf{R}$ .

If  $\text{Sp } \lambda H/\hbar \subseteq \mathbf{N}$  for some positive constant  $\lambda$  then, equivalently, we can define a  $\lambda$ -periodic covariant time observable as a set of operators

$$E_\lambda(T) = \lambda^{-1} \int_T U_\lambda(t) C U_\lambda(t)^* dt, \quad T \subset \mathbf{R} \text{ is bounded,}$$

where  $U_\lambda(t) := e^{itH/\hbar}$  and  $C$  is a kernel of  $E_\lambda$ . We say that  $C$  is a *time kernel*.

Especially we are interested in the case of a  $k$ -mode electromagnetic field. Define a Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$  where  $\mathcal{H}_i$  is a Hilbert space spanned by the number states  $\{|n\rangle_i \mid n \in \mathbf{N}\}$ . Define

a number operator  $N_i := \sum_{n=0}^{\infty} n |n\rangle_i \langle n|$  and (omitting the vacuum energy operator) the total Hamiltonian  $H := \sum_{i=1}^k \hbar \omega_i N_i$  where  $\omega_i$  is the angular frequency of the  $i$ th oscillator so that  $2\pi/\omega_i$  is the period of the oscillation (note the obvious identification of operators  $N_i$  and  $I \otimes \dots \otimes N_i \otimes \dots \otimes I$ ). For any oscillator  $i$  and a time kernel  $C_i := \sum_{n,m=0}^{\infty} c_{n,m}^i |n\rangle_i \langle m|$ , we can define a  $2\pi/\omega_i$ -periodic covariant time observable (associated to the  $i$ th oscillator)

$$E_i(T) = \frac{\omega_i}{2\pi} \int_T U_i(t) C_i U_i(t)^* dt$$

where  $U_i(t) := e^{i\omega_i t N_i}$  and  $T \subset \mathbf{R}$  is bounded. Recall that

$$E_i([t_0, t_0 + k2\pi/\omega_i]) = kI$$

for any  $t_0 \in \mathbf{R}$  (initial time) and  $k \in \mathbf{N}$  (the number of oscillations).

Define then a positive operator  $k$ -measure  $E$  which is defined on the set of (Borel) subsets of  $[0, 2\pi/\omega_1) \times \dots \times [0, 2\pi/\omega_k)$  by equation

$$E(T_1 \times \dots \times T_k) := E_1(T_1) \otimes \dots \otimes E_k(T_k)$$

where  $T_i \subseteq [0, 2\pi/\omega_i)$  for all  $i = 1, 2, \dots, k$ . This measure can be used to define a time difference observables for  $k$  harmonic oscillators. Especially consider the case  $k = 2$  in detail: now

$$E(T_1 \times T_2) = \frac{\omega_1 \omega_2}{4\pi^2} \int_{T_1} U_1(t) C_1 U_1(t)^* dt \otimes \int_{T_2} U_2(t') C_2 U_2(t')^* dt'$$

where  $C_1 = \sum_{n,m=0}^{\infty} c_{n,m}^1 |n\rangle_1 \langle m|$  and  $C_2 = \sum_{n,m=0}^{\infty} c_{n,m}^2 |n\rangle_2 \langle m|$  are time kernels and  $U_1(t) = e^{i\omega_1 t N_1}$  and  $U_2(t) = e^{i\omega_2 t N_2}$  are time shifters; recall that  $E(T_1 \times T_2)$  can be extended to sets of the form  $T_1 \times T_2$  where  $T_1$  and  $T_2$  are bounded subsets of  $\mathbf{R}$ .

Although it is easy to define the phase difference observables of two oscillators a problem arise in the case of time difference. In the case of phase, the value space of a single mode oscillator is always a phase interval, say,  $[0, 2\pi)$ . Hence, the value space of the phase difference observable of two modes is also  $[0, 2\pi)$ . But in the case of time, the value spaces of two single-mode oscillators are  $[t_1, t_1 + 2\pi/\omega_1)$  and  $[t_2, t_2 + 2\pi/\omega_2)$  or, when enlarged,  $\mathbf{R}$  and  $\mathbf{R}$ . So it would be



tempting to define, that the value space of a time difference observable would be  $\mathbf{R}$  also (and then we could restrict it to some time difference interval). But then there exists a difficulty to normalize the time difference observable as the following calculation shows.

Denote  $|n, k\rangle := |n\rangle_1 \otimes |k\rangle_2$ . The density of the bimeasure

$$(T_1, T_2) \mapsto \langle n, k | E(T_1 \times T_2) | m, l \rangle$$

is

$$\begin{aligned} & \frac{\omega_1 \omega_2}{4\pi^2} c_{n,m}^1 c_{k,l}^2 e^{i(n-m)\omega_1 t} e^{i(k-l)\omega_2 t'} = \\ & = \frac{\omega_1 \omega_2}{4\pi^2} c_{n,m}^1 c_{k,l}^2 e^{i[(n-m)\omega_1 + (k-l)\omega_2]t} e^{i(k-l)\omega_1 x}, \end{aligned}$$

where we have introduced the time difference parameter  $x = t' - t$ . The problem is now how to choose the  $[a, b]$  interval of the phase difference  $x$  and the normalization interval  $[c, d]$  such that

$$\frac{\omega_1 \omega_2}{4\pi^2} \int_c^d e^{i[(n-m)\omega_1 + (k-l)\omega_2]t} dt \int_a^b e^{i(k-l)\omega_1 x} dx = \delta_{n,m} \delta_{k,l} \quad (7)$$

for all  $n, m, k, l \in \mathbf{N}$ ; this is the necessary condition for normalization of a time difference observable. (In the case of phase both intervals are  $[0, 2\pi)$  and (formally)  $\omega_1 = \omega_2 = 1$ .) For example, the choice  $\mathbf{R}$  for  $[a, b]$  (the value space of time difference  $x$ ) is not possible since  $\int_{-\infty}^{\infty} e^{i(k-l)\omega_1 x} dx$  does not exist. Moreover, the choice  $[a, b] = [0, 2\pi/\omega_1)$  is not reasonable for the time difference, since the time difference could be out of this interval when  $2\pi/\omega_2 > 2\pi/\omega_1$ . In addition, comparing to the phase difference,  $\int_c^d e^{i[(n-m)\omega_1 + (k-l)\omega_2]t} dt$  would be  $s\delta_{n-m, l-k}$ ,  $s > 0$ , but it is difficult to find an interval  $[c, d]$  for that.

Of course, if the angular frequencies  $\omega_1$  and  $\omega_2$  are the same,  $\omega_1 = \omega_2 = \omega$ , one gets

$$\frac{\omega^2}{4\pi^2} \int_c^d e^{i(n-m+k-l)\omega t} dt \int_a^b e^{i(k-l)\omega x} dx = \delta_{n,m} \delta_{k,l}, \quad (8)$$

when, for example,  $[a, b] = [c, d] = [0, 2\pi/\omega)$ . Then, defining a phase variable  $\theta(t) = \omega t$  we get essentially the same results as in the case

of phase difference introduced in [25], that is, we can define a time difference observable as

$$[0, 2\pi/\omega) \supseteq X \mapsto \sum_{n,m,k,l \in \mathbf{N}} \delta_{n-m,k-l} c_{n,m}^1 c_{k,l}^2 \frac{\omega}{2\pi} \times \\ \times \int_X e^{i(n-m)\omega x} dx |n, k\rangle \langle m, l|.$$

Note that if  $c_{n,m}^1 \equiv 1$  and  $c_{k,l}^2 \equiv 1$  then this observable gives the Pegg-Barnett phase difference distributions [30] in all states.

## 6 Discussion and conclusion

In a popular article [31] W. Heisenberg once mentioned: "Even time (*in QM*, Authors) needs re-interpretation of which the nature is not yet clear...". To our mind, the source of perplexity about time is the deeply ingrained idea of 'instants of time' as of something belonging to a 'river' or 'flow' relentlessly flowing forward. We believe that it is far better to regard 'instants of time' as a theoretical construction. In more fundamental, that is in atomic and subatomic level, it is far better to consider lapses of time as something which could be determined and measured by a proper 'quantum clock'. Such quantum clock is nothing else but the quantum objects, atoms, atomic nuclei etc., in which the transitions between quantum states mimic the periodic motion of a clock hand. In such approach another problem arises however: the problem of proper definition of phase observables in QM. There is an increasing agreement that not all observables of QM can be described by self-adjoint operators. In particular, in this paper to define phase observable we use normalized positive operator measure (POM). It turns out that by means of POMs phase problem can be easily solved, since they are covariant under the phase shifts generated by the number operator.

It is well known that some single-mode phase observables can be measured by using different measurement schemes, for example, eight-port homodyne detection, heterodyne detection, or linear amplification. For example, in eight-port homodyne detection there are two input ports, the port for the signal field and the second port for the reference field, and the fields are combined using beam splitters.

Then photons of output fields are counted using detectors which measure the number of photons arrived to the detector surface in some given time interval of a laboratory clock. The second input field is in a large amplitude coherent state with a fixed phase (strong laser light); it is the reference field which corresponds fixing an origin of the phase interval. Hence, we simply compare the phase of the first field to the phase of the fixed reference field. If we want to measure a phase difference observable of two modes, then we can use two eight-port homodyne detectors which both have the same strong laser field as their reference input field.

If we have some source of (quasi)monochromatic light we could use both homodyne and heterodyne detection schemes to measure its properties. If we have two sources with angular frequencies  $\omega_1$  and  $\omega_2$  then we could use two homodyne or heterodyne detectors for measuring the phase (or time) difference. We should admit however, that it is not yet clear how we could apply these measurements to the case of time difference observables or atomic clock synchronization; the work is under progress and we plan to treat this problem in another publication.

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## Comment on NOTES ON THE THEORY OF QUANTUM CLOCK

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The Authors start their discussion of the notion of quantum time with an extensive reminder of the notion of time in classical physics. They have however forgotten to remind that in classical physics, especially in Special Theory of Relativity, the notion of time  $t$  for each event depends on some "irreducible element of conventionality", namely on the Einstein synchronization condition

$$t = \frac{T_1 + T_2}{2}, \quad (1)$$

where  $T_1$  and  $T_2$  are the measured instants of time when the observer emits and detects the reflected light signals, respectively. H. Reichenbach [1] argued that this Einstein synchronization condition may be replaced by a more general one of the form

$$t = (1 - \epsilon) T_1 + \epsilon T_2, \quad (2)$$

where  $0 < \epsilon < 1$  is an arbitrary parameter. The choice of  $\epsilon$  is a matter of convention and cannot have any physical consequences. Various

physical aspects of such irreducible element of conventionality were discussed in [2] and more recently in [3]. The immediate question arises: **Is the notion of quantum time free from any element of conventionality?** Unfortunately, the Authors do not give an answer to this fundamental question. Let us hope that, in the promised discussion of the synchronization of atomic clocks, the Authors will be able to clarify all elements of conventionality contained in the definitions of quantum time.

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