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EXISTENCE OF TIME OPERATOR FOR A SINGULAR HARMONIC OSCILLATOR

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(Received 21 March 2005; accepted 15 April 2005)

Abstract

The time operator for a quantum singular oscillator of the Calogero-Sutherland type is constructed in terms of the generators of the SU(1,1) group. In the space spanned by the eigenstates of the Hamiltonian, the time operator is not self-adjoint. We show, that the time-energy uncertainty relation can be given the meaning within the Barut-Girardello coherent states defined for the singular oscillator. We have also shown the relationship with the time-of-arrival operator of Aharonov and Bohm.

1 Introduction

The existence of a self-adjoint time operator conjugate to a given Hamiltonian is a longstanding problem of quantum mechanics. The unequal role played by time as an observable in classical and quantum mechanics is the main source of controversy . The problem arises because we expect observables to be represented in quantum mechanics by self-adjoint operators.[1,2]. In an attempt to promote time to be an observable, we have to face a well-known argument of Pauli [3] that such an operator cannot be self-adjoint if the spectrum of a self-adjoint Hamiltonian is bounded from below. As a consequence, the time-energy uncertainty relation cannot be deduced from the same kinematical point of view as the position-momentum uncertainty relation. Nevertheless, the search for various time operators and the analysis of their self-adjointness and associated time-energy uncertainty relations have been the subject of a number of papers [4]. The general consensus seems to be that no such self-adjoint operator exists.

Recently, the validity of Pauli's objections has been critically evaluated [5], with the conclusion that there is no a priori reason to exclude the existence of self-adjoint time operators for semibounded Hamiltonians.

In this work we consider the problem of constructing a self-adjoint time operator for a singular harmonic oscillator.

2 The problem

The singular harmonic oscillator of the Calogero-Sutherland type [6] is described by the Hamiltonian

$$H_{CS} \equiv 2\omega K_3 = \omega^2 K + H, \tag{1}$$

where $K = x^2/2$ and

$$H = \frac{1}{2}(p^2 + \frac{g}{x^2}),\ g > 0. \tag{2}$$

is the Calogero-Moser [7] scale invariant Hamiltonian. We have identified H_{CS} Hamiltonian with the compact generator, K_3, of the SU(1,1) group, which is the dynamical group of this problem. Two other generators of $SU(1,1)$ are

$$\begin{aligned} K_1 &= \frac{1}{2}(\omega K - \frac{1}{\omega}H), \\ K_2 &= D, \end{aligned} \tag{3}$$

where $D = -(xp + px)/4$ is the scale operator.

The group generators K_3 and $K_\pm = K_1 \pm iK_2$ satisfy the standard commutation relations of the su(1,1) algebra:

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3. \tag{4}$$

Our objective is to construct an operator $\hat{T}$ in terms of the generators $K_3, K_\pm$ that is conjugate to the Hamiltonian H_{CS} and satisfies $[H_{CS}, \hat{T}] = i$.

Let us denote by $|n, k>, n = 0, 1, 2, ...$ the complete orthonormal basis states, which diagonalize the compact generator $K_3 = H_{CS}/2\omega$ [8]. The Bargman index $k = \frac{1}{2}(1 + \sqrt{g + \frac{1}{4}})$ is related to the eigenvalue $k(k-1) = g$ of the quadratic Casimir operator $\hat{C}_2 = K_3^2 - K_1^2 - K_2^2$ of the $SU(1,1)$ group. These states are obtained from $|0, k>$ by n-fold application of K_+:

$$\begin{aligned} |n, k> &= \sqrt{\frac{\Gamma(2k)}{\Gamma(2k+n)n!}}(K_+)^n|0, k>, \\ K_-|0, k> &= 0, \\ K_3|n, k> &= (n + k)|n, k>, \ n = 0, 1, 2, ... \end{aligned} \tag{5}$$

In the space spanned by the eigenstates of the generator K_3, we immediately encounter the problem. The matrix elements of $[H_{CS}, \hat{T}]$ in the basis $|n, k>$,

$$< n, k|[H_{CS}, \hat{T}]|m, k >= 2\omega(n - m) < n, k|\hat{T}|m, k > \tag{6}$$

vanish for $n = m$. This implies that $[H_{CS}, \hat{T}] \neq i$ if $<n, k|\hat{T}|m, k> \neq 0$. This relation is correct only if the operation by $\hat{T}$ on a state $|n, k>$ is of the form

$$\hat{T}|n, k >= \sum_m t_{mn}|m, k > . \tag{7}$$

However, $\hat{T}$ does not have that property if it is conjugate to H_{CS}. In the next Section, we shall study the commutator $[H_{CS}, \hat{T}]$ in the time-variable representation, i.e., in the representation in which $\hat{T}$ is diagonal.

3 Construction of $\hat{T}$

Let us observe that from $[K_3, K_-^n] = -nK_-^n$, we can make a simple ansatz that $\hat{T}$ is some power series function of K_- and K_+ such that

$$[K_3, F(K_\pm)] = \pm K_\pm F'(K_\pm) = \frac{i}{4\omega}. \tag{8}$$

A possible solution is

$$\hat{T} = \frac{1}{4i\omega}(lnK_- \; - \; lnK_+), \tag{9}$$

which is easily represented in the coherent state representation of the operator K_-.

States which diagonalize the operator K_-

$$K_-|z,k> = z|z,k>, \tag{10}$$

where z is an arbitrary complex number, are known as the Barut-Girardello (BG) coherent states [9], [10]. The expansion of these states over the orthonormal basis $|n,k>$ is

$$|z,k\rangle = \frac{z^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(2k+n)}}|n,k\rangle, \tag{11}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. The above BG states are normalized to unity, they resolve the identity operator, but are not mutually orthogonal

$$< z_1, k|z_2, k> = I_{2k-1}(2\sqrt{z_1^* z_2})[I_{2k-1}(2|z_1|)I_{2k-1}(2|z_2|)]^{-1/2}. \tag{12}$$

Due to this property any quantum state $|\psi>$ can be represented by the analytic function

$$f_\psi(z) = \sqrt{I_{2k-1}(2|z|)}(z^{1/2-k}) < k, z^*|\psi> . \tag{13}$$

The operators $K_\pm$ and K_3 act in the Hilbert space of analytic functions $f_\psi(z)$ as linear differential operators

$$K_+ = z \ , \quad K_- = 2k\frac{d}{dz} + z\frac{d^2}{dz^2} \ , \quad K_3 = k + z\frac{d}{dz} \ . \qquad (14)$$

In terms of BG coherent states, the time operator $\hat{T}$ is

$$\hat{T} = (4\pi i)^{-1} \int d\mu(z,k) ln(\frac{z}{z*})|z,k><z,k|, \qquad (15)$$

where $d\mu(z,k) = 2K_{2k-1}(2|z|)I_{2k-1}(2|z|)d^2z/\pi$.

4 Discussion

BG coherent states can also be written as an exponential operator acting on the vacuum state of K_-,

$$|z,k>= e^{zK_+(K_3+k)^{-1}}|0,k> . \qquad (16)$$

In deriving this expression, we have used an operator identity

$$[K_+(K_3+k)^{-1}]^n = K_+^n \frac{\Gamma(K_3+k)}{\Gamma(K_3+k+n)}. \qquad (17)$$

Note also that the operator $K_+(K_3+k)^{-1}$ is canonical to K_-:

$$[K_-, K_+(K_3+k)^{-1}] = 1. \qquad (18)$$

It is easy to see that $H = \omega(K_3 - K_1)$ is related to K_-:

$$e^{-\omega K} H e^{\omega K} = -2\omega K_-. \qquad (19)$$

Therefore, the energy eigenstates of $H|E>= E|E>$ are proportional to the BG coherent states [9,11] if $z = -E/2\omega$:

$$|E>= e^{\omega K}|-\frac{E}{2\omega}, k> . \qquad (20)$$

Note also, that the state $<x|E>$, in the limit $E \to 0$, is not normalizable, since $\lim_{E\to 0} <x|E>=<x|e^{\omega K}|0,k>\propto \omega^k x^{2k-1/2}$.
The difficulty arises from the oscillating behavior of $<x|E>$ at large distances [12].

Finally, we consider an explicit construction of time operator for H_{CS} using the method developed in [13,14,15]. We first observe that there exists a singular similarity transformation between H_{CS} and the Hamiltonian of the ordinary harmonic oscillator, $H_h = H_{CS}(g = 0)$:

$$\begin{aligned} H_{CS}S &= SH_h, \\ S &= e^{-K_-}e^{K_-^0}, \end{aligned} \tag{21}$$

where $K_-^0 = K_-(g = 0)$. The time operator for H_h was constructed and discussed in [14, 15, 16]. Its construction is simple if we observe that the Casimir operator with $k = 3/4$ can be used to express the operator K in the form

$$\begin{aligned} K &= T_0H_0T_0 + \frac{1}{16H_0} \\ &= QH_0Q - \frac{i}{2}Q, \\ Q &= -T_0 + \frac{i}{4H_0}. \end{aligned} \tag{22}$$

where

$$T_0 = -\frac{1}{2}(x\frac{1}{p} + \frac{1}{p}x) \tag{23}$$

is the time-of-arrival operator of Aharonov and Bohm [16], and $H_0 = p^2/2$. Then the Hermitian operator

$$T_h = \frac{1}{2}(T_h(Q) + T_h^\dagger(Q)) \tag{24}$$

satisfies $[H_h, T_h] = i$, where

$$T_h(Q) = \frac{1}{\omega}arctg(\omega Q). \tag{25}$$

It is now easy to see that the time operator for the Hamiltonian H_{CS} is

$$T_{CS} = ST_hS^{-1}, \quad S^{-1} \neq S^\dagger. \tag{26}$$

Note that in this construction $T_{CS} \neq T_{CS}^\dagger$. Formally, in the limit $\omega \to 0$ we obtain the time operator for the scale invariant Hamiltonian H [17].

5 Conclusions

In conclusion, we have presented an algebraic method of constructing Hermitian operators conjugate to a Hamiltonian with $SU(1,1)$ dynamical symmetry. In terms of generators of $SU(1,1)$, the time operator for a singular harmonic oscillator is constructed explicitly, and shown that it can be related to the time-of-arrival operator, T_0 of Aharonov and Bohm. The question whether time operators thus constructed are self-adjoint operators in Hilbert space requires a careful examination of their spectra and eigenfunctions. The time-energy commutation relation is studied in the energy and the time domains. The eigenvalue problem of the operator T_{CS} can be solved in the time domain using BG coherent states. It is not self-adjoint and its eigenfunctions are not orthogonal. Therefore, the problem of finding self-adjoint T_h and T_{CS} is still open [5,18]. Let us also mention that the problem of time-operator for a repulsive singular potential of the Calogero-Moser type [7] is interesting for several reasons. It is scale invariant and has the full conformal group $SO(2,1)$ as a dynamical symmetry group [12] with the generators H, D and K. The spectrum of H, for $g > 0$ is positive, continuous, and bounded from below, but with a non-normalizable ground state. H can be easily extended to the well-known one-dimensional N-body problem of Calogero-Moser [7]. Recently, it has been observed that the dynamics of scalar particles near the horizon of a black hole is also associated with this Hamiltonian [17,19,20,21].

It is important to point out here that the solution of the formal equation $[H, \hat{T}] = i$ is not unique. Any $\hat{T}' = \hat{T} + \phi(H)$, with arbitrary ϕ, satisfies the same canonical commutation relation.

Acknowledgment

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under Contract No.0098004.

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Comment on EXISTENCE OF TIME OPERATOR FOR SINGULAR HARMONIC OSCILLATOR

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The paper deals with a construction of time operators for one particle Calogero-Moser (CM) and Calogero-Sutherland (CS) models. By definition the time operator $\hat{T}$ is self-adjoint one canonically conjugate to a given Hamiltonian $\hat{H}$; $[\hat{T}, \hat{H}] = \hat{1}i$. The most serious problem concerning the time operator is its very existence. Due to Pauli's argument no self-adjoint time operator exists for semibounded or discrete quantum mechanical Hamiltonian, otherwise the unitary operator $exp(-ia\hat{T})$,where a is a real number, will map the discrete or bounded spectrum of H into the entire real line. There have been attempts to avoid this conclusion by pointing out a formal character of Pauli's assumptions concerning the domains of the operators involved and trying to arrange situations where these assumptions are not met. However, it seems that instead of trying to provide rather particular and exotic counterexamples to Pauli's arguments it would be more interesting and important to understand and explain their meaning perhaps in a more general framework. In the present article, in spite of the title, the authors consider mainly algebraic and rather formal constructions of operators conjugate to Hamiltonians of CM and CS models. A unitary transformations that connects CM system with free particle and similarity transformation which relates CS model with harmonic oscillator are used to construct relevant operators starting from known ones; T_0 corresponding to free particle and T_h corresponding to harmonic oscillator respectively. It is known that T_0 is not self-adjoint while the question whether T_h is self-adjoint remains at the best open. By very construction the same can be said about operators conjugate to CM and CS Hamiltonians.