

A TIME REPRESENTATION

Lucas Lamata and Juan León

Instituto de Matemáticas y Física Fundamental, CSIC

Serrano 113-bis, 28006 Madrid, Spain

e-mail: lamata@imaff.cfmac.csic.es, leon@imaff.cfmac.csic.es

(Received 17 May 2005; accepted 3 June 2005)

Abstract

The paper contains a proposal for an energy and time representation. We construct modes that correspond to fuzzy distributions around discrete values of energy or time. The modes form an orthogonal and complete set in the space of square integrable functions. Energy and time are self adjoint in the space spanned by the modes. The widths of the modes are analyzed as well as their energy-time uncertainty relations. The lower uncertainty attainable for the modes is shown. We also show times of arrival for massless particles.

The Pauli theorem revisited

Two arbitrary states of an elementary system can be transformed into each other by symmetry operations. This opens the door to express what can be observed of the system, i.e. the system properties, in terms of the generators of these symmetries. In the case of the Poincare group they are the momenta \hat{P}^μ and the angular momenta and boosts $\hat{M}^{\mu\nu}$. For the system to be elementary they are constrained by the mass shell condition $\hat{P}^2 = m^2$ and by the spin condition $\hat{W}^2 = m^2 s(s+1)$. (The Pauli-Lubanski vector is defined as $\hat{W}_\mu = \epsilon_{\mu\nu\alpha\beta} \hat{P}^\nu \hat{M}^{\alpha\beta}$, with $\hbar = 1$ in this paper unless otherwise specified).

Notice now the conjunction of both, the four vector character of the momenta on one side, with the necessity of introducing conjugate operators to the three-momentum to formulate dynamics on the other. This calls for the introduction of a four vector operator \hat{Q}^μ conjugate to \hat{P}^μ such that

$$[\hat{P}^\mu, \hat{Q}^\nu] = ig^{\mu\nu} \quad (1)$$

Regretfully, this is not possible to attain. The reason is that \hat{Q}^μ would produce translations in the momentum. If \hat{P}^μ is defined on the mass shell, then

$$\hat{P}'^\mu = \exp(-i\delta p \hat{Q}) \hat{P}^\mu \exp(i\delta p \hat{Q}) = \hat{P}^\mu + \delta p^\mu \quad (2)$$

and, no matter the value of the four vector parameter δp , $\hat{P}'^2 \neq m^2$. The transformations generated by \hat{Q} pull the momentum \hat{P} out of the particle mass shell.

The problem is independent of the form of dynamics in use. It has far reaching consequences. First, it is necessary to abandon a four vector \hat{Q}^μ and hence the hope of building a covariant form dynamics in terms of momenta and positions [1]. Then, with time and position demoted to the role of mere parameters, it is necessary to introduce quantum fields with creation and annihilation operators playing the role of conjugate pairs. To our accustomed eyes they look like the appropriate recipe for particle number non conservation [2] but, Is it so? At least two important lessons remain: 1. That time is along the direction that would be conjugate to the solution of the mass shell constraint and 2. That time runs due to the constraint.

A time representation

With the usual non covariant choice, where the mass shell condition reads as $\hat{P}^0 = H(\hat{\mathbf{P}})$, the problem turns into a version of the Pauli theorem [3], namely

$$\hat{H}' = \exp(-i\delta E \hat{T}) \hat{H} \exp(i\delta E \hat{T}) = \hat{H} + \delta E, \text{ with } \delta E \in \mathbf{R} \quad (3)$$

This implies that the spectrum of \hat{H} has to be the real line, something that runs again the existence of the physically necessary ground state. So, in the time form of dynamics the problem looks like the incompatibility of \hat{T} with the boundedness of the Hamiltonian. On the other hand, if \hat{T} is not selfadjoint, then $[\hat{H}, \hat{T}] \neq i$ and there are way outs from the problem.

Perhaps the simplest case for an observable time is the time of arrival (TOA) of a particle at a certain position in one space dimension [4]. Classically $t(x) = m(x - q)/p$ where q, p are the dynamical variables of phase space (initial position and momentum) and x the arrival position. Obtaining the quantum version $\hat{t}(x)$ of this TOA is complicated due to the presence of $1/\hat{p}$ and to operator ordering. It is well known that there is no selfadjoint $\hat{t}(x)$, its most symmetrical expression being

$$\hat{t}(x) = -\exp(-ix\hat{p}) \sqrt{\frac{m}{\hat{p}}} \hat{q} \sqrt{\frac{m}{\hat{p}}} \exp(ix\hat{p}) \quad (4)$$

The eigenvectors of this operator, $|t, x, s\rangle$ can be given in the momentum representation where $1/\hat{p}$ is easier to deal with, as

$$\langle p | t, x, s \rangle = \theta(sp) \sqrt{\frac{|p|}{m}} \exp(i\frac{p^2}{2m} t) \langle p | x \rangle \quad (5)$$

The degeneration parameter s can take the values $+1$ for right movers, -1 for left movers. This is the only residue left at one space dimension of the continuous manifold of directions present for higher D.

The bad news come in the form of nonorthogonality of the time eigenstates. This is a consequence of the fact that the Hamiltonian is bounded from below: $\sigma(\hat{H}) = [E_0, \infty)$ with $E_0 = 0$. By using (5) we get

$$\begin{aligned} \langle t, x, s | t', x, s' \rangle &= \frac{1}{2\pi} \delta_{ss'} \int_{E_0=0}^{\infty} \exp(iE(t-t')) = \\ \frac{1}{2\pi} \delta_{ss'} \lim_{\epsilon \rightarrow 0^+} \frac{i}{t' - t + i\epsilon} &= \frac{1}{2} \delta_{ss'} (\delta(t-t') - \frac{i}{\pi} \mathcal{P} \frac{1}{t-t'}) \end{aligned} \quad (6)$$

To get orthogonality it is necessary to move E_0 to $-\infty$ something that resembles the Weisskopf-Wigner trick for resonances. In any case it is clear that the problem source is in $\sigma(\hat{H}) \neq \sigma(\hat{t})$.

Marolf devised a procedure, used in ref [5], to surmount nonorthogonality. The idea is to replace \hat{t} and \hat{H} by “regularized” expressions that avoid the difficulties that arise when $p \rightarrow 0$, namely, to introduce

$$f_\epsilon(p) = \left\{ \frac{m}{|p|} \text{ if } |p| > \epsilon, \text{ else } \frac{m|p|}{\epsilon^2} \right\} \text{ with } \epsilon \text{ small and positive} \quad (7)$$

Then,

$$\hat{t}(x) = -\exp(-ix\hat{p}) \sqrt{f_\epsilon(\hat{p})} \hat{q} \sqrt{f_\epsilon(\hat{p})} \exp(ix\hat{p}) \quad (8)$$

an expression tailored for the momentum representation where

$$\langle p | t, x, s \rangle_\epsilon = \theta(sp) \frac{1}{\sqrt{f_\epsilon(p)}} \exp(iE_\epsilon(p) t) \langle p | x \rangle \quad (9)$$

The “regularized energy” is given by

$$E_\epsilon(p) = \int_{\pm\epsilon}^p \frac{dp'}{f_\epsilon(p')} = \{E(p) - E(\epsilon) \text{ if } |p| > \epsilon, \text{ else } E(\epsilon) \ln\left(\frac{E(p)}{E(\epsilon)}\right)\} \quad (10)$$

Thus, while $E(p) = 0$ when $p = 0$, $E_\epsilon(p) \rightarrow -\infty$ as $p \rightarrow 0$. This solves the problem by removing the lower bound in $\sigma(\hat{H})$. It is straightforward to show that

$$\epsilon \langle t, x, s | t', x, s' \rangle_\epsilon = \delta_{ss'} \delta(t-t') \quad (11)$$

The procedure works fine for the case of free particles. With minor obvious modifications, it works equally well for relativistic particles and for any number of space dimensions. However, it is only suitable for those cases where momentum remains constant. So, its very definition brings this procedure to a dead end.

Soon after the publication of ref [5] Giannitrapani [6] observed that the time of arrival was an instance of generalized observable endowed with a probabilistic interpretation as a positive operator valued (POV) measure. In fact, the time of arrival eigenstates form a complete set:

$$\langle p | \left\{ \sum_s \int dt |t, x, s\rangle \langle t, x, s| \right\} |p'\rangle = \langle p |p'\rangle \quad (12)$$

Not being orthogonal they can not constitute a projector valued measure, but – as pointed out in ref [6] – nothing prevents from using them to construct a POV measure. Giannitrapani showed that the probability that the time of arrival at x of the state ρ be in the range $[T, T']$ is

$$P_x([T, T']) = Tr \left[\rho \left(\sum_s \int_T^{T'} dt |t, x, s\rangle \langle t, x, s| \right) \right] \quad (13)$$

The mean value and variance of the TOA are given in [6], where the problems arising from nonorthogonality when trying to obtain kinematical uncertainty relations between energy and time are discussed. The paper also analyzes the free particle TOA operator of (4) acting on the domain of infinitely differentiable functions over the compact subsets of values of $p \in \mathbf{R} - \{0\}$. In this case the variance of \hat{t} turns out to be computable as for ordinary observables and the Heisenberg uncertainty relation holds. To summarize: Endowed with the POV measure interpretation, the TOA became an useful instrument appropriate to extract physical information from one of the primary laboratory events: “when” a detector clicks. No wonder this is the customary approach in the current literature.

From eigenvectors to wave packets

Wigner introduced [7] in 1932 a distribution at midway between the position and the momentum representation with the aim of describing particle properties in phase space. The non positivity of the distribution was an obstruction to its use as a probability distribution. Later on, Husimi [8] introduced sets of minimal uncertainty

states $|q p\rangle$ in position and momentum:

$$\langle x|q p\rangle = (2\pi\sigma^2)^{-1/4} \exp\left(-\frac{(x-q)^2}{4\sigma^2}\right) \exp(ipx) \quad (14)$$

The distribution is centered at the point (q, p) . Given a system in an arbitrary state $|\psi\rangle$, the probability that the system occupy a region in phase space centered at (q, p) of half widths $(\sigma, 1/2\sigma)$ is given by $\rho_H(q, p) = (2\pi)^{-1} |\langle q p|\psi\rangle|^2$

We now return to the time of arrival of a free particle at a specific position in one space dimension. We learnt the difficulties that arise in this seemingly simple problem. They can be articulated through the Pauli theorem in two complementary ways: The impossibility of finding a conjugate pair of time and energy selfadjoint operators or, the difficulty to connect the two different spectra $\sigma(\hat{t}) = \mathbf{R}$ and $\sigma(\hat{H}) = \mathbf{R}_+$. To avoid them, we will follow a procedure that resembles Husimi's.

We start by introducing two functions $g_\nu(t)$ and $f_\nu(\omega)$ defined over the real line $t \in \mathbf{R} = (-\infty, +\infty)$ and the positive real line $\omega \in \mathbf{R}_+ = (0, +\infty)$ respectively. We take ν as a positive integer $\nu > 1$. Also, we assume g_ν and f_ν to be Fourier transforms of each other. Finally,

$$g_\nu(t) = \frac{1}{(\beta + it)^\nu}, \quad f_\nu(\omega) = \frac{\sqrt{2\pi}}{\Gamma(\nu)} \omega^{\nu-1} \exp(-\beta\omega) \quad (15)$$

These functions on which t (respectively ω) act multiplicatively, present the following nice property under Fourier transformation:

$$\begin{aligned} t g_\nu(t) &\leftrightarrow -i \frac{df_\nu(\omega)}{d\omega} \\ i \frac{dg_\nu(t)}{dt} &\leftrightarrow \omega f_\nu(\omega) \end{aligned} \quad (16)$$

We point out that for $\nu > 1$ both derivatives act on these functions as selfadjoint operators ($f_\nu(0) = 0$). It is also remarkable that $g_\nu \in L^2(\mathbf{R})$ and $f_\nu \in L^2(\mathbf{R}_+)$. So, they can be given a probabilistic interpretation. Finally, to the canonical pair $(t, i \frac{d}{dt})$ acting on t (the would be "Time Representation") corresponds the unitarily equivalent pair $(-i \frac{d}{d\omega}, \omega)$ acting on ω (the would be "Energy Representation").

This looks like a promising starting point for constructing true conjugate representations for time and energy. The obstruction is the need of completeness of the g_ν in \mathbf{R} and of the f_ν in \mathbf{R}_+ to build systems of generators for square integrable functions. This is a necessary condition to arrive at a POV measure. If in addition we find orthogonality, then we would reach a PV measure. Notice however, that these measures would correspond to fuzzy distributions around a central value, in much the same way that the Husimi Gaussian wave packets do. This can be seen with our g_ν and f_ν , which are not eigenstates of time or energy.

$$t g_\nu(t) = i\beta g_\nu(t) - i g_{\nu-1}(t), \quad \omega f_\nu(\omega) = \nu f_{\nu+1}(\omega) \quad (17)$$

The construction of eigenfunctions is hopeless at this stage. For instance, be $\psi(\omega) = \sum_{\nu \geq \nu_0} c_\nu f_\nu(\omega)$ a putative eigenfunction, so that $\omega \psi(\omega) = \lambda \psi(\omega)$. Then $\psi(\omega) \propto \delta(\omega - \lambda)$; expanding this in the $f_\nu(\omega)$ would at least require that they form a complete set, which is not the case. The power of this simple example is that it signals the way to proceed.

Energy and time representations

Orthogonal polynomials are a useful tool to solve a variety of problems in physics. The Laguerre polynomials $L_n^\alpha(x)$ constitute a set of orthogonal polynomials on the interval $(0, \infty)$ with weight $x^\alpha e^{-x}$. Accordingly, we can define the set of orthogonal functions

$$\varphi_n^\alpha(\omega) = \{0 \text{ if } \omega < 0, \text{ else } c_n^\alpha \left(\frac{\omega}{\omega_0}\right)^{\alpha/2} e^{-\omega/2\omega_0} L_n^\alpha(\omega/\omega_0)\} \quad (18)$$

where

$$c_n^\alpha = \left(\frac{\Gamma(n+1)}{\omega_0 \Gamma(\alpha+n+1)}\right)^{1/2} \quad L_n^\alpha(x) = \sum_{m=0}^n n(-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} \quad (19)$$

and ω_0 is a short of width of the exponential distribution that also serves to keep dimensions right (still, $\hbar = 1$). This definition, provides a discrete denumerable set of modes that constitute a basis for all the functions belonging to $L^2(0, \infty)$. These functions comprise all functions with positive frequencies (i.e. $\omega > 0$) that can also

be associated to probabilities (in fact, $\int_0^\infty d\omega |\varphi_n^\alpha(\omega)|^2 = 1$). The orthogonality and completeness relations read explicitly as

$$\int_0^\infty d\omega \overline{\varphi_n^\alpha(\omega)} \varphi_{n'}^\alpha(\omega) = \delta_{nn'} \text{ and } \sum_n \varphi_n^\alpha(\omega) \overline{\varphi_n^\alpha(\omega')} = \delta(\omega - \omega') \quad (20)$$

The physical information contained in the modes is readily obtained.

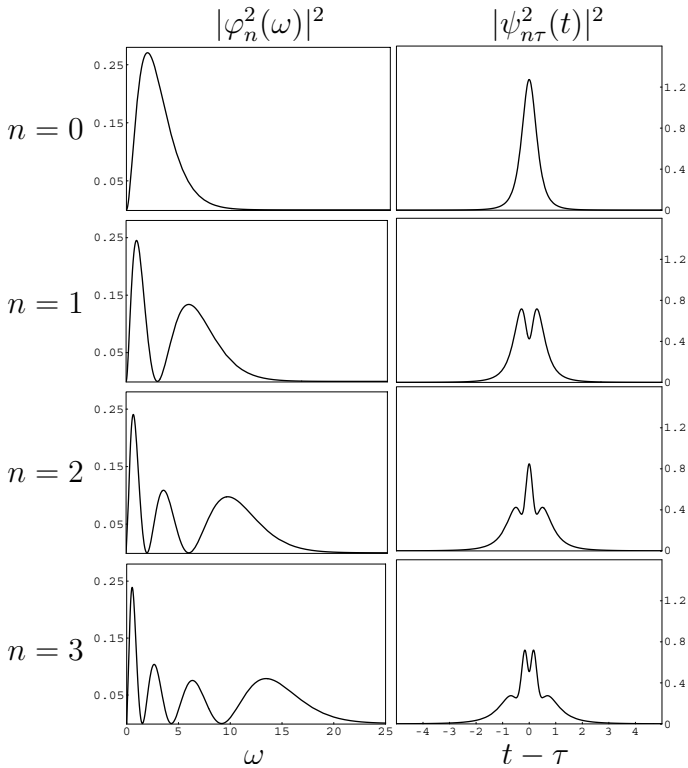


Figure 1: The lowest modes for $\alpha = 2$. ω is given in units of ω_0 and $t - \tau$ in units of ω_0^{-1} . The number of maxima for each mode is $n + 1$. The mode widths grow with increasing n .

In this representation, the Hamiltonian, in spite of its simple form

A time representation

$\hat{H} = \omega$ is a non-diagonal matrix with elements given through

$$\hat{H} |n\rangle = \sum_{m=n-1}^{n+1} d_{mn}^1 \omega_0 |m\rangle \quad (21)$$

In the same way

$$\hat{H}^2 |n\rangle = \sum_{m=n-2}^{n+2} d_{mn}^2 \omega_0^2 |m\rangle \quad (22)$$

The coefficients d_{mn} 's are computable by standard methods. Notice that d_{mn}^2 is not $(d_{mn}^1)^2$. The average value of the energy in the n -th mode is

$$\langle n | \hat{H} | n \rangle = d_{nn}^1 = (\alpha + 2n + 1) \omega_0 \quad (23)$$

and the average of the squared Hamiltonian

$$\langle n | \hat{H}^2 | n \rangle = d_{nn}^2 = [(n+1)(\alpha+n+1) + (\alpha+2n+1)^2 + n(\alpha+n)] \omega_0^2 \quad (24)$$

Finally, the variance of \hat{H} , $(\Delta H)_n^2$ in the n -th mode is given by

$$\begin{aligned} (\Delta H)_n^2 &= \langle n | \hat{H}^2 | n \rangle - \langle n | \hat{H} | n \rangle^2 = d_{nn}^2 - (d_{nn}^1)^2 \\ &= [(n+1)(\alpha+n+1) + n(\alpha+n)] \omega_0^2 \end{aligned} \quad (25)$$

Another question to investigate are the eigenvalues E and eigenfunctions Ψ_E of the Hamiltonian. Ψ_E can be expanded in terms of the modes as $\Psi_E = \sum_n c_n |n\rangle$ which translates into the set of linear equations $\sum_n \omega_0 d_{mn}^1 c_n = E c_m$ left to the interested reader as an exercise.

We take the next step forward and define wave packets centered at the point (τ, E) in time energy space. To simplify the discussion we assume that E is the average energy in the n -th mode $E = (\alpha + 2n + 1)\omega_0$. We assume – for our purposes here – that we can trade (τ, n) by (τ, E) . Finally, the wave packet we are looking for is:

$$\varphi_{n\tau}^\alpha(\omega) = e^{i\omega\tau} \varphi_n^\alpha(\omega) \quad (26)$$

The meaning of this time τ just introduced can best be explored in the time representation. The Fourier transform of the modes $|n, \tau\rangle$ bring them from the energy to the time representation that is

$$\psi_{n\tau}^\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega e^{-i\omega(t-\tau)} \varphi_n^\alpha(\omega) \quad (27)$$

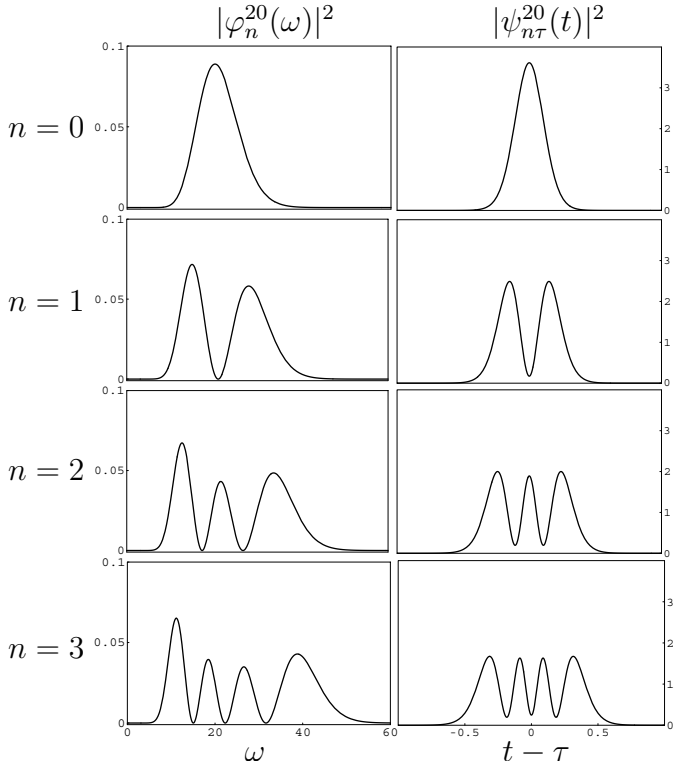


Figure 2: The effect of increasing α . The modes plotted here are for $\alpha = 20$. Notice that here the width in t is one order of magnitude smaller than in Fig. 1

Notice that $\psi_{n\tau}^\alpha(t) = \psi_{n0}^\alpha(t - \tau)$ by construction. Also, as the $\varphi_{n\tau}^\alpha(\omega)$ are a basis in \mathbf{R}_+ they constitute a basis for $L^2(\mathbf{R})$. So, we conclude that they form a system of imprimitivity leading to a PV measure. This is valid for both, the ω and the t representations as they are unitarily equivalent.

The explicit expression of the modes in the t representation is quite cumbersome, but throws some light on their physical content

A time representation

and on the way that the two different spectra are connected. From ref [9] we learn that the integral in (27) can be carried out giving:

$$\begin{aligned} \psi_{n\tau}^\alpha(t) &= \frac{\omega_0}{\sqrt{2\pi}} c_n^\alpha \frac{\Gamma(\alpha/2 + 1)\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + 1)} \\ &\times \frac{{}_2F_1(-n, \alpha/2 + 1; \alpha + 1; (1/2 + i\omega_0(t - \tau))^{-1})}{(1/2 + i\omega_0(t - \tau))^{\alpha/2+1}} \end{aligned} \quad (28)$$

Being the first entry a negative integer $-n$, the hypergeometric function terminates. Explicitly

$$\begin{aligned} {}_2F_1(-n, \alpha/2 + 1; \alpha + 1; (1/2 + i\omega_0(t - \tau))^{-1}) &= \\ \sum_{m=0}^n \frac{(-n)_m (\alpha/2 + 1)_m}{(\alpha + 1)_m m!} (1/2 + i\omega_0(t - \tau))^{-m} \end{aligned} \quad (29)$$

where each term in the sum in (29) comes from the corresponding term in the sum in (19).

Due to the fact that $\psi_{n0}^\alpha(t)^* = \psi_{n0}^\alpha(-t)$, we have $\langle n, \tau | \hat{T} | n, \tau \rangle = \tau$. To compute $\langle n, \tau | \hat{T}^2 | n, \tau \rangle$, we consider the derivative of the mode $\varphi_n^\alpha(\omega)(\omega_0 = 1)$:

$$\frac{d}{d\omega} \varphi_n^\alpha(\omega) = \left(\frac{\alpha}{2\omega} - \frac{1}{2} \right) \varphi_n^\alpha(\omega) - \frac{c_n^\alpha}{c_{n-1}^\alpha} \varphi_{n-1}^\alpha(\omega) \Theta(n, 0), \quad (30)$$

where we define $\Theta(n, 0) = 1$ when $n > 0$ and zero otherwise. On the other hand, the Laguerre polynomials verify

$$L_n^\alpha = \sum_{m=0}^n L_m^{\alpha-1}, \quad (31)$$

and thus

$$\frac{d}{d\omega} \varphi_n^\alpha(\omega) = \sum_{l=0}^n N_{nl}^\alpha \varphi_l^{\alpha-2}(\omega) - \frac{1}{2} \varphi_n^\alpha(\omega) - \frac{c_n^\alpha}{c_{n-1}^\alpha} \varphi_{n-1}^\alpha(\omega) \Theta(n, 0), \quad (32)$$

where

$$N_{nl}^\alpha = \frac{\alpha}{2} \frac{c_n^\alpha}{c_l^{\alpha-2}} (n - l + 1). \quad (33)$$

The matrix element $\langle n, \tau | \hat{T}^2 | n, \tau \rangle$ results, after some straightforward operations,

$$\langle n, \tau | \hat{T}^2 | n, \tau \rangle \equiv \int_{-\infty}^{\infty} dt t^2 \psi_{n\tau}^\alpha(t)^* \psi_{n\tau}^\alpha(t) = \tau^2 + \int_{-\infty}^{\infty} dt t^2 \psi_{n0}^\alpha(t)^* \psi_{n0}^\alpha(t), \quad (34)$$

where we use $\langle n, 0 | \hat{T} | n, 0 \rangle = 0$. The last integral in (34) is given in

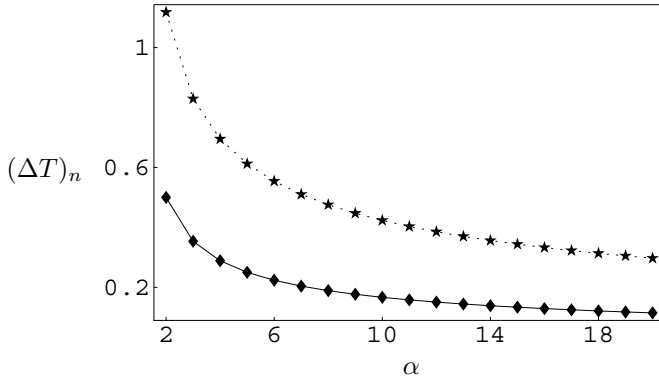


Figure 3: $(\Delta T)_n$ (in units of ω_0^{-1}), as a function of α , for $n = 0$ (continuous) and $n = 3$ (dotted).

energy representation by

$$\int_{-\infty}^{\infty} dt t^2 \psi_{n0}^\alpha(t)^* \psi_{n0}^\alpha(t) = - \int_0^\infty d\omega \varphi_n^\alpha(\omega) \frac{d^2}{d\omega^2} \varphi_n^\alpha(\omega) = \int_0^\infty d\omega \left[\frac{d}{d\omega} \varphi_n^\alpha(\omega) \right]^2. \quad (35)$$

Making use of (32) we get

$$\begin{aligned} \langle n, \tau | \hat{T}^2 | n, \tau \rangle &= \tau^2 + \frac{1}{4} + \sum_{l=0}^n (N_{nl}^\alpha)^2 + \left(\frac{c_n^\alpha}{c_{n-1}^\alpha} \right)^2 \Theta(n, 0) \quad (36) \\ &- \sum_{l=0}^n N_{nl}^\alpha \int_0^\infty d\omega \varphi_l^{\alpha-2}(\omega) \varphi_n^\alpha(\omega) \\ &- 2 \frac{c_n^\alpha}{c_{n-1}^\alpha} \Theta(n, 0) \sum_{l=0}^n N_{nl}^\alpha \int_0^\infty d\omega \varphi_l^{\alpha-2}(\omega) \varphi_{n-1}^\alpha(\omega) \end{aligned}$$

A time representation

These expressions can be readily computed. The results confirm the expectations: The value of \hat{T}^2 is near τ^2 , the closer to it the larger α . We plot this behaviour in Fig. 3. Notice that (35) is nothing else but the variance $(\Delta T)_n^2$ for the mode n , something explicit after (34).

It is possible to build minimal uncertainty packets by combining different modes with appropriate coefficients. Instead, we will show how close the packets are to the lowest uncertainty. We do this in Fig. 4. Perhaps, the most prominent feature of these uncertainty relations is that the lowest uncertainty reached for each mode is bounded by $n + 1/2$, an asymptotic value corresponding to $\alpha \rightarrow \infty$. On the other hand, the uncertainty remains bounded for the cases of physical interest.

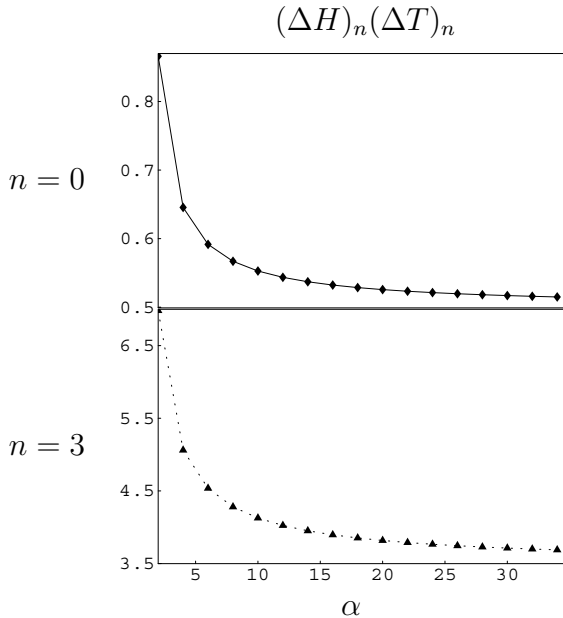


Figure 4: $(\Delta H)_n(\Delta T)_n$ as a function of α , for $n = 0, 3$.

Finally, we look for the time of arrival at x of some state $|\Psi\rangle$. We describe the state by a ket to avoid unwieldy notation, but the dis-

cussion could apply equally to density matrices. Working in parallel to what done to get (4) and (5), we could give the modes for mean time of arrival τ at x with direction $s = \pm 1$ (for right and left movers respectively). They are given by:

$$\langle n \tau x s | \Psi \rangle = \int_0^\infty d\omega \langle x | p(\omega) \rangle \overline{\varphi_{n\tau}^\alpha(\omega)} \langle \omega | \Psi \rangle \quad (37)$$

the notation $p(\omega)$ stands for the dispersion relation governing the system. For a massive non relativistic particle $\omega = p^2/2m$, etc. We

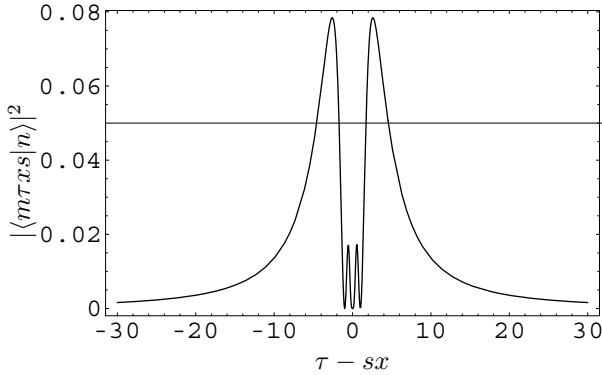


Figure 5: $|\langle m\tau xs|n \rangle|^2$ for $m = 1$, $n = 3$ and $\alpha = 2$.

choose the case of a massless particle where $p(\omega) = s\omega$. This not only avoids square roots but focus on the very interesting case of photons. In fact for a photon [10] we could use $\langle x | \Psi \rangle = \mathbf{F}(x) = \mathbf{E}(x) + i\mathbf{B}(x)$, whose Schrödinger equation reads

$$i \frac{\partial \mathbf{F}(x, t)}{\partial t} = -i (\boldsymbol{\sigma} \wedge \nabla) \mathbf{F}(x, t) \quad (38)$$

where $\boldsymbol{\sigma}$ is the photon spin. Admitting the quantum leap involved in reducing this to one space dimension, it would translate into $i \frac{\partial F}{\partial t} = -i \frac{\partial F}{\partial x}$. This is what we are using for the dispersion relation. By decomposing the state in terms of the modes $\langle \omega | \Psi \rangle = \sum_m \phi_m^\alpha \varphi_m^\alpha(\omega)$, one gets after some computation that

A time representation

$$\begin{aligned} \langle n \tau x s | \Psi \rangle &= \sum_m \psi_{nm}^\alpha \frac{(i\omega_0(\tau - sx))^{m+n}}{(1 + i\omega_0(\tau - sx))^{m+n+1}} \\ &\times {}_2F_1 \left(-m, -n; -m - n - \alpha; \frac{1 + (\omega_0(\tau - sx))^2}{(\omega_0(\tau - sx))^2} \right) \end{aligned} \quad (39)$$

where $\psi_{nm}^\alpha = \frac{\omega_0}{\sqrt{2\pi}} c_n^\alpha c_m^\alpha \phi_m^\alpha \frac{\Gamma(m+n+\alpha+1)}{m!n!}$. In Figs. 5 and 6 we plot these quantities for two cases of interest. Notice the distribution of these values around $\tau = sx$. This was expected. Notice also the zero at this value. This is just the modes orthogonality.

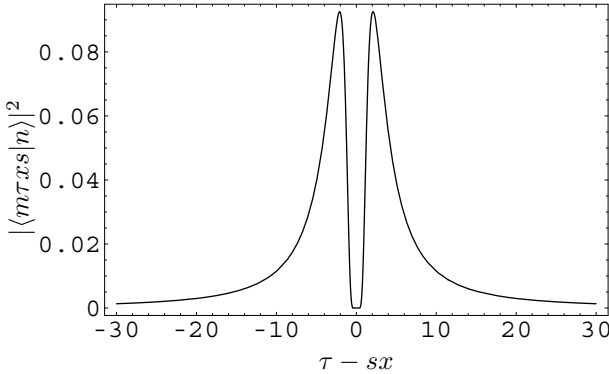


Figure 6: $|\langle m\tau xs|n \rangle|^2$ for $m = 1$, $n = 3$ and $\alpha = 20$.

Conclusions

We have constructed a representation for time and energy in which both operate in a selfadjoint manner. The representation has the nice property of being suitable for probabilistic interpretation. It is given in terms of a set of modes build in terms of Laguerre polynomials and their weight functions. This allows to surmount quite easily the difficulties associated to the different support of the energy and time spectra. These difficulties not only translate into the Pauli theorem, but also prevent some asymptotic behaviour of the modes, precisely

the more interesting from the physical point of view [11]. According to the Paley Wiener theorem XII [12], it is not possible to get exponential asymptotic behaviour for the functions of time that are Fourier transforms of bounded functions of energy. Here, we dealt with the problem by using our modes. They are not exponential in time; this is out of reach. However, they have a controlled variance and uncertainty relations, in a form that makes them suitable for physics.

ACKNOWLEDGMENTS

This work was partially supported by the Ministerio de Educación y Ciencia of Spain under project BMF 2002-00834. The work of L. Lamata was supported by the FPU grant AP2003-0014.

Note added in proof: After the submission of this paper the authors became aware of the interesting article Jos M. Isidro, *Phys. Lett. A* 334 370 (2005), in which a similar problem is studied from a different approach.

References

- [1] P. Teller, *An interpretive introduction to Quantum Field Theory*, Princeton University Press, Princeton 1995.
- [2] R. J. Crewther, *Introduction to Quantum Field Theory*, hep-th/9505152.
- [3] W. Pauli, *General Principles of Quantum Mechanics*, Springer, Berlin, 1980.
- [4] J. G. Muga and C. R. Leavens, *Phys. Rep.* 338, 353, 2000.
- [5] N. Grot, C. Rovelli and R. S. Tate, *Phys. Rev. A* 54, 4676 (1996), quant-ph/9603021
- [6] R. Giannitrapani, *J. Math. Phys.* 39, 5180 (1998). quant-ph/9611015.
- [7] E. P. Wigner, *Phys. Rev.* 40, 749 (1932).

A time representation

- [8] K. Husimi, *Phys. Math. Soc. Japan*, 22, 264 (1940).
- [9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products*, Academic Press, San Diego, 2000. Formulas 7.414.7 and 7.414.4.
- [10] I. Białynicki-Birula, *Progress in Optics*. 36, 245 (1997).
- [11] I. Białynicki-Birula, *Phys. Rev. Lett.* 80, 5247 (1998).
- [12] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, New York 1934.

Comment on A TIME REPRESENTATION

Edward Kapuścik

Department of Physics, University of Łódź

Henryk Niewodniczański Institute of Nuclear Physics PAN

Kraków, Poland

In my opinion the paper goes into the right direction of disjoining the notion of the Hamiltonian and the notion of the generator of the time evolution. It is a common believe that these two notions are identical. As a matter of fact, it is certainly true in the classical Hamilton - Jacobi formalism but has never been proved in quantum mechanics. It was simply transmitted to quantum mechanics by the correspondence principle. The example of the notion of angular momentum and the notion of the generators of rotations explicitly shows that there is a difference between classical and quantum generators of symmetry. In quantum physics apart from the orbital momentum the spin arises. This shows that the relation between classical generators of symmetry and classical physical observables is different from the relation of quantum mechanical generators of symmetry and quantum observables. It is not excluded that the quantum mechanical generator of time translation is not identical with the quantum mechanical Hamiltonian. This statement is strengthened by the fact that already in classical physics we may have many non equivalent Hamiltonians[1] and it is not clear which one to use as a generator

Comment

of time translation in quantum mechanics. Therefore, I strongly welcome the construction presented in the paper where the Authors use two different representations, the time representation and the energy representation, which are only unitary equivalent but not identical.

References

- [1] A. Horzela, *Concepts of Physics 1* (2004) 11 - 50.