

COMBINATORICS OF BOSON NORMAL ORDERING AND SOME APPLICATIONS

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Abstract

In this work we consider the problem of normal ordering of boson creation and annihilation operators. This ordering procedure facilitates quantum mechanical calculations in the coherent states representation and is commonly used in e.g. quantum optics. We provide the solution to the normal ordering problem for powers and exponentials of two classes of operators. The first one consists of boson strings and more generally homogeneous polynomials, while the second one treats operators linear in one of the creation or annihilation operators. Both solutions generalize Bell and Stirling numbers arising in the number operator case. We use the advanced combinatorial analysis to provide closed form expressions, generating func-

tions, recurrences, etc. The analysis is based on the Dobiński-type relations and the umbral calculus methods. As an illustration of this framework we point out the applications to the construction of generalized coherent states, operator calculus and ordering of deformed bosons.

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1 Introduction

In this work we are concerned with the one mode boson creation a^\dagger and annihilation a operators satisfying the commutation relation $[a, a^\dagger] = 1$. We are interested in the combinatorial structures arising in the problem of normal ordering of a wide class of boson expressions. It shall provide us with the effective tools for systematic treatment of these problems.

By normal ordering of an operator expressed through the boson a and a^\dagger operators we mean moving all annihilation operators to the right of all creation operators with the use of commutation relation. This procedure yields an operator which is equivalent (in the operator sense) to the original one but has a different functional representation. It is of both mathematical and physical interest. From the mathematical point of view it allows one to represent the operator by a function of two, in a sense 'commuting', variables. It is also connected with the physicist's perspective of the problem. Commonly used, the so called coherent state representation (see Appendix A) implicitly requires the knowledge of the normally ordered form of the operators in question. Put the other way, for the normally ordered operator its coherent state matrix elements may be immediately read off. This representation is widely used *e.g.* in quantum optics. We mention also that calculation of the vacuum expectation values is much easier for operators in the normally ordered form. For other applications see [KS85].

A standard approach to the normal ordering problem is through the Wick theorem. It directly links the problem to combinatorics, *i.e.* searching for all possible contractions in the boson expression and then summing up the resulting terms. This may be efficiently used for solving problems with finite number of boson operators (especially when one uses the computer algebra packages). Although this looks very simple in that form it is not very constructive in more sophisticated cases. The main disadvantage is that it does not give much help in solving problems concerning operators defined through infinite series expansions. To do this we would have to know the underlying structure of the numbers involved. This still requires a lot of careful analysis (not accessible to computers). In this work we approach these problems using methods of advanced combinatorial analysis [Com74]. It proves to be an efficient way for obtaining com-

compact formulas for normally ordered expansion coefficients and then analyzing their properties.

A great body of work was already put in the field. In his seminal paper [Kat74] Jacob Katriel pointed out that the numbers which come up in the normal ordering problem for $(a^\dagger a)^n$ are the Stirling numbers of the second kind. Later on, the connection between the exponential generating function of the Bell polynomials and the coherent state matrix elements of $e^{\lambda a^\dagger a}$ was provided [Kat00][Kat02]. In Chapter 3 we give a modern review of these results with special emphasis on the Dobiński relations. We also make use of a specific realization of the commutation relation $[a, a^\dagger] = 1$ in terms of the multiplication X and derivative D operators. It may be thought of as an introduction to the methods used later on in this text. Chapter 4 is written in that spirit. We use this methodology to investigate and obtain compact formulas for the coefficients arising in normal ordering of a boson monomial (string of boson creation and annihilation operators). Then we proceed to the normal ordering of powers of a boson string and more generally homogeneous boson polynomial (*i.e.* the combinations of the of the boson strings with the same excess of creation over annihilation operators). The numbers appearing in the solution generalize Stirling and Bell numbers of the second kind. For this problem we also supply the exponential generating functions which are connected with the exponentials of the operators in question. In each case we provide the coherent state matrix elements of the boson expressions.

Recalling the current state of the knowledge in the field we should also mention the approach based on Lie-group methodology [Wil67]. It proves useful for normal ordering problems for the exponentials of expressions which are quadratic in boson operators [Meh77][AM77]. In a series of papers by various authors [Wit75][Mik83][Mik85][Kat83] some effort in extending these results to operators having the specific form $(a^\dagger a + (a^\dagger)^r)^n$ was made. In Chapter 5 we systematically extend this class to operators of the form $(q(a^\dagger)a + v(a^\dagger))^n$ and $e^{\lambda(q(a^\dagger)a + v(a^\dagger))}$ where $q(x)$ and $v(x)$ are arbitrary functions. This is done by the use of umbral calculus methods [Rom84] in finding representations of the monomiality principle (*i.e.* representations of the Heisenberg-Weyl algebra in the space of polynomials) and application of the coherent state methodology. Moreover, we establish a one-to-

one connection between this class of normal ordering problems and the family of Sheffer-type polynomials.

These two classes of problems extend the current state of knowledge on the subject in two different directions in a quite systematic way. We believe this to be a considerable push forward in the normal ordering problem. We also emphasize the use of combinatorial methods which we give explicitly and which prove to be very efficient in this kind of analysis.

We illustrate this approach by examples taken both from combinatorics and physics (*e.g.* for the hamiltonian of a generalized Kerr medium). Moreover we also comment on applications and extensions of the formalism. We give just a few of them in Chapter 6. First we observe how to extend this formalism to the case of deformed commutation relations. In general it can be done at the expense of introducing operator-valued Stirling numbers. Next, we use the fact that the use of the Dobiński relations allow us to find the solutions to the Stieltjes moment problem for the numbers arising in the normal ordering procedure. We use these solutions to construct and analyze new families of generalized coherent states [KS85]. We end by commenting on the specific form of the formulas obtained in Chapter 5 which may be used to derive the explicit action of generalized shift operators, called the substitution theorem. These three examples already show that the methods we use in solving normal ordering problems lead to many diverse applications. At the end this text we append complete list of publications which indicate some other developments on the subject.

2 Preliminaries

In this chapter we give basic notions exploited later on in the text. We start by fixing some conventions. Next we recall the occupation number representation along with the creation a^\dagger and annihilation a operator formalism. This serves to define and comment on the normal ordering problem for boson operators. We end by pointing out a particular representation of the above in terms of the multiplication and derivative operators.

2.1 Conventions

In the mathematical literature there is always a certain freedom in making basic definitions. Sometimes it is confusing, though. For that reason it is reasonable to establish explicitly some conventions in the beginning.

In the following by an indeterminate we primarily mean a formal variable in the context of formal power series (see Appendix B). It may be thought as a real or complex number whenever the analytic properties are assured.

We frequently make use of summation and the product operations. We give the following conventions concerning their limits

$$\sum_{n=N_0}^N \dots = 0 \quad \text{and} \quad \prod_{n=N_0}^N \dots = 1 \quad \text{for } N < N_0.$$

Also the convention $0^0 = 1$ is applied.

Moreover we define the so called *falling factorial* symbol by

$$x^{\underline{k}} = x \cdot (x - 1) \cdot \dots \cdot (x - k + 1), \quad x^{\underline{0}} = 1,$$

for nonnegative integer k and indeterminate x .

By the *ceiling* function $\lceil x \rceil$, for x real, we mean the nearest integer greater or equal to x .

2.2 Occupation number representation: Boson operators

We consider a pair of one mode boson annihilation a and creation a^\dagger operators satisfying the commutation relation

$$[a, a^\dagger] = 1. \quad (1)$$

Together with the identity operator the generators $\{a, a^\dagger, 1\}$ constitute the *Heisenberg-Weyl* algebra.

The *occupation number representation* arises from the interpretation of a and a^\dagger as operators annihilating and creating a particle (object) in a system. From this point of view the Hilbert space \mathcal{H} of states (sometimes called Fock space) is generated by the *number states* $|n\rangle$, where $n = 0, 1, 2, \dots$ count the number of particles (for bosons up to infinity). We assume here the existence of a unique *vacuum* state $|0\rangle$ such that

$$a|0\rangle = 0. \quad (2)$$

Then the number states $\{|n\rangle\}_{n=0}^\infty$ may be taken as an orthonormal basis in \mathcal{H} , *i.e.*

$$\langle n|k\rangle = \delta_{n,k} \quad (3)$$

and

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1. \quad (4)$$

The last operator equality is called a *resolution of unity* which is equivalent to the completeness property.

It may be deduced from Eqs.(1) and (2) that operators a and a^\dagger act on the number states as

$$\begin{aligned} a|n\rangle &= \sqrt{n} |n-1\rangle, \\ a^\dagger|n\rangle &= \sqrt{n+1} |n+1\rangle. \end{aligned} \quad (5)$$

(Indetermined phase factors may be incorporated into the states.)
Then all states may be created from the vacuum through

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \quad (6)$$

It also follows that the *number operator* N counting the number of particles in a system and defined by

$$N|n\rangle = n|n\rangle \quad (7)$$

may be represented as $N = a^\dagger a$ and satisfies the the following commutation relations

$$\begin{aligned} [a, N] &= a, \\ [a^\dagger, N] &= -a^\dagger. \end{aligned} \tag{8}$$

This construction may be easily extended to the multi-boson case. The canonical form of the commutator Eq.(1) originates from the study of a quantum particle in the harmonic oscillator potential and quantization of the electromagnetic field. Any standard textbook on Quantum Mechanics may serve to survey of these topics.

We note that the commutation relations of Eqs.(1) and (8) may be easily extended to the deformed case [Sol94] (see also Section 6.1).

2.3 Normal ordering

The boson creation a^\dagger and annihilation a operators considered in previous section do not commute. This is the reason for some ambiguities in the definitions of the operator functions in Quantum Mechanics. To solve this problem one has to additionally define the order of the operators involved.

These difficulties with operator ordering led to the definition of the *normally ordered* form of the boson operator in which all the creation operators a^\dagger stand to the left of the annihilation operators a .

There are two well defined procedures on the boson expressions yielding a normally ordered form. Namely, the *normal ordering* \mathcal{N} and the *double dot* $: :$ operations.

By normal ordering of a general function $F(a, a^\dagger)$ we mean $\mathcal{N}[F(a, a^\dagger)]$ which is obtained by moving all the annihilation operators a to the right using the commutation relation of Eq.(1). We stress the fact that after this normal ordering procedure, the operator remains the same $\mathcal{N}[F(a, a^\dagger)] = F(a, a^\dagger)$. It is only its *functional* representation which changes.

On the contrary the double dot operation operation $: F(a, a^\dagger) :$ means the same procedure but *without* taking into account the commutation relation of Eq.(1), *i.e.* moving all annihilation operators a to the right as if they commute with the creation operators a^\dagger .

We emphasize that in general $F(a, a^\dagger) \neq :F(a, a^\dagger):$. The equality holds only for operators which are already in normal form (*e.g.* $\mathcal{N}[F(a, a^\dagger)] = :F(a, a^\dagger):$).

Using these two operations we say that the *normal ordering problem* for $F(a, a^\dagger)$ is solved if we are able to find an operator $G(a, a^\dagger)$ for which the following equality is satisfied

$$F(a, a^\dagger) = \mathcal{N}[F(a, a^\dagger)] \equiv :G(a, a^\dagger): \quad (9)$$

This normally ordered form is especially useful in the coherent state representation widely used in quantum optics (see Appendix A). Also calculation of the vacuum expectation values in quantum field theory is immediate whenever this form is known.

Here is an example of the above ordering procedures

$$\begin{array}{ccc}
 aa^\dagger aaa^\dagger a & \xrightarrow[\substack{\mathcal{N} \\ [a, a^\dagger]=1}]{} & \underbrace{(a^\dagger)^2 a^4 + 4 a^\dagger a^3 + 2 a^2}_{\substack{a^\dagger \text{ - to the left} \quad a \text{ - to the right}}} \\
 aa^\dagger aaa^\dagger a & \xrightarrow[\substack{\vdots \vdots \\ a, a^\dagger \text{ - commute} \\ \text{(like numbers)}}]{} & \overbrace{a^\dagger a^\dagger a a a}
 \end{array}$$

Another simple illustration is the ordering of the product $a^k(a^\dagger)^l$ which is in the so called *anti-normal* form (*i.e.* all annihilation operators stand to the left of creation operators). The double dot operation readily gives

$$: a^k(a^\dagger)^l : = (a^\dagger)^l a^k,$$

while the normal ordering procedure \mathcal{N} requires some exercise in the use of Eq.(1) yielding (proof by induction)

$$a^k(a^\dagger)^l = \mathcal{N}[a^k(a^\dagger)^l] \equiv \sum_{p=0}^k \binom{k}{p} l! (a^\dagger)^{l-p} a^{k-p}. \quad (10)$$

These examples explicitly show that these two procedures furnish completely different results (except for the operators which are already in normal form).

There is also a 'practical' difference in their use. That is while the application of the double dot operation $::$ is almost immediate, for the normal ordering procedure \mathcal{N} certain skill in commuting operators a and a^\dagger is needed.

A standard approach to the problem is by the *Wick theorem*. It reduces the normal ordering procedure \mathcal{N} to the double dot operation on the sum over all possible contractions (*contraction* means removal of a pair of annihilation and creation operators in the expression such that a precedes a^\dagger). Here is an example

$$\begin{aligned}
 aa^\dagger aaa^\dagger aaa^\dagger &= \sum : \{ \text{all contractions} \} : \\
 &= : aa^\dagger aaa^\dagger aaa^\dagger : \\
 &+ : aa^\dagger aaa^\dagger a \not{a} \not{a}^\dagger + aa^\dagger aaa^\dagger \not{a} a \not{a}^\dagger + aa^\dagger a \not{a} a^\dagger aa \not{a}^\dagger + \\
 &\quad aa^\dagger \not{a} aa^\dagger aa \not{a}^\dagger + \not{a} a^\dagger aaa^\dagger aa \not{a}^\dagger + aa^\dagger a \not{a} \not{a}^\dagger aaa^\dagger + \\
 &\quad aa^\dagger \not{a} a \not{a}^\dagger aaa^\dagger + \not{a} a^\dagger aa \not{a}^\dagger aaa^\dagger + \not{a} \not{a}^\dagger aaa^\dagger aaa^\dagger : \\
 &+ : aa^\dagger a \not{a} \not{a}^\dagger a \not{a} \not{a}^\dagger + aa^\dagger \not{a} a \not{a}^\dagger a \not{a} \not{a}^\dagger + \not{a} a^\dagger aa \not{a}^\dagger a \not{a} \not{a}^\dagger + \\
 &\quad \not{a} \not{a}^\dagger aaa^\dagger a \not{a} \not{a}^\dagger + aa^\dagger a \not{a} \not{a}^\dagger \not{a} a \not{a}^\dagger + aa^\dagger \not{a} a \not{a}^\dagger \not{a} a \not{a}^\dagger + \\
 &\quad \not{a} a^\dagger aa \not{a}^\dagger \not{a} a \not{a}^\dagger + \not{a} \not{a}^\dagger aaa^\dagger \not{a} a \not{a}^\dagger + aa^\dagger \not{a} \not{a} \not{a}^\dagger aa \not{a}^\dagger + \\
 &\quad \not{a} a^\dagger a \not{a} \not{a}^\dagger aa \not{a}^\dagger + \not{a} \not{a}^\dagger a \not{a} a^\dagger aa \not{a}^\dagger + aa^\dagger \not{a} \not{a} \not{a}^\dagger aa \not{a}^\dagger + \\
 &\quad \not{a} a^\dagger \not{a} a \not{a}^\dagger aa \not{a}^\dagger + \not{a} \not{a}^\dagger \not{a} aa^\dagger aa \not{a}^\dagger + \not{a} a^\dagger a \not{a} \not{a}^\dagger aa \not{a}^\dagger + \\
 &\quad \not{a} a^\dagger \not{a} a \not{a}^\dagger aa \not{a}^\dagger + \not{a} \not{a}^\dagger a \not{a} \not{a}^\dagger aaa^\dagger + \not{a} \not{a}^\dagger \not{a} a \not{a}^\dagger aaa^\dagger : \\
 &+ : \not{a} \not{a}^\dagger a \not{a} \not{a}^\dagger a \not{a} \not{a}^\dagger + \not{a} \not{a}^\dagger \not{a} a \not{a}^\dagger a \not{a} \not{a}^\dagger + \not{a} \not{a}^\dagger a \not{a} \not{a}^\dagger \not{a} a \not{a}^\dagger + \\
 &\quad \not{a} \not{a}^\dagger \not{a} a \not{a}^\dagger \not{a} a \not{a}^\dagger + \not{a} \not{a}^\dagger \not{a} \not{a} \not{a}^\dagger aa \not{a}^\dagger + \not{a} \not{a}^\dagger \not{a} \not{a} \not{a}^\dagger aa \not{a}^\dagger : \\
 &= (a^\dagger)^3 a^5 + 9 (a^\dagger)^2 a^4 + 18 a^\dagger a^3 + 6 a^2.
 \end{aligned}$$

One can easily see that the number of contractions may be quite big. This difficulty for polynomial expressions may be overcome by using modern computer algebra systems. Nevertheless, for nontrivial functions (having infinite expansions) the problem remains open. Also it does not provide the analytic formulas for the coefficients of the normally ordered terms in the final expression. A systematic treatment of a large class of such problems is the subject of this work.

At the end of this Section we recall some formulas connected with

operator reordering. The first one is the exponential mapping formula, sometimes called the Hausdorff transform, which for any well defined function $F(a, a^\dagger)$ yields

$$\begin{aligned} e^{xa} F(a, a^\dagger) e^{-xa} &= F(a, a^\dagger + x), \\ e^{-xa^\dagger} F(a, a^\dagger) e^{xa^\dagger} &= F(a + x, a^\dagger). \end{aligned} \tag{11}$$

It can be used to derive the following commutators

$$\begin{aligned} [a, F(a, a^\dagger)] &= \frac{\partial}{\partial a^\dagger} F(a, a^\dagger), \\ [a^\dagger, F(a, a^\dagger)] &= -\frac{\partial}{\partial a} F(a, a^\dagger). \end{aligned} \tag{12}$$

The proofs may be found in any book on Quantum Mechanics, *e.g.* [Lou64].

Also a well known property of the Heisenberg-Weyl algebra of Eq.(1) is a disentangling formula

$$e^{\lambda(a+a^\dagger)} = e^{\lambda^2/2} e^{\lambda a^\dagger} e^{\lambda a} = e^{-\lambda^2/2} e^{\lambda a} e^{\lambda a^\dagger}, \tag{13}$$

which may serve as an example of the normal ordering procedure

$$e^{\lambda(a+a^\dagger)} = \mathcal{N} \left[e^{\lambda(a+a^\dagger)} \right] \equiv e^{\lambda^2/2} : e^{\lambda(a+a^\dagger)} : . \tag{14}$$

This type of expressions exploits the Lie structure of the algebra and uses a simplified form of Baker-Campbell-Hausdorff formula. For this and other disentangling properties of the exponential operators, see [Wil67][Wit75][MSI93][Das96].

Finally, we must mention other ordering procedures also used in physics, like the anti-normal or the Weyl (symmetric) ordered form. We note that there exist translation formulas between these expressions, see *e.g.* [CG69][ST79].

2.4 X and D representation

The choice of the representation of the algebra may be used to simplify the calculations. In the following we apply this, although, we note that with some effort one could also manage using solely the operator properties of a and a^\dagger .

There are some common choices of the Heisenberg-Weyl algebra representation in Quantum Mechanics. One may take *e.g.* a pair of hermitian operators X and $P = i\frac{d}{dx}$ acting on the dense subspace of square integrable functions or the infinite matrix representation of Eq.(5) defined in the Fock space. This choice is connected with a particular interpretation intimately connected with the quantum mechanical problem to be solved.

Here we choose the simplest possible representation of the commutation relation of Eq.(1) which acts in the space of (formal) polynomials. We make the identification

$$\begin{aligned} a^\dagger &\longleftrightarrow X \\ a &\longleftrightarrow D, \end{aligned} \tag{15}$$

where X and D are formal multiplication and derivative ($D = \frac{d}{dx}$) operators, respectively. They are defined by their action on monomials

$$\begin{aligned} Xx^n &= x^{n+1}, \\ Dx^n &= nx^{n-1}. \end{aligned} \tag{16}$$

(It also defines the action of these operators on the formal power series.)

Note that the commutation relation of Eq. (1) remains the same, *i.e.*

$$[D, X] = 1. \tag{17}$$

For more details see Appendix B and Chapter 5.

This choice of representation is most appropriate, as we note that the problem we shall be concerned with has a purely algebraic background. We are interested in the reordering of operators, and that only depends on the algebraic properties of the commutator of Eqs.(1) or (17). We emphasize that the conjugacy property of the operators or the scalar product do not play any role in that problem. To get rid of these unnecessary constructions we choose the representation of the commutator of Eq.(1) in the space of polynomials defined by Eq.(16) where these properties do not play primary role, however possible to implement (see the Bargmann-Segal representation [Bar61][Seg63]). We benefit from this choice by the resulting increased simplicity of the calculations.

3 Generic example. Stirling and Bell numbers.

We define Stirling and Bell numbers as solutions to the normal ordering problem for powers and the exponential of the number operator $N = a^\dagger a$, where a and a^\dagger are the annihilation and creation operators. All their combinatorial properties are derived using only this definition. Coherent state matrix elements of N^n and $e^{\lambda N}$ are shown to be the Bell polynomials and their exponential generating function, respectively.

This Chapter may be thought of as a simple introduction to some methods and problems encountered later on in the text. We give a simple example on which all the essential techniques of the proceeding sections are used. Although Stirling and Bell numbers have a well established purely combinatorial origin [Com74][Rio84][Wil94][GKP94] we choose here a more physical approach. We define and investigate Stirling and Bell numbers as solutions to the normal ordering problem [Kat74][Kat00][Kat02].

Consider the number operator $N = a^\dagger a$ and search for the normally ordered form of its n -th power (iteration). It can be written as

$$(a^\dagger a)^n = \sum_{k=1}^n S(n, k)(a^\dagger)^k a^k \quad (18)$$

where the integers $S(n, k)$ are the so called *Stirling numbers*. One may also define the so called *Bell polynomials*

$$B(n, x) = \sum_{k=1}^n S(n, k)x^k \quad (19)$$

and *Bell numbers*

$$B(n) = B(n, 1) = \sum_{k=1}^n S(n, k) \quad (20)$$

For convenience we apply the conventions

$$S(n, 0) = \delta_{n,0} \quad \text{and} \quad S(n, k) = 0 \quad \text{for } k > n \quad (21)$$

and

$$B(0) = B(0, x) = 1. \quad (22)$$

(It makes the calculations easier, *e.g.* one may ignore the summation limits when changing their order.)

In the following we are concerned with the properties of these Stirling and Bell numbers.

First we state the *recurrence relation* for Stirling numbers

$$S(n + 1, k) = kS(n, k) + S(n, k - 1), \quad (23)$$

with initial conditions as in Eq.(21). The proof (by induction) can be deduced from the equalities

$$\begin{aligned} & \sum_{k=1}^{n+1} S(n + 1, k) (a^\dagger)^k a^k = (a^\dagger a)^{n+1} = a^\dagger a (a^\dagger a)^n \\ & = \sum_{k=1}^n S(n, k) a^\dagger a (a^\dagger)^k a^k \stackrel{(10)}{=} \sum_{k=1}^n S(n, k) (a^\dagger)^k (a^\dagger a + k) a^k \\ & = \sum_{k=2}^{n+1} S(n, k - 1) (a^\dagger)^k a^k + \sum_{k=1}^n k S(n, k) (a^\dagger)^k a^k \\ & \stackrel{(21)}{=} \sum_{k=1}^{n+1} (S(n, k - 1) + k S(n, k)) (a^\dagger)^k a^k. \end{aligned}$$

We shall not make use of this recurrence relation, preferring the simpler analysis based on the Dobiński relation. To this end we proceed to the essential step which we shall extensively use later on, *i.e. change of representation*. Eq.(18) rewritten in the X and D representation (see Section 2.4) takes the form

$$(XD)^n = \sum_{k=1}^n S(n, k) X^k D^k. \quad (24)$$

With this trick we shall obtain all the properties of Stirling and Bell numbers.

We first act with this equation on the monomial x^m . This gives for m integer

$$m^n = \sum_{k=1}^n S(n, k) m^k,$$

where $x^k = x \cdot (x - 1) \cdot \dots \cdot (x - k + 1)$ is the *falling factorial* ($x^0 = 1$). Observing that a (non zero) polynomial can have only a finite set of zeros justifies the generalization

$$x^n = \sum_{k=1}^n S(n, k) x^k. \tag{25}$$

This equation can be interpreted as a change of basis in the space of polynomials. This gives an interpretation of Stirling numbers as the *connection coefficients* between two bases $\{x^n\}_{n=0}^\infty$ and $\{x^{\underline{n}}\}_{n=0}^\infty$. Now we act with Eq.(24) on the exponential function e^x . We obtain

$$\sum_{k=0}^\infty k^n \frac{x^k}{k!} = e^x \sum_{k=1}^n S(n, k) x^k.$$

Recalling the definition of the Bell polynomials Eq.(19) we get

$$B(n, x) = e^{-x} \sum_{k=0}^\infty k^n \frac{x^k}{k!}, \tag{26}$$

which is a celebrated *Dobiński formula* [Wil94]. It is usually stated for Bell numbers in the form

$$B(n) = \frac{1}{e} \sum_{k=1}^\infty \frac{k^n}{k!}. \tag{27}$$

Note that both series are convergent by the d'Alembert criterion.

The most striking property of the Dobiński formula is the fact that integer numbers $B(n)$ or polynomials $B(n, x)$ can be represented as nontrivial infinite sums. Here we only mention that this remarkable property provides also solutions to the moment problem (see Section 6.2)

In the following we shall exploit the Dobiński formula Eq.(26) to investigate further properties of Stirling and Bell numbers.

Applying the Cauchy multiplication rule Eq.(225) to Eq.(26) and comparing coefficients we obtain *explicit expression* for $S(n, k)$

$$S(n, k) = \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} (-1)^{k-j} j^n. \quad (28)$$

Next, we define the *exponential generating function* of the polynomials $B(n, x)$ (see Appendix B) as

$$G(\lambda, x) = \sum_{n=0}^{\infty} B(n, x) \frac{\lambda^n}{n!}, \quad (29)$$

which contains all the information about the Bell polynomials. Substituting Eq.(26) into Eq.(29), changing the summation order and then identifying the expansions of the exponential functions we obtain

$$\begin{aligned} G(\lambda, x) &= e^{-x} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} k^n \frac{x^k}{k!} \frac{\lambda^n}{n!} = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} k^n \frac{\lambda^n}{n!} \\ &= e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} e^{\lambda k} = e^{-x} e^{x e^{\lambda}}. \end{aligned}$$

Finally we may write the exponential generating function $G(\lambda, x)$ in the compact form

$$G(\lambda, x) = e^{x(e^{\lambda}-1)}. \quad (30)$$

Sometimes in applications the following exponential generating function of the Stirling numbers $S(n, k)$ is used

$$\sum_{n=k}^{\infty} S(n, k) \frac{\lambda^n}{n!} = \frac{(e^{\lambda} - 1)^k}{k!}. \quad (31)$$

It can be derived by comparing expansions in x of Eqs.(29) and (30)

$$\begin{aligned}
 G(\lambda, x) &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n S(n, k) x^k \right) \frac{\lambda^n}{n!} \\
 &= 1 + \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} S(n, k) \frac{\lambda^n}{n!} \right) x^k, \\
 G(\lambda, x) &= e^{x(e^\lambda - 1)} = 1 + \sum_{k=1}^{\infty} \frac{(e^\lambda - 1)^k}{k!} x^k.
 \end{aligned}$$

Bell polynomials share an interesting property called the *Sheffer identity* (note resemblance to the binomial identity)

$$B(n, x + y) = \sum_{k=0}^n \binom{n}{k} B(k, y) B(n - k, x) \tag{32}$$

It is the consequence of the following equalities

$$\begin{aligned}
 \sum_{n=0}^{\infty} B(n, x + y) \frac{\lambda^n}{n!} &= e^{(x+y)(e^\lambda - 1)} = e^{x(e^\lambda - 1)} e^{y(e^\lambda - 1)} \\
 &= \sum_{n=0}^{\infty} B(n, x) \frac{\lambda^n}{n!} \cdot \sum_{n=0}^{\infty} B(n, y) \frac{\lambda^n}{n!} \\
 &\stackrel{(225)}{=} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} B(k, y) B(n - k, x) \right) \frac{\lambda^n}{n!}.
 \end{aligned}$$

By this identity and also by the characteristic exponential generating function of Eq.(30) Bell polynomials $B(n, x)$ are found to be of Sheffer-type (see Appendix B).

Differentiating the exponential generating function of Eqs.(29) and (30) we have (see Eq.(227))

$$\sum_{n=0}^{\infty} B(n + 1, x) \frac{\lambda^n}{n!} = \frac{\partial}{\partial \lambda} G(\lambda, x) = e^\lambda e^{x(e^\lambda - 1)} = e^\lambda \sum_{n=0}^{\infty} B(n, x) \frac{\lambda^n}{n!}$$

which by the Cauchy product rule Eq.(225) yields the *recurrence relation* for the Bell polynomials

$$B(n + 1, x) = \sum_{k=0}^n \binom{n}{k} B(k, x). \tag{33}$$

And consequently for the Bell numbers we have $B(n+1) = \sum_{k=0}^n \binom{n}{k} B(k)$. By the same token applied to the exponential generating function of Stirling numbers $S(n, k)$ of Eq.(31) the recurrence relation of Eq.(23) can be also derived.

Finally, using any of the derived properties of Stirling or Bell numbers one can easily calculate them explicitly. Here are some of them

	$S(n, k), \quad 1 \leq k \leq n$								$B(n)$
$n = 1$	1								1
$n = 2$	1	1							2
$n = 3$	1	3	1						5
$n = 4$	1	7	6	1					15
$n = 5$	1	15	25	10	1				52
$n = 6$	1	31	90	65	15	1			203
$n = 7$	1	63	301	350	140	21	1		877
$n = 8$	1	127	966	1701	1050	266	28	1	4140
...

Now we come back to normal ordering. Using the properties of coherent states (see Appendix A) we conclude from Eqs.(18) and (19) that diagonal coherent state matrix elements generate Bell polynomials [Kat00]

$$\langle z | (a^\dagger a)^n | z \rangle = B(n, |z|^2). \tag{34}$$

Moreover, expanding the exponential $e^{\lambda a^\dagger a}$ and taking the diagonal coherent state matrix element we have

$$\langle z | e^{\lambda a^\dagger a} | z \rangle = \sum_{n=0}^{\infty} \langle z | (a^\dagger a)^n | z \rangle \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} B(n, |z|^2) \frac{\lambda^n}{n!}.$$

We see that the diagonal coherent state matrix elements of $e^{\lambda a^\dagger a}$ yield the exponential generating function of the Bell polynomials (see Eqs.(29) and (30))

$$\langle z | e^{\lambda a^\dagger a} | z \rangle = e^{|z|^2(e^\lambda - 1)}. \tag{35}$$

This result allows us to read off the normally ordered form of $e^{\lambda a^\dagger a}$ (see Appendix A)

$$e^{\lambda a^\dagger a} = \mathcal{N} \left[e^{\lambda a^\dagger a} \right] \equiv : e^{a^\dagger a (e^\lambda - 1)} : . \quad (36)$$

These are more or less all properties of Stirling and Bell numbers. Note that they were defined here as solutions to the normal ordering problem for powers of the number operator $N = a^\dagger a$. The above analysis relied firmly on this definition.

On the other hand these numbers are well known in combinatorial analysis [Com74][Rio84][Wil94][GKP94] where are called Stirling numbers of the second kind and are usually explored using only recurrence relation Eq.(23). Here is their original interpretation in terms of partitions of the set.

- Stirling numbers $S(n, k)$ count the number of ways of putting n different objects into k identical containers (none left empty).
- Bell numbers $B(n)$ count the number of ways of putting n different objects into n identical containers (some may be left empty).

Some other pictorial representations can be also given *e.g.* in terms of graphs [MBP05] or rook numbers [Nav73][SDB⁺04][Var04].

4 Normal Ordering of Boson Expressions

We solve the normal ordering problem for expressions in boson creation a^\dagger and annihilation a operators in the form of a string in the form $(a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1}$. We are especially concerned with its powers (iterations). Next we extend the solution to iterated homogeneous boson polynomials, *i.e.* powers of the operators which are sums of boson strings of the same excess. The numbers obtained in the solutions are generalizations of Stirling and Bell numbers. Recurrence relations, closed-form expressions (including Dobiński-type relations) and generating formulas are derived. Normal ordering of the exponentials of the the aforementioned operators are also treated. Some special cases including Kerr-type hamiltonians are analyzed in detail.

4.1 Introduction

In this chapter we consider expressions in boson creation a^\dagger and annihilation a operators (see Section 2.2). We search for the normally ordered form and show effective ways of finding combinatorial numbers arising in that problem. These numbers generalize Stirling and Bell numbers (see Chapter 3).

The first class of expressions treated in this Chapter are so called *boson strings* or *boson monomials*, see Section 4.2. Here is an example

$$a^\dagger a a^\dagger a^\dagger a a a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a a a a a a^\dagger a a^\dagger a a a^\dagger a^\dagger a^\dagger a^\dagger,$$

which we compactly denote as

$$(a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1}$$

with some nonnegative integers r_m and s_m . By an *excess* of a string we call the difference between the number of creation and annihilation operators, *i.e.* $d = \sum_{m=1}^M (r_m - s_m)$. In the following we assume that the boson strings are of nonnegative excess (for negative excess see Section 4.6).

Boson strings (monomials) can be extended further to polynomials. We treat *homogeneous boson polynomials* which are combinations of boson monomials of the same excess (possibly with some coefficients). Here is an example

$$a a a^\dagger a a^\dagger a^\dagger a a a^\dagger a^\dagger a^\dagger + a^\dagger a a^\dagger a^\dagger a + a^\dagger,$$

which using methods of Section 4.2 can be written in the form

$$(a^\dagger)^d \sum_{k=N_0}^N \alpha_k (a^\dagger)^k a^k,$$

with appropriate d , N , N_0 and α_k 's.

Next, in Sections 4.3 and 4.4 we give recipes on how to approach the problem of normal ordering of *iterated* boson strings and homogeneous boson polynomials, *i.e.* their n -th powers, like

$$(a^\dagger a a^\dagger a^\dagger a a a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a a^\dagger a^\dagger a a a a a a^\dagger a a^\dagger a a a^\dagger a^\dagger a a^\dagger)^n$$

and

$$(a a a^\dagger a a^\dagger a^\dagger a a a^\dagger a^\dagger a^\dagger + a^\dagger a a^\dagger a^\dagger a + a^\dagger)^n.$$

These considerations serve to calculate in Section 4.5 the *exponential generating functions* of generalized Stirling and Bell numbers. This in turn gives the solution to the normal ordering problem of the exponential of the aforementioned boson operators.

In Section 4.6 we comment on operators with negative excess.

Finally in the last Section 4.7 we work out some examples which include generalized Kerr-type hamiltonian.

We refer to [MBP05][BPS+05][SDB+04][Var04] for interpretation of considered combinatorial structures in terms of graphs and rook polynomials.

4.2 General boson strings

In this section we define the generalization of ordinary Bell and Stirling numbers which arise in the solution of the general normal ordering problem for a boson string (monomial) [MBP05][Wit05]. Given two sequences of nonnegative integers $\mathbf{r} = (r_1, r_2, \dots, r_M)$ and $\mathbf{s} = (s_1, s_2, \dots, s_M)$ we define the operator

$$H_{\mathbf{r},\mathbf{s}} = (a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1}. \quad (37)$$

We let $S_{\mathbf{r},\mathbf{s}}(k)$ be the nonnegative integers appearing in the normally ordered expansion

$$H_{\mathbf{r},\mathbf{s}} = (a^\dagger)^{d_M} \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) (a^\dagger)^k a^k, \quad (38)$$

where $d_n = \sum_{m=1}^n (r_m - s_m)$, $n = 1, \dots, M$. Here we assume that the (*overall*) *excess* of a string is nonnegative $d_M \geq 0$ (for negative excess see Section 4.6).

Observe that the r.h.s. of Eq.(38) is already normally ordered and constitutes a homogeneous boson polynomial being the solution to the normal ordering problem. We shall provide explicit formulas for the coefficients $S_{\mathbf{r},\mathbf{s}}(k)$. This is a step further than the Wick's theorem which is nonconstructive in that respect. We call $S_{\mathbf{r},\mathbf{s}}(k)$ *generalized Stirling number*. The *generalized Bell polynomial* is defined as

$$B_{\mathbf{r},\mathbf{s}}(x) = \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k)x^k, \quad (39)$$

and the *generalized Bell number* is the sum

$$B_{\mathbf{r},\mathbf{s}} = B_{\mathbf{r},\mathbf{s}}(1) = \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k). \quad (40)$$

Note that sequence of numbers $S_{\mathbf{r},\mathbf{s}}(k)$ is defined here for $s_1 \leq k \leq s_1 + s_2 + \dots + s_M$. Initial terms may vanish (depending on the structure of \mathbf{r} and \mathbf{s}), all the next are positive integers and the last one is equal to one, $S_{\mathbf{r},\mathbf{s}}(s_1 + s_2 + \dots + s_M) = 1$.

For convenience we apply the convention

$$S_{\mathbf{r},\mathbf{s}}(k) = 0 \quad \text{for } k < s_1 \text{ or } k > s_1 + s_2 + \dots + s_M. \quad (41)$$

We introduce the notation $\mathbf{r} \uplus r_{M+1} = (r_1, r_2, \dots, r_M, r_{M+1})$ and $\mathbf{s} \uplus s_{M+1} = (s_1, s_2, \dots, s_M, s_{M+1})$ and state the *recurrence relation* satisfied by generalized Stirling numbers $S_{\mathbf{r},\mathbf{s}}(k)$

$$S_{\mathbf{r} \uplus r_{M+1}, \mathbf{s} \uplus s_{M+1}}(k) = \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} (d_M + k - j)^{\overline{s_{M+1}-j}} S_{\mathbf{r},\mathbf{s}}(k - j). \quad (42)$$

One can deduce the derivation of Eq.(42) from the following equal-

ities

$$\begin{aligned}
 & (a^\dagger)^{d_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_{M+1}} S_{\mathbf{r} \uplus r_{M+1}, \mathbf{s} \uplus s_{M+1}}(k) (a^\dagger)^k a^k \stackrel{(38)}{=} H_{\mathbf{r} \uplus r_{M+1}, \mathbf{s} \uplus s_{M+1}} \\
 & \stackrel{(37)}{=} (a^\dagger)^{r_{M+1}} a^{s_{M+1}} H_{\mathbf{r}, \mathbf{s}} \stackrel{(38)}{=} (a^\dagger)^{r_{M+1}} a^{s_{M+1}} (a^\dagger)^{d_M} \sum_{k=s_1}^{s_1+\dots+s_M} S_{\mathbf{r}, \mathbf{s}}(k) \times \\
 & (a^\dagger)^k a^k = (a^\dagger)^{r_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_M} S_{\mathbf{r}, \mathbf{s}}(k) a^{s_{M+1}} (a^\dagger)^{d_M+k} a^k \\
 & \stackrel{(10)}{=} (a^\dagger)^{r_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_M} S_{\mathbf{r}, \mathbf{s}}(k) \cdot \\
 & \quad \cdot \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} (d_M + k)^{\underline{j}} (a^\dagger)^{d_M+k-j} a^{s_{M+1}+k-j} \\
 & = (a^\dagger)^{d_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_M} S_{\mathbf{r}, \mathbf{s}}(k) \cdot \\
 & \quad \cdot \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} (d_M + k)^{\underline{j}} (a^\dagger)^{s_{M+1}+k-j} a^{s_{M+1}+k-j} \\
 & = (a^\dagger)^{d_{M+1}} \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} \cdot \\
 & \quad \cdot \sum_{k=s_1+s_{M+1}-j}^{s_1+\dots+s_{M+1}-j} S_{\mathbf{r}, \mathbf{s}}(k) (d_M + k - s_{M+1} + j)^{\underline{j}} (a^\dagger)^k a^k \\
 & \stackrel{(42)}{=} (a^\dagger)^{d_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_M} \\
 & \quad \cdot \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} S_{\mathbf{r}, \mathbf{s}}(k - s_{M+1} + j) (d_M + k - s_{M+1} + j)^{\underline{j}} (a^\dagger)^k a^k \\
 & = (a^\dagger)^{d_{M+1}} \sum_{k=s_1}^{s_1+\dots+s_M} \\
 & \quad \cdot \sum_{j=0}^{s_{M+1}} \binom{s_{M+1}}{j} S_{\mathbf{r}, \mathbf{s}}(k - j) (d_M + k - j)^{\underline{s_{M+1}-j}} (a^\dagger)^k a^k
 \end{aligned}$$

The problem stated in Eq.(38) can be also formulated in terms of multiplication X and derivative D operators (see Section 2.4)

$$X^{r_M} D^{s_M} \dots X^{r_2} D^{s_2} X^{r_1} D^{s_1} = X^{d_M} \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) X^k D^k \quad (43)$$

Acting with the r.h.s. of Eq.(43) on the monomial x^l we get $x^{l+d_M} \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) l^k$. On the other hand action of the l.h.s. on x^l gives $\prod_{m=1}^M (d_{m-1} + l)^{s_m} x^{l+d_M}$. Equating these two results (see Eq.(43)) we have

$$\prod_{m=1}^M (d_{m-1} + l)^{s_m} = \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) l^k$$

By invoking the fact that the only polynomial with an infinite number of zeros is the zero polynomial we justify the generalization

$$\prod_{m=1}^M (d_{m-1} + x)^{s_m} = \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) x^k. \quad (44)$$

This allows us to interpret the numbers $S_{\mathbf{r},\mathbf{s}}(k)$ as the *expansion coefficients* of the polynomial $\{\prod_{m=1}^M (d_{m-1} + x)^{s_m}\}$ in the basis $\{x^k\}_{k=0}^\infty$. Now, replacing monomials in the above considerations with the exponential e^x we conclude that

$$X^{r_M} D^{s_M} \dots X^{r_2} D^{s_2} X^{r_1} D^{s_1} e^x = x^{d_M} e^x B_{\mathbf{r},\mathbf{s}}(x)$$

$$X^{d_M} \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) X^k D^k e^x = \sum_{k=s_1}^\infty \left[\prod_{m=1}^M (d_{m-1} + k)^{s_m} \right] \frac{x^{k+d_M}}{k!}$$

With these two observations together and Eq.(43) we arrive at the *Dobiński-type relation* for the generalized Bell polynomial

$$B_{\mathbf{r},\mathbf{s}}(x) = e^{-x} \sum_{k=s_1}^\infty \left[\prod_{m=1}^M (d_{m-1} + k)^{s_m} \right] \frac{x^k}{k!} \quad (45)$$

Observe that the d'Alembert criterion assures the convergence of the series.

Direct multiplication of series in Eq.(45) using the Cauchy rule of Eq.(225) gives the *explicit formula* for generalized Stirling numbers $S_{\mathbf{r},\mathbf{s}}(k)$

$$S_{\mathbf{r},\mathbf{s}}(k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \prod_{m=1}^M (d_{m-1} + j)^{s_m} . \quad (46)$$

We finally return to normal ordering and observe that the coherent state matrix element (see Appendix A) of the boson string yields the generalized Bell polynomial (see Eqs.(37) and (38))

$$\langle z | H_{\mathbf{r},\mathbf{s}} | z \rangle = (z^*)^{d_M} B_{\mathbf{r},\mathbf{s}}(|z|^2) . \quad (47)$$

4.3 Iterated homogeneous polynomials in boson operators

Consider an operator H_{α}^d of the form

$$H_{\alpha}^d = (a^{\dagger})^d \sum_{k=N_0}^N \alpha_k (a^{\dagger})^k a^k , \quad (48)$$

where the α_k 's are constant coefficients (for convenience we take $\alpha_{N_0} \neq 0$ and $\alpha_N \neq 0$) and the nonnegative integer d is the excess of the polynomial. Note that this is a homogeneous polynomial in boson operators. It is already in a normal form. If the starting homogeneous boson polynomial is not normally ordered then the methods of Section 4.2 provide proper tools to put it in the shape of Eq.(48). For boson polynomials of negative degree see Section 4.6.

Suppose we want to calculate the normally ordered form of $(H_{\alpha}^d)^n$. It can be written as

$$(H_{\alpha}^d)^n = (a^{\dagger})^{nd} \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) (a^{\dagger})^k a^k , \quad (49)$$

with $S_{\alpha}(n, k)$ to be determined. These coefficients *generalize Stirling numbers* (see Eq.(18)). In the same manner as in Eqs.(19) and (20) we define *generalized Bell polynomials*

$$B_{\alpha}^d(n, x) = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) x^k , \quad (50)$$

and *generalized Bell numbers*

$$B_{\alpha}^d(n) = B_{\alpha}^d(n, 1) = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k). \quad (51)$$

Note that $S_{\alpha}^d(n, k)$, $B_{\alpha}^d(n, x)$ and $B_{\alpha}^d(n)$ depend on the set of α_k 's defined in Eq.(48). For convenience we make the conventions

$$\begin{aligned} S_{\alpha}^d(0, 0) &= 1, \\ S_{\alpha}^d(n, k) &= 0 \quad \text{for } k > nN, \\ S_{\alpha}^d(n, k) &= 0 \quad \text{for } k < N_0 \quad \text{and } n > 0, \end{aligned} \quad (52)$$

and

$$B_{\alpha}^d(0) = B_{\alpha}^d(0, x) = 1. \quad (53)$$

In the following we show how to calculate generalized Stirling and Bell numbers explicitly. First we state the *recurrence relation*

$$\begin{aligned} S_{\alpha}^d(n+1, k) &= \\ &= \sum_{l=N_0}^N \alpha_l \sum_{p=0}^l \binom{l}{p} (nd + k - l + p)^p S_{\alpha}^d(n, k - l + p), \end{aligned} \quad (54)$$

with initial conditions as in Eq.(52).

It can be deduced from the following equalities

$$\begin{aligned}
 & (a^\dagger)^{(n+1)d} \sum_{k=N_0}^{(n+1)N} S_\alpha^d(n+1, k) (a^\dagger)^k a^k \stackrel{(49)}{=} (H_\alpha^d)^{n+1} = H_\alpha(H_\alpha^d)^n \\
 & \stackrel{(49)}{=} (a^\dagger)^d \sum_{l=N_0}^N \alpha_l (a^\dagger)^l a^l (a^\dagger)^{nd} \sum_{k=N_0}^{nN} S_\alpha^d(n, k) (a^\dagger)^k a^k \\
 & = \sum_{k=N_0}^{nN} S_\alpha^d(n, k) \sum_{l=N_0}^N \alpha_l (a^\dagger)^{d+l} a^l (a^\dagger)^{nd+k} a^k \\
 & \stackrel{(10)}{=} \sum_{k=N_0}^{nN} S_\alpha^d(n, k) \sum_{l=N_0}^N \alpha_l (a^\dagger)^{d+l} \cdot \\
 & \quad \cdot \left(\sum_{p=0}^l \binom{l}{p} (nd+k)^p (a^\dagger)^{nd+k-p} a^{l-p} \right) a^k \\
 & = (a^\dagger)^{(n+1)d} \sum_{k=N_0}^{nN} S_\alpha^d(n, k) \cdot \\
 & \quad \cdot \sum_{l=N_0}^N \alpha_l \sum_{p=0}^l \binom{l}{p} (nd+k)^p (a^\dagger)^{l+k-p} a^{l+k-p} \\
 & = (a^\dagger)^{(n+1)d} \sum_{l=N_0}^N \alpha_l \sum_{p=0}^l \binom{l}{p} \cdot \\
 & \quad \cdot \sum_{k=N_0+l-p}^{nN+l-p} (nd+k-l+p)^p S_\alpha^d(n, k-l+p) (a^\dagger)^k a^k \\
 & \stackrel{(52)}{=} (a^\dagger)^{(n+1)d} \sum_{k=N_0}^{nN} \\
 & \quad \cdot \sum_{l=N_0}^N \alpha_l \sum_{p=0}^l \binom{l}{p} (nd+k-l+p)^p S_\alpha^d(n, k-l+p) (a^\dagger)^k a^k.
 \end{aligned}$$

Eq.(49) rewritten in the X and D representation takes the form

$$(H_{\alpha}^d(D, X))^n = X^{nd} \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) X^k D^k, \quad (55)$$

where

$$H_{\alpha}^d(D, X) = X^d \sum_{k=N_0}^N \alpha_k X^k D^k. \quad (56)$$

We shall act with both sides of the Eq.(55) on the monomials x^l . The r.h.s. yields

$$X^{nd} \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) X^k D^k x^l = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) l^k x^{l+nd}.$$

On the other hand iterating the action of Eq.(56) on monomials x^l we have

$$(H_{\alpha}^d(D, X))^n x^l = \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k \right] x^{l+nd}.$$

Then when substituted in Eq.(55) we get

$$\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) l^k.$$

Recalling the fact that a nonzero polynomial can have only a finite set of zeros we obtain

$$\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (x + (i-1)d)^k = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) x^k. \quad (57)$$

This provides interpretation of the generalized Stirling numbers $S_{\alpha}^d(n, k)$ as the *connection coefficients* between two sets of polynomials $\{x^k\}_{k=0}^{\infty}$ and $\{\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (x + (i-1)d)^k\}_{n=0}^{\infty}$.

Now acting with both sides of Eq.(55) on the exponential function we get

$$(H_{\alpha}^d(D, X))^n e^x = e^x x^{nd} \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) X^k = e^x x^{nd} B_{\alpha}^d(n, x)$$

$$(H_{\alpha}^d(D, X))^n e^x = x^{nd} \sum_{l=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i - 1)d)^k \right] \frac{x^l}{l!}$$

Taking these equations together we arrive at the *Dobiński-type relation*

$$B_{\alpha}^d(n, x) = e^{-x} \sum_{l=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i - 1)d)^k \right] \frac{x^l}{l!}. \quad (58)$$

Note that by the d'Alembert criterion this series is convergent.

Now multiplying the series in Eq.(58) and using the Cauchy multiplication rule of Eq.(225) we get the explicit expression for generalized Stirling numbers

$$S_{\alpha}^d(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \prod_{i=1}^n \sum_{l=N_0}^N \alpha_l (j + (i - 1)d)^l. \quad (59)$$

We end this section observing that the diagonal coherent state matrix elements (see Appendix A) of $(H_{\alpha})^n$ generate generalized Bell polynomials (see Eq.(49))

$$\langle z | (H_{\alpha}^d)^n | z \rangle = (z^*)^{nd} B_{\alpha}^d(n, |z|^2). \quad (60)$$

Other properties of generalized Stirling and Bell numbers will be investigated in subsequent Sections.

4.4 Iterated boson string

In Section 4.2 it was indicated that every boson string can be put in normally ordered form which is a homogeneous polynomial in a and a^{\dagger} . (Note, the reverse statement is evidently not true, *i.e.* 'most' homogeneous polynomials in a and a^{\dagger} can not be written as a boson

string.) This means that the results of Section 4.3 may be applied to calculate the normal form of iterated boson string. We take an operator

$$H_{\mathbf{r},\mathbf{s}} = (a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1} \quad (61)$$

where $\mathbf{r} = (r_1, r_2, \dots, r_M)$ and $\mathbf{s} = (s_1, s_2, \dots, s_M)$ are fixed integer vectors and search for its n -th power normal form

$$(H_{\mathbf{r},\mathbf{s}})^n = (a^\dagger)^{nd} \sum_{k=s_1}^{n(s_1+s_2+\dots+s_M)} S_{\mathbf{r},\mathbf{s}}(n, k) (a^\dagger)^k a^k, \quad (62)$$

where $d = \sum_{i=1}^M (r_i - s_i)$.

Then the procedure is straightforward. In the first step we use results of Section 4.2 to obtain

$$H_{\mathbf{r},\mathbf{s}} = (a^\dagger)^d \sum_{k=s_1}^{s_1+s_2+\dots+s_M} S_{\mathbf{r},\mathbf{s}}(k) (a^\dagger)^k a^k.$$

One can use any of the formulas in the Eqs.(42), (45) or (46).

Once the numbers $S_{\mathbf{r},\mathbf{s}}(k)$ are calculated they play the role of the coefficients in the homogeneous polynomial in a^\dagger and a which was treated in preceding Section. Taking $\alpha_k = S_{\mathbf{r},\mathbf{s}}(k)$, $N_0 = s_1$ and $N = s_1 + s_2 + \dots + s_M$ in Eq.(48) we arrive at the normal form using methods of Section 4.3.

We note that analogous scheme can be applied to a homogeneous boson polynomial which is not originally in the normally ordered form.

4.5 Generating functions

In the preceding Sections we have considered generalized Stirling numbers as solutions to the normal ordering problem. We have given recurrence relations and closed form expressions including Dobiński-type formulas. Another convenient way to describe them is through their generating functions. We define *exponential generating functions* of generalized Bell polynomials $B_\alpha^d(n, x)$ as

$$G_\alpha^d(\lambda, x) = \sum_{n=0}^{\infty} B_\alpha^d(n, x) \frac{\lambda^n}{n!} \quad (63)$$

They are usually formal power series. Because $B_{\alpha}^d(n)$ grows too rapidly with n these series are divergent (except in the case $\alpha_k = 0$ for $k > 1$ treated in Chapters 3 and 5). In Section 4.7 we give a detailed discussion and show how to improve the convergence by the use of *hypergeometric generating functions*.

In spite of this 'inconvenience', knowing that formal power series can be rigorously handled (see Appendix B), in the following we give some useful (formal) expressions.

Substituting the Dobiński-type formula of Eq.(58) into definition Eq.(63)

$$G_{\alpha}^d(\lambda, x) = e^{-x} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k \right] \frac{x^l \lambda^n}{l! n!}$$

which changing the summation order yields

$$G_{\alpha}^d(\lambda, x) = e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{n=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k \right] \frac{\lambda^n}{n!}. \quad (64)$$

Using the Cauchy multiplication rule of Eq.(225) we get the expansion in x :

$$\begin{aligned} G_{\alpha}^d(\lambda, x) &= \\ &= \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{l=0}^j \binom{j}{l} (-1)^{j-l} \sum_{n=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k \right] \frac{\lambda^n}{n!}, \end{aligned} \quad (65)$$

which by comparison with

$$\begin{aligned} G_{\alpha}^d(\lambda, x) &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n S_{\alpha}^d(n, k) x^k \right) \frac{\lambda^n}{n!} \\ &= 1 + \sum_{k=N_0}^{\infty} \left(\sum_{n=k}^{\infty} S_{\alpha}^d(n, k) \frac{\lambda^n}{n!} \right) x^k \end{aligned}$$

gives the exponential generating function of the generalized Stirling

numbers $S_{\alpha}^d(n, k)$ in the form (compare with Eq.(59))

$$\begin{aligned} \sum_{n=k}^{\infty} S_{\alpha}^d(n, k) \frac{\lambda^n}{n!} &= \\ &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \sum_{n=0}^{\infty} \left[\prod_{i=1}^n \sum_{k=N_0}^N \alpha_k (l + (i-1)d)^k \right] \frac{\lambda^n}{n!}. \end{aligned} \tag{66}$$

We note that for specific cases Eqs.(64), (65) and (66) simplify a lot, see Section 4.7 for some examples.

This kind of generating functions are connected with the diagonal coherent state elements of the operator $e^{\lambda H_{\alpha}^d}$. Using observation Eq.(60) we get

$$\langle z | e^{\lambda H_{\alpha}^d} | z \rangle = \sum_{n=0}^{\infty} \langle z | (H_{\alpha}^d)^n | z \rangle \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} B_{\alpha}^d(n, |z|^2) \frac{((z^*)^d \lambda)^n}{n!} \tag{67}$$

which yields

$$\langle z | e^{\lambda H_{\alpha}^d} | z \rangle = G_{\alpha}^d((z^*)^d \lambda, |z|^2). \tag{68}$$

This in turn lets us write (see Appendix A) the normally ordered expression

$$e^{\lambda H_{\alpha}^d} = : G_{\alpha}^d((a^{\dagger})^d \lambda, a^{\dagger} a) : . \tag{69}$$

Observe that the exponential generating functions $G_{\alpha}^d(\lambda, x)$ in general are not analytical around $\lambda = 0$ (they converge only in the case $\alpha_k = 0$ for $k > 1$, see Chapters 3 and 5). Nevertheless, *e.g.* for negative values of $\lambda \alpha_N$ and $d = 0$ the expressions in Eqs.(64) or (65) converge and may be used for explicit calculations. The convergence properties can be handled with the d'Alembert or Cauchy criterion. We also note that the number state matrix elements of $e^{\lambda H_{\alpha}^d}$ are finite (the operator is well defined on the dense subset generated by the number states). It can be seen and explicitly calculated from Eqs.(69), (64) or (65).

4.6 Negative excess

We have considered so far the generalized Stirling and Bell numbers arising in the normal ordering problem of boson expressions and their powers (iterations) with nonnegative excess. Actually when the excess is negative, *i.e.* there are more annihilation a than creation a^\dagger operators in a string, the problem is dual and does not lead to new integer sequences.

To see this we first take a string defined by two vectors $\mathbf{r} = (r_1, r_2, \dots, r_M)$ and $\mathbf{s} = (s_1, s_2, \dots, s_M)$ with a negative (overall) excess $d_M = \sum_{m=1}^M (r_m - s_m) < 0$. The normal ordering procedure extends the definition of the numbers $S_{\mathbf{r},\mathbf{s}}(k)$ for the case of negative excess through

$$\begin{aligned} H_{\mathbf{r},\mathbf{s}} &= (a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1} \\ &= \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\mathbf{r},\mathbf{s}}(k) (a^\dagger)^k a^k a^{-d_M}. \end{aligned} \quad (70)$$

Definition of $B_{\mathbf{r},\mathbf{s}}(x)$ and $B_{\mathbf{r},\mathbf{s}}$ is analogous to Eqs.(39) and (40):

$$B_{\mathbf{r},\mathbf{s}}(x) = \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\mathbf{r},\mathbf{s}}(k) x^k, \quad (71)$$

and

$$B_{\mathbf{r},\mathbf{s}} = B_{\mathbf{r},\mathbf{s}}(1) = \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\mathbf{r},\mathbf{s}}(k). \quad (72)$$

Note that the limits in the above equations are different from those in Section 4.2.

By taking the hermitian conjugate we have

$$\begin{aligned}
 & \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\mathbf{r},\mathbf{s}}(k)(a^\dagger)^k a^k a^{-d_M} \\
 \stackrel{(70)}{=} & \left(\left((a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1} \right)^\dagger \right)^\dagger \\
 = & \left((a^\dagger)^{s_1} a^{r_1} (a^\dagger)^{s_2} a^{r_2} \dots (a^\dagger)^{s_M} a^{r_M} \right)^\dagger \\
 \stackrel{(38)}{=} & \left((a^\dagger)^{-d_M} \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\bar{\mathbf{s}},\bar{\mathbf{r}}}(k)(a^\dagger)^k a^k \right)^\dagger \\
 = & \sum_{k=r_M}^{r_1+r_2+\dots+r_M} S_{\bar{\mathbf{s}},\bar{\mathbf{r}}}(k)(a^\dagger)^k a^k a^{-d_M},
 \end{aligned}$$

where $\bar{\mathbf{s}} = (s_M, \dots, s_2, s_1)$ and $\bar{\mathbf{r}} = (r_M, \dots, r_2, r_1)$ correspond to the string $H_{\bar{\mathbf{s}},\bar{\mathbf{r}}}$ with positive overall excess equal to $-d_M > 0$.

This leads to the symmetry property

$$S_{\mathbf{r},\mathbf{s}}(k) = S_{\bar{\mathbf{s}},\bar{\mathbf{r}}}(k) \tag{73}$$

as well as

$$B_{\mathbf{r},\mathbf{s}}(n, x) = B_{\bar{\mathbf{s}},\bar{\mathbf{r}}}(n, x) \quad \text{and} \quad B_{\mathbf{r},\mathbf{s}}(n) = B_{\bar{\mathbf{s}},\bar{\mathbf{r}}}(n) \tag{74}$$

for any integer vectors \mathbf{r} and \mathbf{s} .

Formulas in Section 4.2 may be used to calculate the generalized Stirling and Bell numbers in this case if the symmetry properties Eqs.(73) and (74) are taken into account. We conclude by writing explicitly the coherent state matrix element of a string for $d_M < 0$:

$$\langle z | (a^\dagger)^{r_M} a^{s_M} \dots (a^\dagger)^{r_2} a^{s_2} (a^\dagger)^{r_1} a^{s_1} | z \rangle = B_{\mathbf{r},\mathbf{s}}(|z|^2) z^{-d_M}. \tag{75}$$

Now we proceed to homogeneous polynomials of Section 4.3. In the case of the negative excess we define

$$H_\alpha^{-d} = \sum_{k=N_0}^N \alpha_k (a^\dagger)^k a^k a^d, \tag{76}$$

with some constant coefficients α_k 's (for convenience we take $\alpha_{N_0} \neq 0$ and $\alpha_N \neq 0$) and nonnegative integer d . The excess of this homogeneous boson polynomial is $-d$. We search for the numbers $S_{\alpha}^{-d}(n, k)$ defined by the normal form of the n -th power of Eq.(76) as

$$(H_{\alpha}^{-d})^n = \sum_{k=N_0}^{nN} S_{\alpha}^{-d}(n, k) (a^{\dagger})^k a^k a^{nd}. \quad (77)$$

Generalized Bell polynomials $B_{\alpha}^{-d}(n, x)$ and Bell numbers $B_{\alpha}^{-d}(n)$ are defined in the usual manner.

In the same way, taking the hermitian conjugate, we have

$$\begin{aligned} & \sum_{k=N_0}^{nN} S_{\alpha}^{-d}(n, k) (a^{\dagger})^k a^k a^{nd} \stackrel{(76)}{=} \left((H_{\alpha}^{-d})^n \right)^{\dagger} = \left(\left((H_{\alpha}^{-d})^{\dagger} \right)^n \right)^{\dagger} \\ & = \left(\left((a^{\dagger})^d \sum_{k=N_0}^N \alpha_k^* (a^{\dagger})^k a^k \right)^n \right)^{\dagger} \stackrel{(49)}{=} \left((H_{\alpha^*}^d)^n \right)^{\dagger} \\ & = \sum_{k=N_0}^{nN} S_{\alpha}^d(n, k) (a^{\dagger})^k a^k a^{nd} \end{aligned}$$

(Note that $(S_{\alpha^*}^d)^* = S_{\alpha}^d$, see any formula of Section 4.3.) Consequently the following symmetry property holds true

$$S_{\alpha}^{-d}(n, k) = S_{\alpha}^d(n, k) \quad (78)$$

and consequently

$$B_{\alpha}^{-d}(n, x) = B_{\alpha}^d(n, x) \quad \text{and} \quad B_{\alpha}^{-d}(n) = B_{\alpha}^d(n) \quad (79)$$

for any vector α and integer d .

Again any of the formulas of Section 4.3 may be used in calculations for the case of negative excess. Coherent state matrix element of Eq.(76) takes the form

$$\langle z | (H_{\alpha}^{-d})^n | z \rangle = B_{\alpha}^{-d}(n, |z|^2) z^d. \quad (80)$$

4.7 Special cases

The purpose of this section is to illustrate the above formalism with some examples. First we consider iteration of a simple string of the form $(a^\dagger)^r a^s$ and work out some special cases in detail. Next we proceed to expressions involving homogeneous boson polynomials with excess zero $d = 0$. This provides the solution to the normal ordering of a generalized Kerr-type hamiltonian.

4.7.1 Case $((a^\dagger)^r a^s)^n$

Here we consider the boson string in the form

$$H_{r,s} = (a^\dagger)^r a^s \tag{81}$$

for $r \geq s$ (the symmetry properties provide the opposite case, see Section 4.6). It corresponds in the previous notation to $H_{\mathbf{r},\mathbf{s}}$ with $(\mathbf{r}, \mathbf{s}) = (r, s)$ or H_α^d with $d = r - s$ and $\alpha_s = 1$ (zero otherwise).

The n -th power in the normally ordered form defines generalized Stirling numbers $S_{r,s}(n, k)$ as

$$(H_{r,s}^d)^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k) (a^\dagger)^k a^k. \tag{82}$$

Consequently we define the generalized Bell polynomials $B_{r,s}(n, x)$ and generalized Bell numbers $B_{r,s}(n)$ as

$$B_{r,s}(n, x) = \sum_{k=s}^{ns} S_{r,s}(n, k) x^k \tag{83}$$

and

$$B_{r,s}(n) = B_{r,s}(n, 1) = \sum_{k=s}^{ns} S_{r,s}(n, k). \tag{84}$$

The following convention is assumed:

$$\begin{aligned} S_{r,s}(0, 0) &= 1, \\ S_{r,s}(n, k) &= 0 \text{ for } k > ns, \\ S_{r,s}(n, k) &= 0 \text{ for } k < s \text{ and } n > 0, \end{aligned} \tag{85}$$

and

$$B_{r,s}(0) = B_{r,s}(0, x) = 1. \quad (86)$$

As pointed out these numbers and polynomials are special cases of those considered in Section 4.3, *i.e.*

$$\begin{aligned} S_{r,s}(n, k) &= S_{\alpha}^{(d)}(n, k) \\ B_{r,s}(n, x) &= B_{\alpha}^{(d)}(n, x) \\ B_{r,s}(n) &= S_{\alpha}^{(d)}(n) \end{aligned}$$

for $d = r - s$ and $\alpha_s = 1$ (zero otherwise).

In the following the formalism of Section 4.3 is applied to investigate the numbers in this case. By this we demonstrate that the formulas simplify considerably when the special case is considered. The discussion also raises some new points concerning the connection with special functions.

Below we choose the way of increasing simplicity, *i.e.* we start with $r > s$ and then restrict our attention to $r = s$ and $s = 1$. For $r = 1$ and $s = 1$ we end up with conventional Stirling and Bell numbers (see Chapter 3). Where possible we just state the results without comment.

Before proceeding to the program as sketched, we state the essential results which are the same for each case. We define the exponential generating function as

$$G_{r,s}(\lambda, x) = \sum_{n=0}^{\infty} B_{r,s}(n, x) \frac{\lambda^n}{n!}. \quad (87)$$

The coherent state matrix elements can be read off as

$$\langle z | (H_{r,s})^n | z \rangle = B_{r,s}(\lambda(z^*)^{r-s}, |z|^2) \quad (88)$$

and

$$\langle z | e^{\lambda H_{r,s}} | z \rangle = G_{r,s}(\lambda(z^*)^{r-s}, |z|^2). \quad (89)$$

This provides the normally ordered forms

$$(H_{r,s})^n = : B_{r,s}(\lambda(a^\dagger)^{r-s}, a^\dagger a) : \quad (90)$$

and

$$e^{\lambda H_{r,s}} = : G_{r,s}(\lambda(a^\dagger)^{r-s}, a^\dagger a) : . \quad (91)$$

We shall see that usually the generating function of Eq.(87) is formal and Eq.(89) is not analytical around $\lambda = 0$. No matter of the convergence subtleties operator equations Eqs.(90) and (91) hold true in the occupation number representation.

$r > s$

Recurrence relation

$$S_{r,s}(n+1, k) = \sum_{p=0}^s \binom{s}{p} (n(r-s) + k - r + p)^p S_{r,s}(n, k - s + p) \quad (92)$$

with initial conditions as in Eq.(85).

Connection property

$$\prod_{j=1}^n (x + (j-1)(r-s))^s = \sum_{k=s}^{ns} S_{r,s}(n, k) x^k. \quad (93)$$

The Dobiński-type relation

$$B_{r,s}(n, x) = e^{-x} \sum_{k=s}^{\infty} \prod_{j=1}^n [k + (j-1)(r-s)]^s \frac{x^k}{k!} \quad (94)$$

$$= (r-s)^{s(n-1)} e^{-x} \sum_{k=0}^{\infty} \left[\prod_{j=1}^s \frac{\Gamma(n + \frac{k+j}{r-s})}{\Gamma(1 + \frac{k+j}{r-s})} \right] \frac{x^k}{k!}. \quad (95)$$

Explicit expression

$$S_{r,s}(n, k) = \frac{(-1)^k}{k!} \sum_{p=s}^k (-1)^p \binom{k}{p} \prod_{j=1}^n (p + (j-1)(r-s))^s. \quad (96)$$

The “non-diagonal” generalized Bell numbers $B_{r,s}(n)$ can always be expressed as special values of generalized hypergeometric functions

${}_pF_q$, e.g. the series $B_{2r,r}(n)$ can be written down in a compact form using the confluent hypergeometric function of Kummer:

$$B_{2r,r}(n) = \frac{(rn)!}{e \cdot r!} {}_1F_1(rn + 1, r + 1; 1) \tag{97}$$

and still more general family of sequences has the form ($p, r = 1, 2 \dots$):

$$B_{pr+p,pr}(n) = \frac{1}{e} \left[\prod_{j=1}^r \frac{(p(n-1) + j)!}{(pj)!} \right].$$

$${}_rF_r(pn + 1, pn + 1 + p, \dots, pn + 1 + p(r-1); 1 + p, 1 + 2p, \dots, 1 + rp; 1),$$

etc...

The exponential generating function takes the form

$$\begin{aligned} G_{r,s}(\lambda, x) &= e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{n=0}^{\infty} \left[\prod_{i=1}^n (l + (i-1)(r-s))^{\underline{s}} \right] \frac{\lambda^n}{n!} \\ &= 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \sum_{l=0}^j \binom{j}{l} (-1)^{j-l} \sum_{n=1}^{\infty} \left[\prod_{i=1}^n (l + (i-1)(r-s))^{\underline{s}} \right] \frac{\lambda^n}{n!}. \end{aligned}$$

For $s > 1$ it is purely formal series (not analytical around $\lambda = 0$).

A well-defined and convergent procedure for such sequences is to consider what we call *hypergeometric generating functions*, which are the exponential generating function for the ratios $B_{r,s}/(n!)^t$, where t is an appropriately chosen integer. A case in point is the series $B_{3,2}(n)$ which may be written explicitly from Eq.(94) as:

$$B_{3,2}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{(n+k)!(n+k+1)!}{k!(k+1)!(k+2)!}.$$

Its hypergeometric generating function $\tilde{G}_{3,2}(\lambda)$ is then:

$$\tilde{G}_{3,2}(\lambda) = \sum_{n=0}^{\infty} \left[\frac{B_{3,2}(n)}{n!} \right] \frac{\lambda^n}{n!} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+2)!} {}_2F_1(k+2, k+1; 1; \lambda).$$

Similarly for $G_{4,2}(\lambda)$ one obtains:

$$\tilde{G}_{4,2}(\lambda) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+2)!} {}_2F_1\left(\frac{k+2}{2}, \frac{k}{2} + 1; 1; 4\lambda\right).$$

Eq.(97) implies more generally:

$$\begin{aligned} \tilde{G}_{2r,r}(\lambda) &= \sum_{n=0}^{\infty} \left[\frac{B_{2r,r}(n)}{(n!)^{r-1}} \right] \frac{\lambda^n}{n!} \\ &= \begin{cases} \frac{1}{1!e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} {}_2F_1 \left(\frac{k+1}{2}, \frac{k}{2} + 1; 1; 4\lambda \right), & r = 2, \\ \frac{1}{2!e} \sum_{k=0}^{\infty} \frac{1}{(k+2)!} {}_3F_2 \left(\frac{k+1}{3}, \frac{k+2}{3}, \frac{k+3}{3}; 1, 1; 27\lambda \right), & r = 3, \\ \frac{1}{3!e} \sum_{k=0}^{\infty} \frac{1}{(k+3)!} {}_4F_3 \left(\frac{k+1}{4}, \dots, \frac{k+4}{4}; 1, 1, 1; 256\lambda \right), & r = 4 \dots \end{cases} \\ &\text{etc...} \end{aligned}$$

See [JPS01] for other instances where this type of hypergeometric generating functions appear.

r=s

Recurrence relation

$$S_{r,r}(n+1, k) = \sum_{p=0}^r \binom{r}{p} (k-r+p)^p S_{r,r}(n, k-r+p) \quad (98)$$

with initial conditions as in Eq.(85).

Connection property

$$[x^r]^n = \sum_{k=r}^{nr} S_{r,r}(n, k) x^k. \quad (99)$$

The Dobiński-type relation has the form:

$$B_{r,r}(n, x) = e^{-x} \sum_{k=s}^{\infty} k^r \frac{x^k}{k!} \quad (100)$$

$$= e^{-x} \sum_{k=0}^{\infty} \left[\frac{(k+r)!}{k!} \right]^{n-1} \frac{x^k}{k!}. \quad (101)$$

Explicit expression

$$S_{r,r}(n, k) = \frac{(-1)^k}{k!} \sum_{p=r}^k (-1)^p \binom{k}{p} [p^r]^n. \quad (102)$$

Using Eq.(102) we can find the following exponential generating function of $S_{r,r}(n, k)$:

$$\sum_{n=\lceil k/r \rceil} \frac{x^n}{n!} S_{r,r}(n, k) = \frac{(-1)^k}{k!} \sum_{p=r}^k (-1)^p \binom{k}{p} \left(e^{xp(p-1)\dots(p-r+1)} - 1 \right). \quad (103)$$

We refer to Section 4.7.2 for considerations of the exponential generating functions.

s=1

Recurrence relation

$$S_{r,1}(n + 1, k) = (n(r - 1) + k - r + 1)S_{r,1}(n, k) + S_{r,1}(n, k - 1) \quad (104)$$

with initial conditions as in Eq.(85).

Connection property

$$\prod_{j=1}^n (x + (j - 1)(r - s)) = \sum_{k=s}^{ns} S_{r,1}(n, k) x^k. \quad (105)$$

The Dobiński-type relation has the form:

$$B_{r,1}(n, x) = e^{-x} \sum_{k=1}^{\infty} \prod_{j=1}^n [k + (j - 1)(r - 1)] \frac{x^k}{k!}. \quad (106)$$

Explicit expression

$$S_{r,1}(n, k) = \frac{(-1)^k}{k!} \sum_{p=s}^k (-1)^p \binom{k}{p} \prod_{j=1}^n (p + (j - 1)(r - 1)). \quad (107)$$

The generalized Bell numbers $B_{r,1}(n)$ can always be expressed as a combination of $r - 1$ different hypergeometric functions of type ${}_1F_{r-1}(\dots; x)$, each of them evaluated at the same value of argument

$x = (r - 1)^{1-r}$; here are some lowest order cases:

$$B_{2,1}(n) = \frac{n!}{e} {}_1F_1\left(n + 1; 2; 1\right) = (n - 1)! L_{n-1}^{(1)}(-1), \tag{108}$$

$$B_{3,1}(n) = \frac{2^{n-1}}{e} \left(\frac{2\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}} {}_1F_2\left(n + \frac{1}{2}; \frac{1}{2}, \frac{3}{2}; \frac{1}{4}\right) + n! {}_1F_2\left(n + 1; \frac{3}{2}, 2; \frac{1}{4}\right) \right),$$

$$B_{4,1}(n) = \frac{3^{n-1}}{2e} \left(\frac{3^{3/2}\Gamma\left(\frac{2}{3}\right)\Gamma\left(n + \frac{1}{3}\right)}{\pi} {}_1F_3\left(n + \frac{1}{3}; \frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \frac{1}{27}\right) + \frac{3\Gamma\left(n + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} {}_1F_3\left(n + \frac{2}{3}; \frac{2}{3}, \frac{4}{3}, \frac{5}{3}; \frac{1}{27}\right) + n! {}_1F_3\left(n + 1; \frac{4}{3}, \frac{5}{3}, 2; \frac{1}{27}\right) \right),$$

etc...

In Eq.(108) $L_m^{(\alpha)}(y)$ is the associated Laguerre polynomial.

In this case the exponential generating function of Eq.(87) converges (see also Chapter 5)

$$G_{r,1}(\lambda, x) = e^{x\left(\frac{1}{r-\sqrt{1-(r-1)\lambda}}-1\right)}. \tag{109}$$

The exponential generating function of $S_{r,1}(n, k)$ takes the form

$$\sum_{n=\lceil k/r \rceil}^{\infty} \frac{x^n}{n!} S_{r,1}(n, k) = \frac{1}{k!} \left[(1 - (r - 1)x)^{-\frac{1}{r-1}} - 1 \right]^k. \tag{110}$$

See also Chapter 5 for detailed discussion of this case.

We end this section by writing down some triangles of generalized Stirling and Bell numbers, as defined by Eqs.(82) and (84).

r=1, s=1

	$S_{1,1}(n, k), 1 \leq k \leq n$	$B_{1,1}(n)$
$n = 1$	1	1
$n = 2$	1 1	2
$n = 3$	1 3 1	5
$n = 4$	1 7 6 1	15
$n = 5$	1 15 25 10 1	52
$n = 6$	1 31 90 65 15 1	203

r=2, s=1

	$S_{2,1}(n, k), 1 \leq k \leq n$						$B_{2,1}(n)$
$n = 1$	1						1
$n = 2$	2	1					3
$n = 3$	6	6	1				13
$n = 4$	24	36	12	1			73
$n = 5$	120	240	120	20	1		501
$n = 6$	720	1800	1200	300	30	1	4051

r=3, s=1

	$S_{3,1}(n, k), 1 \leq k \leq n$						$B_{3,1}(n)$
$n = 1$	1						1
$n = 2$	3	1					4
$n = 3$	15	9	1				25
$n = 4$	105	87	18	1			211
$n = 5$	945	975	285	30	1		2236
$n = 6$	10395	12645	4680	705	45	1	28471

4.7.2 Generalized Kerr Hamiltonian

In this section we consider the case of the homogeneous boson polynomial with excess zero ($d = 0$). This illustrates the formalism of Sections 4.3 and 4.5 on the example which constitutes the solution to the normal ordering problem for the generalized Kerr medium. The Kerr medium is described by the hamiltonian $H = a^\dagger a^\dagger a a$, [MW95]. By generalization we mean the hamiltonian in the form

$$H_\alpha = \sum_{k=N_0}^N \alpha_k (a^\dagger)^k a^k. \tag{111}$$

Observe that this is exactly the form of the operators (with excess $d = 0$) considered in Section 4.3, *i.e.*

$$H_{\alpha} \equiv H_{\alpha}^0. \tag{112}$$

To simplify the notation we skip the index $d = 0$ in $S_{\alpha}^d(n, k)$, $B_{\alpha}^d(n, x)$, $B_{\alpha}^d(n)$ and $G_{\alpha}^d(\lambda, x)$.

We are now ready to write down the solution to the normal ordering problem in terms of generalized Stirling numbers

$$(H_{\alpha})^n = \sum_{k=N_0}^{nN} S_{\alpha}(n, k) (a^{\dagger})^k a^k, \tag{113}$$

generalized Bell polynomials

$$B_{\alpha}(n, x) = \sum_{k=N_0}^{nN} S_{\alpha}(n, k) x^k, \tag{114}$$

and Bell numbers

$$B_{\alpha}(n) = B_{\alpha}^d(n, 1) = \sum_{k=N_0}^{nN} S_{\alpha}(n, k). \tag{115}$$

Recurrence relation

$$S_{\alpha}(n + 1, k) = \sum_{l=N_0}^N \alpha_l \sum_{p=0}^l \binom{l}{p} (k - l + p)^p S_{\alpha}(n, k - l + p), \tag{116}$$

with initial conditions as in Eq.(52).

Connection property

$$\left(\sum_{k=N_0}^N \alpha_k x^k \right)^n = \sum_{k=N_0}^{nN} S_{\alpha}(n, k) x^k. \tag{117}$$

The Dobiński-type relation has the form:

$$B_{\alpha}(n, x) = e^{-x} \sum_{l=0}^{\infty} \left(\sum_{k=N_0}^N \alpha_k l^k \right)^n \frac{x^l}{l!}. \tag{118}$$

Explicit expression

$$S_{\alpha}(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(\sum_{l=N_0}^N \alpha_l j^l \right)^n. \quad (119)$$

Exponential generating function

$$G_{\alpha}(\lambda, x) = \sum_{n=0}^{\infty} B_{\alpha}(n, x) \frac{\lambda^n}{n!} \quad (120)$$

$$= e^{-x} \sum_{n=0}^{\infty} e^{\lambda \sum_{r=N_0}^N \alpha_r n^r} \frac{x^n}{n!} \quad (121)$$

$$= \sum_{m=0}^{\infty} \left[\sum_{l=0}^m \binom{m}{l} (-1)^{m-l} e^{\lambda \sum_{r=N_0}^N \alpha_r l^r} \right] \frac{x^m}{m!}. \quad (122)$$

Again, the exponential generating function of Eq.(120) is formal around $\lambda = 0$ (for $N > 1$). Although we note that by the d'Alembert criterion for $\lambda \alpha_N < 0$ the the series of Eq.(121) converges.

The generating function of the Stirling numbers is

$$\sum_{n=k}^{\infty} S_{\alpha}(n, k) \frac{\lambda^n}{n!} = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{\lambda \sum_{k=N_0}^N \alpha_k l^k}. \quad (123)$$

Now, returning to the coherent state representation we have

$$\langle z | e^{\lambda H_{\alpha}} | z \rangle = G_{\alpha}(\lambda, |z|^2). \quad (124)$$

This provides the normally ordered form of the exponential

$$e^{\lambda H_{\alpha}} = : G_{\alpha}(\lambda, a^{\dagger} a) : . \quad (125)$$

These considerations may serve as a tool in investigations in quantum optics whenever the coherent state representation is needed, *e.g.* they allow one to calculate the Husimi functions or other phase space pictures for thermal states [BHPS05].

5 Monomiality principle and normal ordering

We solve the boson normal ordering problem for $(q(a^\dagger)a + v(a^\dagger))^n$ with arbitrary functions $q(x)$ and $v(x)$. This consequently provides the solution for the exponential $e^{\lambda(q(a^\dagger)a + v(a^\dagger))}$ generalizing the shift operator. In the course of these considerations we define and explore the monomiality principle and find its representations. We exploit the properties of Sheffer-type polynomials which constitute the inherent structure of this problem. In the end we give some examples illustrating the utility of the method and point out the relation to combinatorial structures.

5.1 Introduction

In Chapter 4 we treated the normal ordering problem of powers and exponentials of boson strings and homogeneous polynomials. Here we shall extend these results in a very particular direction. We consider operators linear in the annihilation a or creation a^\dagger operators. More specifically, we consider operators which, say for linearity in a , have the form

$$q(a^\dagger)a + v(a^\dagger)$$

where $q(x)$ and $v(x)$ are arbitrary functions. Passage to operators linear in a^\dagger is immediate through conjugacy operation.

We shall find the normally ordered form of the n -th power (iteration)

$$(q(a^\dagger)a + v(a^\dagger))^n$$

and then the exponential

$$e^{\lambda(q(a^\dagger)a + v(a^\dagger))}.$$

This is a far reaching generalization of the results of [Mik83][Mik85][Kat83] where a special case of the operator $a^\dagger a + a^r$ was considered.

In this approach we use methods which are based on the monomiality principle [BDHP05]. First, using the methods of umbral calculus, we

find a wide class of representations of the canonical commutation relation Eq.(17) in the space of polynomials. This establishes the link with Sheffer-type polynomials. Next the specific matrix elements of the above operators are derived and then, with the help of coherent state theory, extended to the general form. Finally we obtain the normally ordered expression for these operators. It turns out that the exponential generating functions in the case of linear dependence on the annihilation (or creation) operator are of Sheffer-type, and that assures their convergence.

In the end we give some examples with special emphasis on their Sheffer-type origin. We also refer to Section 6.3 for other application of derived formulas.

5.2 Monomiality principle

Here we introduce the concept of monomiality which arises from the action of the multiplication and derivative operators on monomials. Next we provide a wide class of representations of that property in the framework of Sheffer-type polynomials. Finally we establish the correspondence to the occupation number representation.

5.2.1 Definition and general properties

Let us consider the Heisenberg-Weyl algebra satisfying the commutation relation

$$[P, M] = 1. \quad (126)$$

In Section 2.4 we have already mentioned that a convenient representation of Eq.(126) is the derivative D and multiplication X representation defined in the space of polynomials. Action of these operators on the monomials is given by Eq.(16). Here we extend this framework.

Suppose one wants to find the representations of Eq.(126) such that the action of M and P on certain polynomials $s_n(x)$ is analogous to the action of X and D on monomials. More specifically one searches for M and P and their associated polynomials $s_n(x)$ (of degree n , $n = 0, 1, 2, \dots$) which satisfy

$$\begin{aligned} Ms_n(x) &= s_{n+1}(x), \\ Ps_n(x) &= n s_{n-1}(x). \end{aligned} \quad (127)$$

The rule embodied in Eq.(127) is called the *monomiality principle*. The polynomials $s_n(x)$ are then called *quasi-monomials* with respect to operators M and P . These operators can be immediately recognized as raising and lowering operators acting on the $s_n(x)$'s.

The definition Eq.(127) implies some general properties. First the operators M and P obviously satisfy Eq.(126). Further consequence of Eq.(127) is the eigenproperty of MP , *i.e.*

$$MPs_n(x) = ns_n(x). \tag{128}$$

The polynomials $s_n(x)$ may be obtained through the action of M^n on s_0

$$s_n(x) = M^n s_0 \tag{129}$$

and consequently the exponential generating function of $s_n(x)$'s is

$$G(\lambda, x) \equiv \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = e^{\lambda M} s_0. \tag{130}$$

Also, if we write the quasimonomial $s_n(x)$ explicitly as

$$s_n(x) = \sum_{k=0}^n s_{n,k} x^k, \tag{131}$$

then

$$s_n(x) = \left(\sum_{k=0}^n s_{n,k} X^k \right) 1. \tag{132}$$

Several types of such polynomial sequences were studied recently using this monomiality principle [DOTV97][GDM97][Dat99][DSC01][Ces00].

5.2.2 Monomiality principle representations: Sheffer-type polynomials

Here we show that if $s_n(x)$ are of *Sheffer-type* then it is possible to give explicit representations of M and P . Conversely, if $M =$

$M(X, D)$ and $P = P(D)$ then $s_n(x)$ of Eq.(127) are necessarily of Sheffer-type.

Properties of Sheffer-type polynomials are naturally handled within the so called *umbral calculus* [Rom84][RR78][Buc98] (see Appendix B). Here we put special emphasis on their ladder structure. Suppose we have a polynomial sequence $s_n(x)$, $n = 0, 1, 2, \dots$ ($s_n(x)$ being a polynomial of degree n). It is called of a Sheffer A-type zero [She39],[Rai65] (which we shall call Sheffer-type) if there exists a function $f(x)$ such that

$$f(D)s_n(x) = ns_{n-1}(x). \tag{133}$$

Operator $f(D)$ plays the role of the lowering operator. This characterization is not unique, *i.e.* there are many Sheffer-type sequences $s_n(x)$ satisfying Eq.(133) for given $f(x)$. We can further classify them by postulating the existence of the associated raising operator. A general theorem [Rom84][Che03] states that a polynomial sequence $s_n(x)$ satisfying the monomiality principle Eq.(127) with an operator P given as a function of the derivative operator only $P = P(D)$ is *uniquely* determined by two (formal) power series $f(\lambda) = \sum_{n=0}^{\infty} f_n \frac{\lambda^n}{n!}$ and $g(\lambda) = \sum_{n=0}^{\infty} g_n \frac{\lambda^n}{n!}$ such that $f(0) = 0$, $f'(0) \neq 0$ and $g(0) \neq 0$. The exponential generating function of $s_n(x)$ is then equal to

$$G(\lambda, x) = \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = \frac{1}{g(f^{-1}(\lambda))} e^{xf^{-1}(\lambda)}, \tag{134}$$

and their associated raising and lowering operators of Eq.(127) are given by

$$\begin{aligned} P &= f(D), \\ M &= \left[X - \frac{g'(D)}{g(D)} \right] \frac{1}{f'(D)}. \end{aligned} \tag{135}$$

Observe the important fact that X enters M only linearly. Note also that the order of X and D in $M(X, D)$ matters.

The above also holds true for $f(x)$ and $g(x)$ which are formal power series.

By direct calculation one may check that any pair M, P from Eq.(135) automatically satisfies Eq.(126). The detailed proof can be found in *e.g.* [Rom84], [Che03].

Here are some examples we have obtained of representations of the monomiality principle Eq.(127) and their associated Sheffer-type polynomials:

- a) $M(X, D) = 2X - D, \quad P(D) = \frac{1}{2}D,$
 $s_n(x) = H_n(x)$ - Hermite polynomials;
 $G(\lambda, x) = e^{2\lambda x - \lambda^2}.$
- b) $M(X, D) = -XD^2 + (2X - 1)D - X + 1, \quad P(D) = -\sum_{n=1}^{\infty} D^n,$
 $s_n(x) = n!L_n(x)$ - where $L_n(x)$ are Laguerre polynomials;
 $G(\lambda, x) = \frac{1}{1-\lambda} e^{x \frac{\lambda}{1-\lambda}}.$
- c) $M(X, D) = X \frac{1}{1-D}, \quad P(D) = -\frac{1}{2}D^2 + D,$
 $s_n(x) = P_n(x)$ - Bessel polynomials [Gro78];
 $G(\lambda, x) = e^{x(1-\sqrt{1-2\lambda})}.$
- d) $M(X, D) = X(1 + D), \quad P(D) = \ln(1 + D),$
 $s_n(x) = B_n(x)$ - Bell polynomials;
 $G(\lambda, x) = e^{x(e^\lambda - 1)}.$
- e) $M(X, D) = X e^{-D}, \quad P(D) = e^D - 1,$
 $s_n(x) = x^{\underline{n}}$ - the lower factorial polynomials [TS95];
 $G(\lambda, x) = e^{x \ln(1+\lambda)}.$
- f) $M(X, D) = (X - \tan(D)) \cos^2(D), \quad P(D) = \tan(D),$
 $s_n(x) = R_n(x)$ - Hahn polynomials [Ben91];
 $G(\lambda, x) = \frac{1}{\sqrt{1+\lambda^2}} e^{x \arctan(\lambda)}.$
- g) $M(X, D) = X \frac{1+W_L(D)}{W_L(D)} D, \quad P(D) = W_L(D),$
 where $W_L(x)$ is the Lambert W function [CGH⁺96];
 $s_n(x) = I_n(x)$ - the idempotent polynomials [Com74];
 $G(\lambda, x) = e^{x\lambda e^\lambda}.$

5.2.3 Monomiality vs Fock space representations

We have already called operators M and P satisfying Eq.(127) the rising and lowering operators. Indeed, their action rises and lowers index n of the quasimonomial $s_n(x)$. This resembles the property of creation a^\dagger and annihilation a operators in the Fock space (see Section 2.2) given by

$$\begin{aligned} a|n\rangle &= \sqrt{n} |n-1\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1} |n+1\rangle. \end{aligned} \quad (136)$$

These relations are almost the same as Eq.(127). There is a difference in coefficients. To make them analogous it is convenient to redefine the number states $|n\rangle$ as

$$|\widetilde{n}\rangle = \sqrt{n!} |n\rangle. \quad (137)$$

(Note that $|\widetilde{0}\rangle \equiv |0\rangle$).

Then the creation and annihilation operators act as

$$\begin{aligned} a|\widetilde{n}\rangle &= n|\widetilde{n-1}\rangle \\ a^\dagger|\widetilde{n}\rangle &= |\widetilde{n+1}\rangle. \end{aligned} \quad (138)$$

Now this exactly mirrors the the relation in Eq.(127). So, we make the correspondence

$$\begin{aligned} P &\longleftrightarrow a \\ M &\longleftrightarrow a^\dagger \\ s_n(x) &\longleftrightarrow |\widetilde{n}\rangle, \quad n = 0, 1, 2, \dots \end{aligned} \quad (139)$$

We note that this identification is purely algebraic, *i.e.* we are concerned here only with the commutation relation Eqs.(126), (17) or (1). We do not impose the scalar product in the space of polynomials nor consider the conjugacy properties of the operators. These properties are irrelevant for our proceeding discussion. We note only that they may be rigorously introduced, see *e.g.* [Rom84].

5.3 Normal ordering via monomiality

In this section we shall exploit the correspondence of Section 5.2.3 to obtain the normally ordered expression of powers and exponential of the operators $a^\dagger q(a) + v(a)$ and (by the conjugacy property)

$q(a^\dagger)a + v(a^\dagger)$. To this end shall apply the results of Section 5.2.2 to calculate some specific coherent state matrix elements of operators in question and then through the exponential mapping property we will extend it to a general matrix element. In conclusion we shall also comment on other forms of linear dependence on a or a^\dagger .

We use the correspondence of Section 5.2.3 cast in the simplest form for $M = X$, $P = D$ and $s_n(x) = x^n$, *i.e.*

$$\begin{aligned} D &\longleftrightarrow a \\ X &\longleftrightarrow a^\dagger \\ x^n &\longleftrightarrow \widetilde{|n\rangle}, \quad n = 0, 1, 2, \dots \end{aligned} \tag{140}$$

Then we recall the representation Eq.(135) of operators M and P in terms of X and D . Applying the correspondence of Eq.(140) it takes the form

$$\begin{aligned} P(a) &= f(a), \\ M(a, a^\dagger) &= \left[a^\dagger - \frac{g'(a)}{g(a)} \right] \frac{1}{f'(a)}. \end{aligned} \tag{141}$$

Recalling Eqs.(129),(131) and (132) we get

$$[M(a, a^\dagger)]^n |0\rangle = \sum_{k=0}^n s_{n,k} (a^\dagger)^k |0\rangle. \tag{142}$$

In the coherent state representation it yields

$$\langle z | [M(a, a^\dagger)]^n |0\rangle = s_n(z^*) \langle z | 0 \rangle. \tag{143}$$

Exponentiating $M(a, a^\dagger)$ and using Eq.(134) we obtain

$$\langle z | e^{\lambda M(a, a^\dagger)} |0\rangle = \frac{1}{g(f^{-1}(\lambda))} e^{z^* f^{-1}(\lambda)} \langle z | 0 \rangle. \tag{144}$$

Observe that in Eq.(143) we obtain Sheffer-type polynomials (modulus coherent states overlapping factor $\langle z | 0 \rangle$). This connection will be explored in detail in Section 5.4.2.

The result of Eq.(144) can be further extended to the general matrix element $\langle z | e^{\lambda M(a, a^\dagger)} |z'\rangle$. To this end recall Eq.(215) and write

$$\begin{aligned} \langle z | e^{\lambda M(a, a^\dagger)} |z'\rangle &= e^{-\frac{1}{2}|z'|^2} \langle z | e^{\lambda M(a, a^\dagger)} e^{z' a^\dagger} |0\rangle \\ &= e^{-\frac{1}{2}|z'|^2} \langle z | e^{z' a^\dagger} e^{-z' a^\dagger} e^{\lambda M(a, a^\dagger)} e^{z' a^\dagger} |0\rangle \\ &= e^{z^* z' - \frac{1}{2}|z'|^2} \langle z | e^{-z' a^\dagger} e^{\lambda M(a, a^\dagger)} e^{z' a^\dagger} |0\rangle. \end{aligned}$$

Next, using the exponential mapping property Eq.(11) we arrive at

$$\begin{aligned} \langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle &= e^{z^*z' - \frac{1}{2}|z'|^2} \langle z|e^{\lambda M(a+z',a^\dagger)}|0\rangle \\ &= e^{z^*z' - \frac{1}{2}|z'|^2} \langle z|e^{\lambda\left(a^\dagger - \frac{g'(a+z')}{g(a+z')}\right) \frac{1}{f'(a+z')}}|0\rangle. \end{aligned}$$

Now we are almost ready to apply Eq.(144) to evaluate the matrix element on the r.h.s. of the above equation. Before doing so we have to appropriately redefine functions $f(x)$ and $g(x)$ in the following way (z' - fixed)

$$\begin{aligned} f(x) &\rightarrow \tilde{f}(x) = f(x+z') - f(z'), \\ g(x) &\rightarrow \tilde{g}(x) = g(x+z')/g(z'). \end{aligned}$$

Then $\tilde{f}(0) = 0$, $\tilde{f}'(0) \neq 0$ and $\tilde{g}(0) = 1$ as required by Sheffer property for $\tilde{f}(x)$ and $\tilde{g}(x)$. Observe that these conditions are not fulfilled by $f(x+z')$ and $g(x+z')$. This step imposes (analytical) constraints on z' , *i.e.* it is valid whenever $\tilde{f}'(z') \neq 0$ (although, we note that for formal power series approach this does not present any difficulty). Now we can write

$$\begin{aligned} \langle z|e^{\lambda\left(a^\dagger - \frac{g'(a+z')}{g(a+z')}\right) \frac{1}{f'(a+z')}}|0\rangle &= \langle z|e^{\lambda\left(a^\dagger - \frac{\tilde{g}'(a)}{\tilde{g}(a)}\right) \frac{1}{\tilde{f}'(a)}}|0\rangle \\ &\stackrel{(144)}{=} \frac{1}{\tilde{g}(\tilde{f}^{-1}(\lambda))} e^{z^*\tilde{f}^{-1}(\lambda)} \langle z|0\rangle. \end{aligned}$$

By going back to the initial functions $f(x)$ and $g(x)$ this readily gives the final result

$$\langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle = \frac{g(z')}{g(f^{-1}(\lambda + f(z')))} e^{z^*[f^{-1}(\lambda + f(z')) - z']} \langle z|z'\rangle, \quad (145)$$

where $\langle z|z'\rangle = e^{z^*z' - \frac{1}{2}|z'|^2 - \frac{1}{2}|z|^2}$ is the coherent states overlapping factor (see Eq.(213)).

To obtain the normally ordered form of $e^{\lambda M(a,a^\dagger)}$ we apply the crucial property of the coherent state representation of Eqs.(221) and (222). Then Eq.(145) provides the central result

$$e^{\lambda M(a,a^\dagger)} = : e^{a^\dagger[f^{-1}(\lambda + f(a)) - a]} \frac{g(a)}{g(f^{-1}(\lambda + f(a)))} : . \quad (146)$$

Let us point out again that a^\dagger appears linearly in $M(a, a^\dagger)$, see Eq.(141). We also note that the constraints for functions $f(x)$ and $g(x)$, *i.e.* $f(0) = 0$, $f'(0) \neq 0$ and $g(0) \neq 0$, play no important role. For convenience (simplicity) we put

$$\begin{aligned} q(x) &= \frac{1}{f'(x)}, \\ v(x) &= \frac{g'(x)}{g(x)} \frac{1}{f'(x)}, \end{aligned}$$

and define

$$\begin{aligned} T(\lambda, x) &= f^{-1}(\lambda + f(x)), \\ G(\lambda, x) &= \frac{g(x)}{g(T(\lambda, x))}. \end{aligned}$$

This allows us to rewrite the main normal ordering formula of Eq.(146) as

$$e^{\lambda(a^\dagger q(a) + v(a))} = : e^{a^\dagger [T(\lambda, a) - a]} G(\lambda, a) : \quad (147)$$

where the functions $T(\lambda, x)$ and $G(\lambda, x)$ fulfill the following differential equations

$$\frac{\partial T(\lambda, x)}{\partial \lambda} = q(T(\lambda, x)), \quad T(0, x) = x, \quad (148)$$

$$\frac{\partial G(\lambda, x)}{\partial \lambda} = v(T(\lambda, x)) \cdot G(\lambda, x), \quad G(0, x) = 1. \quad (149)$$

In the coherent state representation it takes the form

$$\langle z' | e^{\lambda(a^\dagger q(a) + v(a))} | z \rangle = \langle z' | z \rangle e^{z'^* [T(\lambda, z) - z]} G(\lambda, z). \quad (150)$$

We conclude by making a comment on other possible forms of linear dependence on a or a^\dagger .

By hermitian conjugation of Eq.(147) we obtain the expression for the normal form of $e^{\lambda(q(a^\dagger)a + v(a))}$. This amounts to the formula

$$e^{\lambda(q(a^\dagger)a + v(a))} = : G(\lambda, a^\dagger) e^{[T(\lambda, a^\dagger) - a^\dagger] a} : \quad (151)$$

with the same differential equations Eqs.(148) and (149) for functions $T(\lambda, x)$ and $G(\lambda, x)$. In the coherent state representation it yields

$$\langle z'|e^{\lambda(q(a^\dagger)a+v(a))}|z\rangle = \langle z'|z\rangle G(\lambda, z'^*)e^{[T(\lambda, z'^*)-z'^*]z} \quad (152)$$

We also note that all other operators linearly dependent on a or a^\dagger may be written in just considered forms with the use of Eq.(12), *i.e.* $aq(a^\dagger) + v(a^\dagger) = q(a^\dagger)a + q'(a^\dagger) + v(a^\dagger)$ and $q(a)a^\dagger + v(a) = a^\dagger q(a) + q'(a) + v(a)$.

Observe that from the analytical point of view certain limitations on the domains of z , z' and λ should be put in some specific cases (locally around zero all the formulas hold true). Also we point out the fact that functions $q(x)$ and $v(x)$ (or equivalently $f(x)$ and $g(x)$) may be taken as the formal power series. Then one stays on the ground of formal power expansions.

In the end we refer to Section 6.3 where these results are applied to derive the substitution formula. We note that the reverse process, *i.e.* derivation of the normally ordered form from the substitution theorem, is also possible, see [BHP⁺05].

5.4 Sheffer-type polynomials and normal ordering: Examples

We now proceed to examples. We will put special emphasis on their Sheffer-type origin.

5.4.1 Examples

We start with enumerating some examples of the evaluation of the coherent state matrix elements of Eqs.(143) and (150). We choose the $M(a, a^\dagger)$'s as in the list a) - g) in Section 5.2.2:

a) $\langle z|(-a + 2a^\dagger)^n|0\rangle = H_n(z^*)\langle z|0\rangle$, Hermite polynomials;

$$\langle z|e^{\lambda(-a+2a^\dagger)}|z'\rangle = e^{\lambda(2z^*-z')-\lambda^2}\langle z|z'\rangle.$$

b) $\langle z|[-a^\dagger a^2 + (2a^\dagger - 1)a - a^\dagger + 1]^n |0\rangle = n!L_n(z^*)\langle z|0\rangle$,

Laguerre polynomials;

$$\langle z|e^{\lambda[-a^\dagger a + (2a^\dagger - 1)a - a^\dagger + 1]}|z'\rangle = \frac{z'^2 - \lambda z' + 1}{(1-z')(1-\lambda(1-z'))} e^{z^* \lambda \frac{(1-z')^2}{\lambda(1-z')-1}} \langle z|z'\rangle.$$

- c) $\langle z | \left(a^\dagger \frac{1}{1-a} \right)^n | 0 \rangle = P_n(z^*) \langle z | 0 \rangle$, Bessel polynomials;
 $\langle z | e^{\lambda \left(a^\dagger \frac{1}{1-a} \right)} | z' \rangle = e^{z^* [1 - \sqrt{1 - 2(\lambda + z' - \frac{1}{2} z'^2)} - z']} \langle z | z' \rangle$.
- d) $\langle z | (a^\dagger a + a^\dagger)^n | 0 \rangle = B_n(z^*) \langle z | 0 \rangle$, Bell polynomials;
 $\langle z | e^{\lambda (a^\dagger a + a^\dagger)} | z' \rangle = e^{z^* (z' + 1) (e^\lambda - 1)} \langle z | z' \rangle$.
- e) $\langle z | (a^\dagger e^{-a})^n | 0 \rangle = (z^*)^n \langle z | 0 \rangle$, the lower factorial polynomials;
 $\langle z | e^{\lambda (a^\dagger e^{-a})} | z' \rangle = e^{z^* [\ln(e^{z'} + \lambda) - z']} \langle z | z' \rangle$.
- f) $\langle z | [(a^\dagger - \tan(a)) \cos^2(a)]^n | 0 \rangle = R_n(z^*) \langle z | 0 \rangle$,
 Hahn polynomials;
 $\langle z | e^{\lambda (a^\dagger - \tan(a)) \cos^2(a)} | z' \rangle =$
 $= \frac{\cos[\arctan(\lambda + \tan(z'))]}{\cos(z')} e^{z^* [\arctan(\lambda + \tan(z')) - z']} \langle z | z' \rangle$.
- g) $\langle z | \left[a^\dagger \frac{1 + W_L(a)}{W_L(a)} a \right]^n | 0 \rangle = I_n(z^*) \langle z | 0 \rangle$,
 the idempotent polynomials;
 $\langle z | e^{\lambda a^\dagger \frac{1 + W_L(a)}{W_L(a)} a} | z' \rangle = e^{z^* [\lambda e^{\lambda + W_L(z')} + z' (e^\lambda - 1)]} \langle z | z' \rangle$.

Note that for $z' = 0$ we obtain the exponential generating functions of appropriate polynomials multiplied by the coherent states overlapping factor $\langle z | 0 \rangle$, see Eq.(145).

These examples show how the Sheffer-type polynomials and their exponential generating functions arise in the coherent state representation. This generic structure is the consequence of Eqs.(143) and (145) or in general Eqs.(150) or (152) and it will be investigated in more detail now. Afterwards we shall provide more examples of combinatorial origin.

5.4.2 Sheffer polynomials and normal ordering

First recall the definition of the family of Sheffer-type polynomials $s_n(z)$ defined (see Appendix B) through the exponential generating function as

$$G(\lambda, z) = \sum_{n=0}^{\infty} s_n(z) \frac{\lambda^n}{n!} = A(\lambda) e^{zB(\lambda)} \quad (153)$$

where functions $A(\lambda)$ and $B(\lambda)$ satisfy: $A(0) \neq 0$ and $B(0) = 0$, $B'(0) \neq 0$.

Returning to normal ordering, recall that the coherent state expectation value of Eq.(151) is given by Eq.(152). When one fixes z' and takes λ and z as indeterminates, then the r.h.s. of Eq.(152) may be read off as an exponential generating function of Sheffer-type polynomials defined by Eq.(153). The correspondence is given by

$$A(\lambda) = g(\lambda, z'^*), \tag{154}$$

$$B(\lambda) = [T(\lambda, z'^*) - z'^*]. \tag{155}$$

This allows us to make the statement that the coherent state expectation value $\langle z' | \dots | z \rangle$ of the operator $\exp [\lambda(q(a^\dagger)a + v(a^\dagger))]$ for any fixed z' yields (up to the overlapping factor $\langle z' | z \rangle$) the exponential generating function of a certain sequence of Sheffer-type polynomials in the variable z given by Eqs.(154) and (155). The above construction establishes the connection between the coherent state representation of the operator $\exp [\lambda(q(a^\dagger)a + v(a^\dagger))]$ and a family of Sheffer-type polynomials $s_n^{(q,v)}(z)$ related to q and v through

$$\langle z' | e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} | z \rangle = \langle z' | z \rangle \left(1 + \sum_{n=1}^{\infty} s_n^{(q,v)}(z) \frac{\lambda^n}{n!} \right), \tag{156}$$

where explicitly (again for z' fixed):

$$s_n^{(q,v)}(z) = \langle z' | z \rangle^{-1} \langle z' | [q(a^\dagger)a + v(a^\dagger)]^n | z \rangle. \tag{157}$$

We observe that Eq.(157) is an extension of the seminal formula of J. Katriel [Kat74],[Kat00] where $v(x) = 0$ and $q(x) = x$. The Sheffer-type polynomials are in this case Bell polynomials expressible through the Stirling numbers of the second kind (see Section 3).

Having established relations leading from the normal ordering problem to Sheffer-type polynomials we may consider the reverse approach. Indeed, it turns out that for any Sheffer-type sequence generated by $A(\lambda)$ and $B(\lambda)$ one can find functions $q(x)$ and $v(x)$ such that the coherent state expectation value $\langle z' | \exp [\lambda(q(a^\dagger)a + v(a^\dagger))] | z \rangle$ results in a corresponding exponential generating function of Eq.(153) in indeterminates z and λ (up to the overlapping factor $\langle z' | z \rangle$ and

z' fixed). Appropriate formulas can be derived from Eqs.(154) and (155) by substitution into Eqs.(148) and (149):

$$q(x) = B'(B^{-1}(x - z'^*)), \quad (158)$$

$$v(x) = \frac{A'(B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}. \quad (159)$$

One can check that this choice of $q(x)$ and $v(x)$, if inserted into Eqs. (148) and (149), results in

$$T(\lambda, x) = B(\lambda + B^{-1}(x - z'^*)) + z'^*, \quad (160)$$

$$g(\lambda, x) = \frac{A(\lambda + B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}, \quad (161)$$

which reproduce

$$\langle z' | e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} | z \rangle = \langle z' | z \rangle A(\lambda) e^{zB(\lambda)}. \quad (162)$$

The result summarized in Eqs.(154) and (155) and in their 'dual' forms Eqs.(158)-(161) provide us with a considerable flexibility in conceiving and analyzing a large number of examples.

5.4.3 Combinatorial examples

In this section we will work out examples illustrating how the exponential generating function $G(\lambda) = \sum_{n=0}^{\infty} a_n \frac{\lambda^n}{n!}$ of certain combinatorial sequences $(a_n)_{n=0}^{\infty}$ appear naturally in the context of boson normal ordering. To this end we shall assume specific forms of $q(x)$ and $v(x)$ thus specifying the operator that we exponentiate. We then give solutions to Eqs.(148) and (149) and subsequently through Eqs.(154) and (155) we write the exponential generating function of combinatorial sequences whose interpretation will be given.

- a) Choose $q(x) = x^r$, $r > 1$ (integer), $v(x) = 0$ (which implies $g(\lambda, x) = 1$). Then $T(\lambda, x) = x [1 - \lambda(r - 1)x^{r-1}]^{\frac{1}{1-r}}$. This gives

$$\mathcal{N} \left[e^{\lambda(a^\dagger)^r a} \right] \equiv : \exp \left[\left(\frac{a^\dagger}{(1 - \lambda(r - 1)(a^\dagger)^{r-1})^{\frac{1}{r-1}}} - 1 \right) a \right] :$$

as the normally ordered form. Now we take $z' = 1$ in Eqs.(154) and (155) and from Eq.(162) we obtain

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda(a^\dagger)^r a}|z\rangle = \exp \left[z \left(\frac{1}{(1 - \lambda(r-1))^{\frac{1}{r-1}}} - 1 \right) \right],$$

which for $z = 1$ generates the following sequences:

$$\begin{aligned} r = 2 : & \quad a_n = 1, 1, 3, 13, 73, 501, 4051, \dots \\ r = 3 : & \quad a_n = 1, 1, 4, 25, 211, 2236, 28471, \dots \quad , \text{ etc.} \end{aligned}$$

These sequences enumerate r -ary forests [Slo05][Sta99][FS05].

- b) For $q(x) = x \ln(ex)$ and $v(x) = 0$ (implying $g(\lambda, x) = 1$) we have $T(\lambda, x) = e^{e^\lambda - 1} x e^\lambda$. This corresponds to

$$\mathcal{N} \left[e^{\lambda a^\dagger \ln(e a^\dagger) a} \right] \equiv : \exp \left[\left(e^{e^\lambda - 1} (a^\dagger)^{e^\lambda} - 1 \right) a \right] :,$$

whose coherent state matrix element with $z' = 1$ is equal to

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda a^\dagger \ln(e a^\dagger) a}|z\rangle = \exp \left[z \left(e^{e^\lambda - 1} - 1 \right) \right],$$

which for $z = 1$ generates $a_n = 1, 1, 3, 12, 60, 385, 2471, \dots$ enumerating partitions of partitions [Sta99], [Slo05], [FS05].

The following two examples refer to the reverse procedure, see Eqs.(158)-(161). We choose first a Sheffer-type exponential generating function and deduce $q(x)$ and $v(x)$ associated with it.

- c) $A(\lambda) = \frac{1}{1-\lambda}$, $B(\lambda) = \lambda$, see Eq.(153). This exponential generating function for $z = 1$ counts the number of arrangements $a_n = n! \sum_{k=0}^n \frac{1}{k!} = 1, 2, 5, 65, 326, 1957, \dots$ of the set of n elements [Com74]. The solutions of Eqs.(158) and (159) are: $q(x) = 1$ and $v(x) = \frac{1}{2-x}$. In terms of bosons it corresponds to

$$\mathcal{N} \left[e^{\lambda \left(a + \frac{1}{2-a^\dagger} \right)} \right] \equiv : \frac{2 - a^\dagger}{2 - a^\dagger - \lambda} e^{\lambda a} : = \frac{2 - a^\dagger}{2 - a^\dagger - \lambda} e^{\lambda a}.$$

- d) For $A(\lambda) = 1$ and $B(\lambda) = 1 - \sqrt{1 - 2\lambda}$ one gets the exponential generating function of the Bessel polynomials [Gro78]. For $z = 1$ they enumerate special paths on a lattice [Pit99]. The corresponding sequence is $a_n = 1, 1, 7, 37, 266, 2431, \dots$. The solutions of Eqs.(158) and (159) are: $q(x) = \frac{1}{2-x}$ and $v(x) = 0$. It corresponds to

$$\mathcal{N} \left[e^{\lambda \frac{1}{2-a^\dagger} a} \right] \equiv : e^{(1 - \sqrt{(2-a^\dagger) - 2\lambda}) a} :$$

in the boson formalism.

These examples show that any combinatorial structure which can be described by a Sheffer-type exponential generating function can be cast in boson language. This gives rise to a large number of formulas of the above type which put them in a quantum mechanical setting.

6 Miscellany: Applications

We give three possible extensions of the above formalism. First we show how this approach is modified when one considers deformations of the canonical boson algebra. Next we construct and analyze the generalized coherent states based on some number series which arise in the ordering problems for which we also furnish the solution to the moment problem. We end by using results of Chapter 5 to derive a substitution formula.

6.1 Deformed bosons

We solve the normal ordering problem for $(A^\dagger A)^n$ where A and A^\dagger are one mode deformed boson ladder operators ($[A, A^\dagger] = [N + 1] - [N]$). The solution generalizes results known for canonical bosons (see Chapter 3). It involves combinatorial polynomials in the number operator N for which the generating function and explicit expressions are found. Simple deformations provide illustration of the method.

6.1.1 Introduction

We consider the general deformation of the boson algebra [Sol94] in the form

$$\begin{aligned} [A, N] &= A, \\ [A^\dagger, N] &= -A^\dagger, \\ [A, A^\dagger] &= [N + 1] - [N]. \end{aligned} \tag{163}$$

In the above A and A^\dagger are the (deformed) annihilation and creation operators, respectively, while the number operator N counts particles. It is defined in the occupation number basis as $N|n\rangle = n|n\rangle$ and commutes with $A^\dagger A$ (which is the consequence of the first two commutators in Eq.(163)). Because of that in any representation of (163) N can be written in the form $A^\dagger A = [N]$, where $[N]$ denotes an arbitrary function of N , usually called the “box” function. Note that this is a generalization of the canonical boson algebra of Eqs.(1) and (8) described in Section 2.2 (it is recovered for $[N] = N + const$). For general considerations we do not assume any realization of the

number operator N and we treat it as an independent element of the algebra. Moreover, we do not assume any particular form of the “box” function $[N]$. Special cases, like the $so(3)$ or $so(2, 1)$ algebras, provide us with examples showing how such a general approach simplifies if an algebra and its realization are chosen.

In the following we give the solution to the problem of normal ordering of a monomial $(A^\dagger A)^n$ in deformed annihilation and creation operators. It is an extension of the problem for canonical pair $[a, a^\dagger] = 1$ described in Chapter 3 where we have considered Stirling numbers $S(n, k)$ arising from

$$(a^\dagger a)^n = \sum_{k=1}^n S(n, k) (a^\dagger)^k a^k \quad (\text{for canonical bosons}). \quad (164)$$

In the case of deformed bosons we cannot express the monomial $(A^\dagger A)^n$ as a combination of normally ordered expressions in A^\dagger and A only. This was found for q -deformed bosons some time ago, [KK92][Kat02][KD95][Sch03], and recently for the R -deformed Heisenberg algebra related to the Calogero model, [BN05]. The number operator N occurs in the final formulae because on commuting creation operators to the left we cannot get rid of N if it is assumed to be an independent element of the algebra. In general we can look for a solution of the form

$$(A^\dagger A)^n = \sum_{k=1}^n \mathcal{S}_{n,k}(N) (A^\dagger)^k A^k, \quad (165)$$

where coefficients $\mathcal{S}_{n,k}(N)$ are functions of the number operator N and their shape depend on the box function $[N]$. Following [KK92] we will call them *operator-valued deformed Stirling numbers* or just *deformed Stirling numbers*.

In the sequel we give a comprehensive analysis of this generalization. We shall give recurrences satisfied by $\mathcal{S}_{n,k}(N)$ of (165) and shall construct their generating functions. This will enable us to write down $\mathcal{S}_{n,k}(N)$ explicitly and to demonstrate how the method works on examples of simple Lie-type deformations of the canonical case.

6.1.2 Recurrence relations

One checks by induction that for each $k \geq 1$ the following relation holds

$$[A^k, A^\dagger] = ([N + k] - [N])A^{k-1}. \quad (166)$$

Using this relation it is easy to check by induction that deformed Stirling numbers satisfy the recurrences¹

$$\mathcal{S}_{n+1,k}(N) = \mathcal{S}_{n,k-1}(N) + ([N] - [N - k])\mathcal{S}_{n,k}(N), \quad \text{for } 1 < k < n \quad (167)$$

with initial values

$$\mathcal{S}_{n,1}(N) = ([N] - [N - 1])^{n-1}, \quad \mathcal{S}_{n,n}(N) = 1. \quad (168)$$

The proof can be deduced from the equalities

$$\begin{aligned} & \sum_{k=1}^{n+1} \mathcal{S}_{n+1,k}(N) (A^\dagger)^k A^k = (A^\dagger A)^{n+1} = (A^\dagger A)^n A^\dagger A \\ & = \sum_{k=1}^n \mathcal{S}_{n,k}(N) (A^\dagger)^k A^k A^\dagger A \\ & \stackrel{(166)}{=} \sum_{k=1}^n \mathcal{S}_{n,k}(N) (A^\dagger)^k (A^\dagger A + [N + k] - [N]) A^k \\ & \stackrel{(163)}{=} \sum_{k=2}^{n+1} \mathcal{S}_{n,k-1}(N) (A^\dagger)^k A^k + \sum_{k=1}^n ([N] - [N - k])\mathcal{S}_{n,k}(N) (A^\dagger)^k A^k. \end{aligned}$$

It can be shown that each $\mathcal{S}_{n,k}(N)$ is a homogeneous polynomial of order $n - k$ in variables $[N], \dots, [N - k]$. For a polynomial box function one obtains $\mathcal{S}_{n,k}(N)$ as a polynomial in N . As we shall show, a simple example of the latter is the deformation given by the $so(3)$ or $so(2, 1)$ Lie algebras.

Note that for the “box” function $[N] = N + \text{const}$ (*i.e.*, for the canonical algebra) we get the conventional Stirling numbers of Chapter 3.

¹The recurrence relation Eq.(167) holds for all n and k if one puts the following “boundary conditions”: $\mathcal{S}_{i,j}(N) = 0$ for $i = 0$ or $j = 0$ or $i < j$, except $\mathcal{S}_{0,0}(N) = 1$.

6.1.3 Generating functions and general expressions

We define the set of *ordinary generating functions* of polynomials $\mathcal{S}_{n,k}(N)$, for $k \geq 1$, in the form

$$P_k(x, N) := \sum_{n=k}^{\infty} \mathcal{S}_{n,k}(N) x^n. \quad (169)$$

The initial conditions (168) for $k = 1$ give

$$P_1(x, N) = \frac{x}{1 - ([N] - [N - 1]) x}. \quad (170)$$

Using the recurrences (167) completed with (168) one finds the relation

$$P_k(x, N) = \frac{x}{1 - ([N] - [N - k]) x} P_{k-1}(x, N), \quad \text{for } k > 1, \quad (171)$$

whose proof is provided by the equalities

$$\begin{aligned} P_k(x, N) &\stackrel{(167)}{=} \sum_{n=k}^{\infty} (\mathcal{S}_{n-1,k-1}(N) + ([N] - [N - k]) \mathcal{S}_{n-1,k}(N)) x^n \\ &= x \sum_{n=k}^{\infty} \mathcal{S}_{n-1,k-1}(N) x^{n-1} + ([N] - [N - k]) x \sum_{n=k}^{\infty} \mathcal{S}_{n-1,k}(N) x^{n-1} \\ &= x P_{k-1}(x, N) + ([N] - [N - k]) x P_k(x, N). \end{aligned}$$

The expressions (170) and (171) give explicit formula for the ordinary generating function

$$P_k(x, N) = \prod_{j=1}^k \frac{x}{1 - ([N] - [N - j]) x}. \quad (172)$$

For the canonical algebra it results in the ordinary generating function for Stirling numbers $S(n, k)$:

$$P_k(x) = \frac{x}{(1 - x)(1 - 2x) \cdots (1 - kx)}. \quad (173)$$

Explicit knowledge of the ordinary generating functions (172) enables us to find the $\mathcal{S}_{n,k}(N)$ in a compact form. As a rational function of x it can be expressed as a sum of partial fractions

$$P_k(x, N) = x^k \sum_{r=1}^k \frac{\alpha_r}{1 - ([N] - [N - r]) x}, \quad (174)$$

where

$$\alpha_r = \frac{1}{\prod_{j=1, j \neq r}^k \left(1 - \frac{[N] - [N - j]}{[N] - [N - r]}\right)}. \quad (175)$$

From the definition (169) we have that $\mathcal{S}_{n,k}(N)$ is the coefficient multiplying x^n in the formal Taylor expansion of $P_k(x, N)$. Expanding the fractions in above equations and collecting the terms we get $\mathcal{S}_{n,k}(N) = \sum_{r=1}^k \alpha_r ([N] - [N - r])^{n-k}$. Finally, using (175), we arrive at

$$\mathcal{S}_{n,k}(N) = \sum_{r=1}^k \frac{([N] - [N - r])^{n-1}}{\prod_{j=1, j \neq r}^k ([N - j] - [N - r])} \quad (176)$$

where monotonicity of $[N]$ was assumed.

For the canonical case it yields the conventional Stirling numbers $S(n, k) = \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} (-1)^{k-j} j^n$, see Eq.(28).

The first four families of general deformed Stirling numbers (polynomials) defined by (165) read

$$\mathcal{S}_{1,1}(N) = 1,$$

$$\mathcal{S}_{2,1}(N) = [N] - [N - 1],$$

$$\mathcal{S}_{2,2}(N) = 1,$$

$$\mathcal{S}_{3,1}(N) = ([N] - [N - 1])^2,$$

$$\mathcal{S}_{3,2}(N) = 2[N] - [N - 1] - [N - 2],$$

$$\mathcal{S}_{3,3}(N) = 1,$$

$$\mathcal{S}_{4,1}(N) = ([N] - [N - 1])^3,$$

$$\mathcal{S}_{4,2}(N) = 3[N]^2 - 3[N][N - 1] - 3[N][N - 2] + [N - 1]^2 + [N - 2][N - 1] + [N - 2]^2,$$

$$\mathcal{S}_{4,3}(N) = 3[N] - [N - 1] - [N - 2] - [N - 3],$$

$$\mathcal{S}_{4,4}(N) = 1.$$

6.1.4 Examples of simple deformations

If we fix the "box" function as $[N] = \mp \frac{N(N-1)}{2}$ then (163) become structural relations of $so(3)$ and $so(2, 1)$ algebras, respectively. The first four families of polynomials are (note that $\mathcal{S}_{n,k}^{so(2,1)}(N) = (-1)^{n-k} \mathcal{S}_{n,k}^{so(3)}(N)$)

so(3)

so(2,1)

$$\mathcal{S}_{1,1}(N) = 1$$

$$\mathcal{S}_{1,1}(N) = 1$$

$$\mathcal{S}_{2,1}(N) = -N + 1$$

$$\mathcal{S}_{2,1}(N) = N - 1$$

$$\mathcal{S}_{2,2}(N) = 1$$

$$\mathcal{S}_{2,2}(N) = 1$$

$$\mathcal{S}_{3,1}(N) = (N - 1)^2$$

$$\mathcal{S}_{3,1}(N) = (N - 1)^2$$

$$\mathcal{S}_{3,2}(N) = -3N + 4$$

$$\mathcal{S}_{3,2}(N) = 3N - 4$$

$$\mathcal{S}_{3,3}(N) = 1$$

$$\mathcal{S}_{3,3}(N) = 1$$

$$\mathcal{S}_{4,1}(N) = -(N - 1)^3$$

$$\mathcal{S}_{4,1}(N) = (N - 1)^3$$

$$\mathcal{S}_{4,2}(N) = 7N^2 - 19N + 13$$

$$\mathcal{S}_{4,2}(N) = 7N^2 - 19N + 13$$

$$\mathcal{S}_{4,3}(N) = -6N + 10$$

$$\mathcal{S}_{4,3}(N) = 6N - 10$$

$$\mathcal{S}_{4,4}(N) = 1$$

$$\mathcal{S}_{4,4}(N) = 1$$

Up to now we have been considering A , A^\dagger , and N as independent elements of the algebra. Choosing the representation and expressing N in terms of A and A^\dagger we arrive at expressions like (165) which can be normally ordered further. We may consider a realization of the $so(2, 1)$ in terms of canonical operators a and a^\dagger

$$A = \frac{1}{2\sqrt{2}}aa \quad A^\dagger = \frac{1}{2\sqrt{2}}a^\dagger a^\dagger \quad N = \frac{1}{2} \left(a^\dagger a + \frac{1}{2} \right). \quad (177)$$

The solution of the normal ordering problem for $((a^\dagger)^2 a^2)^n$ was already considered in Section 4.7.1, and leads to

$$((a^\dagger)^2 a^2)^n = \sum_{k=2}^{2n} S_{2,2}(n, k) a^{\dagger k} a^k, \quad (178)$$

where $S_{2,2}(n, k)$ are generalized Stirling numbers for which we have (see Eqs.(98) and (102))

$$S_{2,2}(n, k) = \frac{(-1)^k}{k!} \sum_{p=2}^k (-1)^p \binom{k}{p} [p(p-1)]^n \quad (179)$$

$$= \sum_{l=0}^n (-1)^l \binom{n}{l} S(2n-l, k). \quad (180)$$

Terms in (178) with k even are directly expressible by A and A^\dagger . If k is odd then the factor $a^\dagger a$ may be commuted to the left and we get (the convention of Eq.(85) is used)

$$((a^\dagger)^2 a^2)^n = \sum_{k=1}^n (S_{2,2}(n, 2k) + S_{2,2}(n, 2k+1)(a^\dagger a - 2k)) a^{\dagger 2k} a^{2k}.$$

Finally, we get

$$(A^\dagger A)^n = \sum_{k=1}^n 8^{k-n} [S_{2,2}(n, 2k) + S_{2,2}(n, 2k+1)(2N-2k-1/2)] A^{\dagger k} A^k.$$

This indicates that the general solution involving higher order polynomials in N can be simplified when a particular realization of the algebra is postulated.

6.2 Generalized coherent states

We construct and analyze a family of coherent states built on sequences of integers originating from the solution of the boson normal ordering problem investigated in Section 4.7.1. These sequences generalize the conventional combinatorial Bell numbers and are shown to be moments of positive functions. Consequently, the resulting coherent states automatically satisfy the resolution of unity condition. In addition they display such non-classical fluctuation properties as super-Poissonian statistics and squeezing.

6.2.1 Introduction

Since their introduction in quantum optics many generalizations of standard coherent states have been proposed (see Appendix A). The main purpose of such generalizations is to account for a full description of interacting quantum systems. The conventional coherent states provide a correct description of a typical non-interacting system, the harmonic oscillator. One formal approach to this problem is to redefine the standard boson creation a^\dagger and annihilation a operators, satisfying $[a, a^\dagger] = 1$, to $A = af(a^\dagger a)$, where the function $f(N)$, $N = a^\dagger a$, is chosen to adequately describe the interacting problem. Any deviation of $f(x)$ from $f(x) = \text{const}$ describes a non-linearity in the system. This amounts to introducing the modified (deformed) commutation relations [Sol94][MMSZ97][MM98] (see also Section 6.1)

$$[A, A^\dagger] = [N + 1] - [N], \quad (181)$$

where the “box” function $[N]$ is defined as $[N] = Nf^2(N) > 0$. Such a way of generalizing the boson commutator naturally leads to generalized “nonlinear” coherent states in the form ($[n]! = [0][1] \dots [n]$, $[0] = 1$)

$$|z\rangle = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle, \quad (182)$$

which are eigenstates of the “deformed” boson annihilation operator A

$$A|z\rangle = z|z\rangle. \quad (183)$$

It is worth pointing out that such nonlinear coherent states have been successfully applied to a large class of physical problems in quantum optics [MV96][KVD01]. The comprehensive treatment of coherent states of the form of Eq. (182) can be found in [Sol94][MMSZ97][MM98]. An essential ingredient in the definition of coherent states is the completeness property (or the resolution of unity condition) [KS85][Kla63a][Kla63b][KPS01]. A guideline for the construction of coherent states in general has been put forward in [Kla63a] as a minimal set of conditions. Apart from the conditions of normalizability and continuity in the complex label z , this set reduces to satisfaction of the resolution of unity condition. This implies the existence of a positive function $\tilde{W}(|z|^2)$ satisfying [KPS01]

$$\int_{\mathbb{C}} d^2z |z\rangle\tilde{W}(|z|^2)\langle z| = I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \quad (184)$$

which reflects the completeness of the set $\{|z\rangle\}$.

In Eq.(184) I is the unit operator and $|n\rangle$ is a complete set of orthonormal eigenvectors. In a general approach one chooses strictly positive parameters $\rho(n)$, $n = 0, 1, \dots$ such that the state $|z\rangle$ which is normalized, $\langle z|z\rangle = 1$, is given by

$$|z\rangle = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle, \quad (185)$$

with normalization

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)} > 0, \quad (186)$$

which we assume here to be a convergent series in $|z|^2$ for all $z \in \mathbb{C}$. In view of Eqs. (181) and (182) this corresponds to $\rho(n) = [n]!$ or $[n] = \rho(n)/\rho(n-1)$ for $n = 1, 2, \dots$

Condition (184) can be shown to be equivalent to the following infinite set of equations [KPS01]:

$$\int_0^{\infty} x^n \left[\pi \frac{\tilde{W}(x)}{\mathcal{N}(x)} \right] dx = \rho(n), \quad n = 0, 1, \dots, \quad (187)$$

which is a Stieltjes moment problem for $W(x) = \pi \frac{\tilde{W}(x)}{\mathcal{N}(x)}$.

Recently considerable progress was made in finding explicit solutions of Eq.(187) for a large set of $\rho(n)$'s, generalizing the conventional choice $|z\rangle_c$ for which $\rho_c(n) = n!$ with $\mathcal{N}_c(x) = e^x$ (see Refs. [KPS01][PS99][SPK01][SP00][Six01] and references therein), thereby extending the known families of coherent states. This progress was facilitated by the observation that when the moments form certain combinatorial sequences a solution of the associated Stieltjes moment problem may be obtained explicitly [PS01][BPS03a][BPS04].

In the following we make contact with the combinatorial sequences appearing in the solution of the boson normal ordering problem considered in Section 4.7.1. These sequences have the very desirable property of being moments of Stieltjes-type measures and so automatically fulfill the resolution of unity requirement which is a consequence of the Dobiński-type relations. It is therefore natural to use these sequences for the coherent states construction, thereby providing a link between the quantum states and the combinatorial structures.

6.2.2 Moment problem

We recall the definition of the generalized Stirling numbers $S_{r,1}(n, k)$ of Section 4.7.1, defined through ($n, r > 0$ integers):

$$[(a^\dagger)^r a]^n = (a^\dagger)^{n(r-1)} \sum_{k=1}^n S_{r,1}(n, k) (a^\dagger)^k a^k, \quad (188)$$

as well as generalized Bell numbers $B_{r,1}(n)$

$$B_{r,1}(n) = \sum_{k=1}^n S_{r,1}(n, k). \quad (189)$$

For both $S_{r,1}(n, k)$ and $B_{r,1}(n)$ exact and explicit formulas have been given in Section 4.7.1.

For series $B_{r,1}(n)$ a convenient infinite series representation may be given by the Dobiński-type relations (see Eqs.(106), (95) and (27))

$$B_{r,1}(n) = \frac{(r-1)^{n-1}}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(n + \frac{k+1}{r-1})}{\Gamma(1 + \frac{k+1}{r-1})}, \quad r > 1, \quad (190)$$

and

$$B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{191}$$

From Eqs.(188) and (189) one directly sees that $B_{r,1}(n)$ are integers.

Explicitly: $B_{1,1}(n) = 1, 2, 5, 15, 52, 203, \dots$;

$B_{2,1}(n) = 1, 3, 13, 73, 501, 4051 \dots$;

$B_{3,1}(n) = 1, 4, 25, 211, 2236, 28471, \dots$;

$B_{4,1}(n) = 1, 5, 41, 465, 6721, 117941 \dots$ etc.

It is essential for our purposes to observe that the integer $B_{r,1}(n+1)$, $n = 0, 1, \dots$ is the n -th moment of a positive function $W_{r,1}(x)$ on the positive half-axis. For $r = 1$ we see from Eq.(191) that $B_{1,1}(n+1)$ is the n -th moment of a discrete distribution $W_{1,1}(x)$ located at positive integers, a so-called *Dirac comb*:

$$B_{1,1}(n+1) = \int_0^{\infty} x^n \left[\frac{1}{e} \sum_{k=1}^{\infty} \frac{\delta(x-k)}{(k-1)!} \right] dx. \tag{192}$$

For every $r > 1$ a continuous distribution $W_{r,1}(x)$ will be obtained by excising $(r-1)^n \Gamma(n + \frac{k+1}{r-1})$ from Eq.(190), performing the inverse Mellin transform on it and inserting the result back in the sum of Eq.(190), see [KPS01][Six01] for details. (Note that $B_{r,1}(0) = \frac{e-1}{e}$, $r = 2, 3, \dots$ is no longer integral). In this way we obtain

$$B_{r,1}(n+1) = \int_0^{\infty} x^n W_{r,1}(x) dx, \tag{193}$$

which yields for $r = 2, 3, 4$:

$$W_{2,1}(x) = e^{-x-1} \sqrt{x} I_1(2\sqrt{x}), \tag{194}$$

$$W_{3,1}(x) = \frac{1}{2} \sqrt{\frac{x}{2}} e^{-\frac{x}{2}-1} \left(\frac{2}{\sqrt{\pi}} {}_0F_2\left(\frac{1}{2}, \frac{3}{2}; \frac{x}{8}\right) + \frac{x}{\sqrt{2}} {}_0F_2\left(\frac{3}{2}, 2; \frac{x}{8}\right) \right), \tag{195}$$

$$W_{4,1}(x) = \frac{1}{18\pi\Gamma(\frac{2}{3})} e^{-\frac{x}{3}-1} \left(3^{\frac{13}{6}} \Gamma^2\left(\frac{2}{3}\right) x^{\frac{1}{3}} {}_0F_3\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \frac{x}{81}\right) + 3^{\frac{4}{3}} \pi x^{\frac{2}{3}} {}_0F_3\left(\frac{2}{3}, \frac{4}{3}, \frac{5}{3}; \frac{x}{81}\right) + \pi \Gamma\left(\frac{2}{3}\right) x {}_0F_3\left(\frac{4}{3}, \frac{5}{3}, 2; \frac{x}{81}\right) \right). \tag{196}$$

In Eqs.(194), (196) and (196) $I_\nu(y)$ and ${}_0F_p(\dots; y)$ are modified Bessel and hypergeometric functions, respectively. Other $W_{r,1}(x) > 0$ for $r > 4$ can be generated by essentially the same procedure.

In Figure 2 we display the weight functions $W_{r,1}(x)$ for $r = 1 \dots 4$; all of them are normalized to one. In the inset the height of the vertical line at $x = k$ symbolizes the strength of the delta function $\delta(x - k)$, see Eq.(192). For further properties of $W_{1,1}(x)$ and more generally of $W_{r,r}(x)$ associated with Eq.(189), see [BPS04].

6.2.3 Construction and properties

A comparison of Eqs.(185), (186) and (193) indicates that the normalized states defined through $\rho(n) = B_{r,1}(n + 1)$ as

$$|z\rangle_r = \mathcal{N}_r^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{B_{r,1}(n+1)}} |n\rangle, \quad (197)$$

with normalization

$$\mathcal{N}_r(x) = \sum_{n=0}^{\infty} \frac{x^n}{B_{r,1}(n+1)} > 0, \quad (198)$$

automatically satisfy the resolution of unity condition of Eq.(184), since for $W_{r,1}(x) = \pi \frac{\tilde{W}_{r,1}(x)}{\mathcal{N}_r(x)}$:

$$\int_{\mathbb{C}} d^2 z |z\rangle_r \tilde{W}_{r,1}(|z|^2)_r \langle z| = I = \sum_{n=0}^{\infty} |n\rangle \langle n|. \quad (199)$$

Note that Eq.(197) is equivalent to Eq.(182) with the definition $[n]_r = B_{r,1}(n+1)/B_{r,1}(n)$, $n = 0, 1, 2, \dots$

Having satisfied the completeness condition with the functions $W_{r,1}(x)$, $r = 1, 2, \dots$ we now proceed to examine the quantum-optical fluctuation properties of the states $|z\rangle_r$. From now on we consider the $|n\rangle$'s to be eigenfunctions of the boson number operator $N = a^\dagger a$, i.e. $N|n\rangle = n|n\rangle$. The Mandel parameter [KPS01]

$$Q_r(x) = x \left(\frac{\mathcal{N}_r''(x)}{\mathcal{N}_r'(x)} - \frac{\mathcal{N}_r'(x)}{\mathcal{N}_r(x)} \right), \quad (200)$$

allows one to distinguish between the sub-Poissonian (antibunching effect, $Q_r < 0$) and super-Poissonian (bunching effect, $Q_r > 0$) statistics of the beam. In Figure 3 we display $Q_r(x)$ for $r = 1 \dots 4$. It can be seen that all the states $|z\rangle_r$ in question are super-Poissonian in nature, with the deviation from $Q_r = 0$, which characterizes the conventional coherent states, diminishing for r increasing.

In Figure 4 we show the behavior of

$$S_{Q,r}(z) = \frac{r\langle z|(\Delta Q)^2|z\rangle_r}{2}, \quad (201)$$

and

$$S_{P,r}(z) = \frac{r\langle z|(\Delta P)^2|z\rangle_r}{2}, \quad (202)$$

which are the measures of squeezing in the coordinate and momentum quadratures respectively. In the display we have chosen the section along $Re(z)$. All the states $|z\rangle_r$ are squeezed in the momentum P and dilated in the coordinate Q . The degree of squeezing and dilation diminishes with increasing r . By introducing the imaginary part in z the curves of $S_Q(z)$ and $S_P(z)$ smoothly transform into one another, with the identification $S_Q(i\alpha) = S_P(\alpha)$ and $S_P(i\alpha) = S_Q(\alpha)$ for any positive α .

In Figure 5 we show the signal-to-quantum noise ratio [Yue76] relative to $4\langle c|z|N|z\rangle_c = 4|z|^2$, its value in conventional coherent states; i.e the quantity $\bar{\sigma}_r = \sigma_r - 4\langle c|z|N|z\rangle_c$, where

$$\sigma_r = \frac{[r\langle z|Q|z\rangle_r]^2}{(\Delta Q)^2}, \quad (203)$$

with $(\Delta Q)^2 = {}_r\langle z|Q^2|r\rangle_r - ({}_r\langle z|Q|r\rangle_r)^2$. Again only the section $Re(z)$ is shown. We conclude from Figure 5 that the states $|z\rangle_r$ are more "noisy" than the standard coherent states with $\rho_c(n) = n!$.

In Figure 6 we give the metric factors

$$\omega_r(x) = \left[x \frac{\mathcal{N}_r'(x)}{\mathcal{N}_r(x)} \right]', \quad (204)$$

which describe the geometrical properties of embedding the surface of coherent states in Hilbert space, or equivalently a measure of a distortion of the complex plane induced by the coherent states [KPS01].

Here, as far as r is concerned, the state $|z\rangle_1$ appears to be most distant from the $|z\rangle_c$ coherent states for which $\omega_c = 1$.

6.2.4 Remarks

The use of sequences $B_{r,1}(n)$ to construct coherent states is not limited to the case exemplified by Eq.(197). In fact, any sequence of the form $B_{r,1}(n+p)$, $p = 0, 1, \dots$ will also define a set of coherent states, as then their respective weight functions will be $V_{r,1}^{(p)}(x) = x^{p-1}W_{r,1}(x) > 0$. Will the physical properties of coherent states defined with $\rho_p(n) = B_{r,1}(n+p)$ depend sensitively on p ? A case in point is that of $p = 0$ for which qualitative differences from Figure 3 ($p = 1$) appear. In Figure 7 we present the Mandel parameter for these states. Whereas for $r = 1$ the state is still super-Poissonian, for $r = 2, 3, 4$ one observes novel behavior, namely a crossover from sub- to super-Poissonian statistics for finite values of x . Also, as indicated in Figure 8, we note a cross-over between squeezing and dilating behavior for different r . These curious features merit further investigation.

6.3 Substitution theorem

In Chapter 5 we have demonstrated how to find the normally ordered form of the exponential of the operator linear in the annihilation or creation operator. This representation is of fundamental meaning in the coherent state representation used in physics.

As a demonstration of possible other utility of such formulas we show the direct proof of the *substitution theorem*. More specifically we will demonstrate that for any function $F(x)$ the following equality holds

$$e^{\lambda(q(X)D+v(X))}F(x) = G(\lambda, x) \cdot F(T(\lambda, x)) \quad (205)$$

where functions $T(\lambda, x)$ and $G(\lambda, x)$ may be found from the following equations

$$\frac{\partial T(\lambda, x)}{\partial \lambda} = q(T(\lambda, x)) , \quad T(0, x) = x , \quad (206)$$

$$\frac{\partial G(\lambda, x)}{\partial \lambda} = v(T(\lambda, x)) \cdot G(\lambda, x) , \quad G(0, x) = 1 . \quad (207)$$

First observe from Eq.(205) that the action of $e^{\lambda(q(X)D+v(X))}$ on a function $F(x)$ amounts to:

- a) change of argument $x \rightarrow T(\lambda, x)$ in $F(x)$ which is in fact a substitution;
- b) multiplication by a prefactor $G(\lambda, x)$ which we call a *prefunction*. We also see from Eq.(207) that $G(\lambda, x) = 1$ for $v(x) = 0$. Finally, note that $e^{\lambda(q(X)D+v(X))}$ with λ real generates an abelian, one-parameter group, implemented by Eq.(205); this gives the following group composition law for $T(\lambda, x)$ and $G(\lambda, x)$:

$$\begin{aligned} T(\lambda + \theta, x) &= T(\theta, T(\lambda, x)), \\ G(\lambda + \theta, x) &= G(\lambda, x) \cdot G(\theta, T(\lambda, x)). \end{aligned} \tag{208}$$

Now we proceed to the proof of Eqs.(205)-(207). Using the multiplication X and derivative D representation we may rewrite Eq.(151) as

$$e^{\lambda(q(X)D+v(X))} = : G(\lambda, X) \cdot e^{[T(\lambda, X)-X]D} : \tag{209}$$

with accompanying differential Eqs. (148) and (149) for functions $T(\lambda, x)$ and $G(\lambda, x)$ which are exactly the same as Eqs.(206) and (207).

Now, recalling the Taylor formula $e^{\alpha D}F(x) = F(x + \alpha)$ we may generalize it to

$$: e^{\alpha(X)D} : F(x) = F(x + \alpha(x)). \tag{210}$$

It can be seen from the power series expansion of the exponential. This allows to write

$$\begin{aligned} e^{\lambda(q(X)D+v(X))}F(x) &= : G(\lambda, X) \cdot e^{[T(\lambda, X)-X]D} : F(x) \\ &= G(\lambda, X) : e^{[T(\lambda, X)-X]D} : F(x) \\ &\stackrel{(210)}{=} G(\lambda, X) F(x + [T(\lambda, X) - x]) \\ &= G(\lambda, x) \cdot F(T(\lambda, x)). \end{aligned}$$

This completes the proof.

Here we list several examples illustrating Eqs.(205)-(207) for some choices of $q(x)$ and $v(x)$. Since Eqs.(206) and (207) are first order linear differential equations we shall simply write down their solutions without dwelling on details. First we treat the case of $v(x) = 0$, which implies $g(\lambda, x) \equiv 1$:

Ex.1:

$$q(x) = x, \quad T(\lambda, x) = xe^\lambda$$

which gives $\exp\left(\lambda x \frac{d}{dx}\right) F(x) = F(xe^\lambda)$, a well known illustration of the Euler dilation operator $\exp\left(\lambda x \frac{d}{dx}\right)$.

Ex.2:

$$q(x) = x^r, \quad r > 1, \quad T(\lambda, x) = \frac{x}{(1 - \lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}$$

The above examples were considered in the literature [DOTV97][Lan00][BPS03b][BPS03c].

We shall go on to examples with $v(x) \neq 0$ leading to nontrivial prefuctions:

Ex.3:

$$\begin{aligned} q(x) &= 1 \quad \text{and} \quad v(x) - \text{arbitrary,} \\ T(\lambda, x) &= x + \lambda, \\ g(\lambda, x) &= e^{\int_0^\lambda du v(x+u)}. \end{aligned}$$

Ex.4:

$$\begin{aligned} q(x) &= x, \quad \text{and} \quad v(x) = x^2 \\ T(\lambda, x) &= xe^\lambda, \\ g(\lambda, x) &= \exp\left[\frac{x^2}{2}(e^{2\lambda} - 1)\right]. \end{aligned}$$

Ex.5:

$$\begin{aligned} q(x) &= x^r \quad r > 1 \quad \text{and} \quad v(x) = x^s \\ T(\lambda, x) &= \frac{x}{(1 - \lambda(r-1)x^{r-1})^{\frac{1}{r-1}}} \\ g(\lambda, x) &= \exp\left[\frac{x^{s-r+1}}{1-r} \left(\frac{1}{(1 - \lambda(r-1)x^{r-1})^{\frac{s-r+1}{r-1}}} - 1\right)\right]. \end{aligned}$$

Closer inspection of the above examples (or at any other example which the reader may easily construct) indicates that from the analytical point of view the substitution theorem should be supplemented by some additional assumptions, like restrictions on λ . In general we can say that it can be valid only locally [DPS⁺04]. However, we note that a formulation in the language of formal power series does not require such detailed analysis and the resulting restrictions may be checked afterwards.

We finally note that the substitution theorem can be used to find the normally ordered form of an exponential operator linear in a or a^\dagger (as in Chapter 5). This idea has been exploited in [BHP⁺05].

7 Conclusions

In this work we have considered the boson normal ordering problem for powers and exponentials of two wide classes of operators. The first one consists of boson strings and more generally homogenous polynomials, while the second one treats operators linear in one of the creation or annihilation operators. We have used the methods of advanced combinatorial analysis to obtain a thorough understanding and efficient use of the proposed formalism. In all cases we provided closed form expressions, generating functions, recurrences etc. The analysis was based on the Dobiński-type relations and the umbral calculus methods. In general, the combinatorial analysis is shown to be an effective and flexible tool in this kind of problem.

We also provided a wealth of examples and pointed out possible applications. The advantages may be neatly seen from the coherent state perspective (*e.g.* we may use it for construction of the phase space pictures of quantum mechanics). We may also obtain solutions of the moment problem, enabling us to construct new families of generalized coherent states. Moreover application to the operator calculus is noted and exemplified in the substitution theorem. We also observe that the normal ordering problems for deformed algebras may be handled within that setting.

These few remarks on the possible applications and extensions of the methods used in this work (others may be found in the supplied references) yield a potentially wide field of possible future research.

We would like to point out some other interesting features. We believe that similar combinatorial methods may be used to find the

normally ordered forms of other classes of operators, such as the exponential of a general boson polynomial. Moreover, the extension to the multi-mode boson and fermion case should be susceptible to explicit analysis. This program would open a wide arena for immediate application to variety of physical models.

In conclusion, we would like to recall that the use of advanced combinatorics provides a wealth of interpretative tools for the problems under consideration. Combinatorial objects may be interpreted in terms of graphs, rook polynomials, partitions, correlations etc. We have not touched upon this aspect in this work but we consider it as a very promising one.

A Coherent states

Here we define and review some properties of coherent states [KS85][ZFR90].

We define coherent states $|z\rangle$ as the eigenstates of the annihilation operator

$$a|z\rangle = z|z\rangle, \quad (211)$$

where $z \in \mathbb{C}$ is a complex number. They can be written explicitly as

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (212)$$

These states are normalized but not orthogonal. The coherent states *overlapping factor* is

$$\langle z|z'\rangle = e^{z^*z' - \frac{1}{2}|z'|^2 - \frac{1}{2}|z|^2}, \quad (213)$$

The set of coherent states $\{|z\rangle : z \in \mathbb{C}\}$ constitute an overcomplete basis in the Hilbert space \mathcal{H} . The following *resolution of unity* property holds

$$\frac{1}{\pi} \int_{\mathbb{C}} |z\rangle\langle z| d^2z = 1. \quad (214)$$

Eq.(212) may be rewritten in the form (see Eqs.(6))

$$|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger} |0\rangle, \quad (215)$$

or with the use of Eq.(13) as

$$|z\rangle = e^{za^\dagger - z^*a} |0\rangle. \quad (216)$$

The operator $D(z) = e^{za^\dagger - z^*a}$ is called a *displacement operator* and therefore coherent states are sometimes called displaced vacuum. This group property may be taken as the definition of the coherent states also for other (than Heisenberg-Weyl) groups, see [Per86].

Let us define two self-adjoint operators Q and P by

$$\begin{aligned} Q &= \frac{a^\dagger + a}{\sqrt{2}}, \\ P &= i \frac{a^\dagger - a}{\sqrt{2}}. \end{aligned} \quad (217)$$

They satisfy the Heisenberg-Weyl commutation relation $[Q, P] = i$. These operators play various roles in quantum mechanical problems. They may be interpreted as the position and momentum operators for the quantum particle in the harmonic oscillator potential, in quantum optics they play the role of the field quadratures and are also used in the phase space formulation of quantum mechanics. It can be shown that the coherent states are the *minimum uncertainty states* for operators Q and P , *i.e.*

$$\Delta_{|z\rangle} Q \cdot \Delta_{|z\rangle} P = \frac{1}{2}, \quad (218)$$

where $\Delta_{\psi} A = \sqrt{\langle \psi | (A - \langle \psi | A | \psi \rangle)^2 | \psi \rangle}$ is the uncertainty of the operator A . Moreover they are the only states if one additionally imposes the condition $\Delta_{|z\rangle} Q = \Delta_{|z\rangle} P$ (otherwise the family of *squeezed states* is obtained) which serves as another possible definition of coherent states.

Finally we mention that the resolution of unity of Eq.(214) together with the continuity of the mapping $z \rightarrow |z\rangle$ are sometimes taken as the minimum requirements for the coherent states. However, this definition is not unique and leads to other families of so called *generalized coherent states* [Kla63a][Kla63b][KS85] (see also Section 6.2). We note that, due to their special features, coherent states are widely used in quantum optics [Gla63][KS68] as well as in other areas of physics [KS85].

In this text we especially exploit the property of Eq.(211). It is because for an operator $F(a, a^\dagger)$ which is in the normal form, $F(a, a^\dagger) \equiv \mathcal{N} [F(a, a^\dagger)] \equiv : F(a, a^\dagger) :$, its coherent state matrix elements may be readily written as

$$\langle z | F(a, a^\dagger) | z' \rangle = \langle z | z' \rangle F(z', z^*). \quad (219)$$

Also for the double dot operation it immediately yields

$$\langle z | : G(a, a^\dagger) : | z' \rangle = \langle z | z' \rangle G(z', z^*). \quad (220)$$

Unfortunately for the general operator none of these formulae hold. Nevertheless, there is a very useful property which is true; that is, if for an arbitrary operator $F(a, a^\dagger)$ we have

$$\langle z | F(a, a^\dagger) | z' \rangle = \langle z | z' \rangle G(z^*, z') \quad (221)$$

then the normally ordered form of $F(a, a^\dagger)$ is given by

$$\mathcal{N} [F(a, a^\dagger)] = : G(a^\dagger, a) : . \quad (222)$$

For other properties of coherent states we refer to [KS85][ZFR90].

B Formal power series. Umbral calculus

Here we recall the basic definitions and theorems concerning the formal power series calculus. As an illustration some topics of the umbral calculus are reviewed. For a detailed discussion see [Niv69], [Com74], [Rio84], [Wil94], [GKP94], [FS05], [Rom84], [Rai65], [RR78], [Buc98].

Formal power series

Suppose we are given a series of numbers $(f_n)_{n=0}^{\infty}$. We define a *formal power series* in indeterminate x as

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}. \quad (223)$$

The set of formal power series constitutes a ring when the following operations are imposed

- addition:

$$F(x) + G(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (f_n + g_n) \frac{x^n}{n!} \quad (224)$$

- multiplication (Cauchy product rule):

$$F(x) \cdot G(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} f_k g_{n-k} \frac{x^n}{n!} \quad (225)$$

Both operations have an inverses. The inverse of the series $F(x)$ with respect to addition is the series with negative coefficients $-F(x) = \sum_{n=0}^{\infty} -f_n \frac{x^n}{n!}$. The (unique) multiplicative inverse on the other hand is well defined only when $f_0 \neq 0$ and may be given recursively, see e.g. [Wil94].

One can also define the *substitution* of formal power series by

$$F(G(x)) = \sum_{n=0}^{\infty} f_n \frac{G(x)^n}{n!} \quad (226)$$

when $g_0 = 0$ in the expansion $G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$. Explicit expression for the coefficients is given by the Faà di Bruno formula [Com74][Ald01]. A series $F(x)$ may be shown to have a compositional inverse $F^{-1}(x)$ iff $f_0 = 0$ and $f_1 \neq 0$.

Other definitions and properties can be further given. Here is the example of the *derivative* and *integral* of the formal power series (note that these operators act like shift operators on the sequence $(f_n)_{n=0}^{\infty}$)

$$DF(x) = F'(x) = \sum_{n=1}^{\infty} f_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!} \quad (227)$$

$$\int F(x) dx = \sum_{n=0}^{\infty} f_n \frac{x^{n+1}}{(n+1)!} = \sum_{n=1}^{\infty} f_{n-1} \frac{x^n}{n!} \quad (228)$$

Observe that these definitions mirror the corresponding operations on analytic functions. Nevertheless the limiting operations are not needed in these definitions. The ring of formal power series can be also be given the structure of a complete metric space (see *e.g.* [FS05], [Rom84]).

We note that $(f_n)_{n=0}^{\infty}$ may be also a sequence of polynomials, functions or anything else.

The use of power series is well suited in the context of generating functions. The series as in Eq.(223) is called the *exponential generating function* (because of the factor $1/n!$ in the expansion). Functions $G(x) = \sum_{n=0}^{\infty} g_n x^n$ are called *ordinary generating functions*. The use of generating functions is especially useful in solving recurrences and enumerating combinatorial objects, see *eg.* [Wil94][GKP94][FS05].

All these definitions and properties may be naturally extended to the multivariable case.

For a systematic description of the formal power series and applications see [Niv69], [Com74], [Rio84], [Wil94], [GKP94], [FS05], [Rom84].

Umbral calculus. Sheffer-type polynomials

Subject of the umbral calculus is the study of a Sheffer A-type zero polynomials, called here briefly Sheffer-type [She39], [Rai65]. Without going into details we recall some basic theorems and definitions

[Rom84], [RR78], [Buc98].

Suppose we have a polynomial sequence $s_n(x)$, $n = 0, 1, 2, \dots$ ($s_n(x)$ being the polynomial of degree n). It is called of a *Sheffer-type* if it possesses an exponential generating function of the form

$$G(\lambda, x) = \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = A(\lambda)e^{xB(\lambda)}. \quad (229)$$

for some (possibly formal) functions $A(\lambda)$ and $B(\lambda)$ such that $B(0) = 0$, $B'(0) \neq 0$ and $A(0) \neq 0$. When $B(\lambda) = 1$ it is called *Apple sequence* for $A(\lambda)$. For $A(\lambda) = 1$ it is known as *associated sequence* for $B(\lambda)$.

Yet another definition of the Sheffer-type sequences can be given through their lowering and rising operators, *i.e.* the polynomial sequence $s_n(x)$ is of Sheffer-type iff there exist some functions $f(x)$ and $g(x)$ (possibly formal) such that $f(0) = 0$, $f'(0) \neq 0$ and $g(0) \neq 0$ which satisfy

$$f(D)s_n(x) = ns_{n-1}(x), \quad (230)$$

$$\left[X - \frac{g'(D)}{g(D)} \right] \frac{1}{f'(D)} s_n(x) = s_{n+1}(x). \quad (231)$$

These two definitions describe the Sheffer-type sequence uniquely and the correspondence is given by

$$A(x) = f^{-1}(x), \quad (232)$$

$$B(x) = \frac{1}{g(f^{-1}(x))}. \quad (233)$$

Many curious properties of the Sheffer-type polynomials can be worked out. We quote only one of them, called the *Sheffer identity*, to show the simplicity of formal manipulations. Using definition Eq.(229) we have

$$\begin{aligned} \sum_{n=0}^{\infty} s_n(x+y) \frac{\lambda^n}{n!} &= A(\lambda)e^{(x+y)B(\lambda)} = A(\lambda)e^{xB(\lambda)}e^{yB(\lambda)} \\ &= \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} \cdot \sum_{n=0}^{\infty} p_n(y) \frac{\lambda^n}{n!}. \end{aligned}$$

By the Cauchy product rule Eq.(225) we obtain the Sheffer identity:

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(y) s_{n-k}(x) \quad (234)$$

where $p_n(x)$ is the associated sequence for $B(\lambda)$. Observe that for associated sequences ($A(\lambda) = 1$) this property generalizes the binomial identity and therefore sometimes they are called of *binomial type*.

The Sheffer-type polynomials constitute a basis in the space of polynomials and through that property any formal power series can be developed in that basis also. On the other hand, formal power series can be treated as functionals on the space of polynomials. Investigation of this connection is also the subject of umbral calculus in which the Sheffer-type polynomials play a prominent role [Rom84].

We emphasize that all the functions and series here can be formal and then the operations are meant in the sense of the previous Section.

Thorough discussion of the umbral calculus, Sheffer-type polynomials and applications can be found in [Rom84][RR78][Buc98][Rai65].

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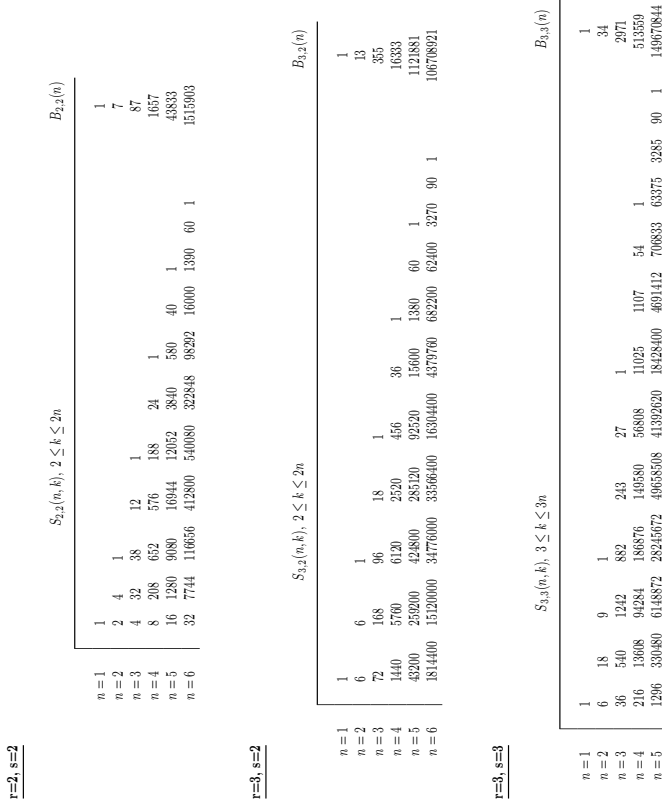


Figure 1: The weight functions $W_{r,1}(x)$, ($x = |z|^2$), in the resolution of unity for $r = 2, 3, 4$ (continuous curves) and for $r = 1$, a Dirac's comb (in the inset), as a function of x .

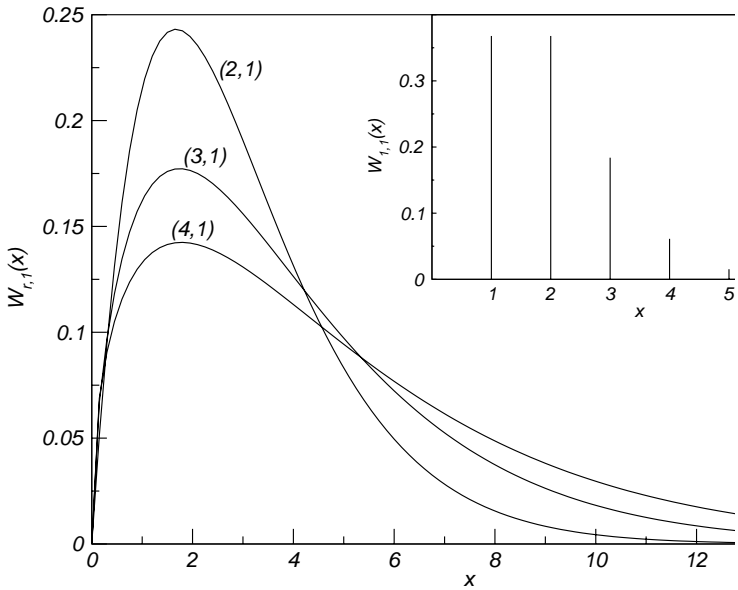


Figure 2: The weight functions $W_{r,1}(x)$, ($x = |z|^2$), in the resolution of unity for $r = 2, 3, 4$ (continuous curves) and for $r = 1$, a Dirac's comb (in the inset), as a function of x .

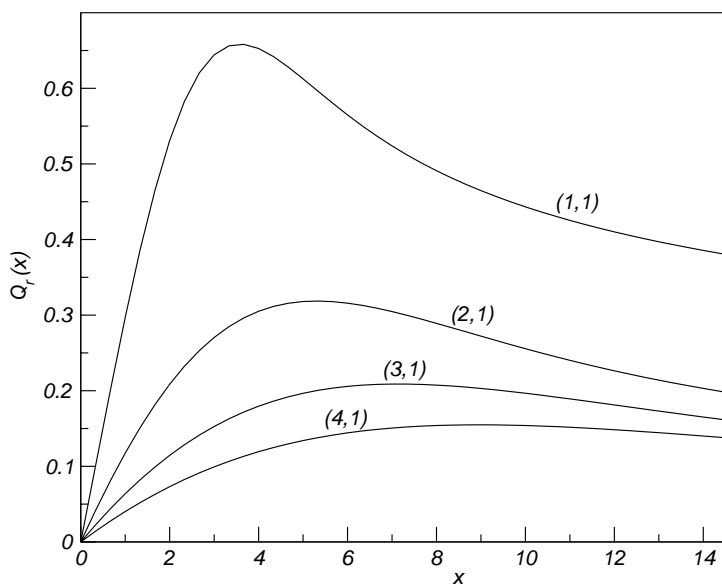


Figure 3: Mandel parameters $Q_r(x)$, for $r = 1 \dots 4$, as a function of $x = |z|^2$, see Eq.(200).

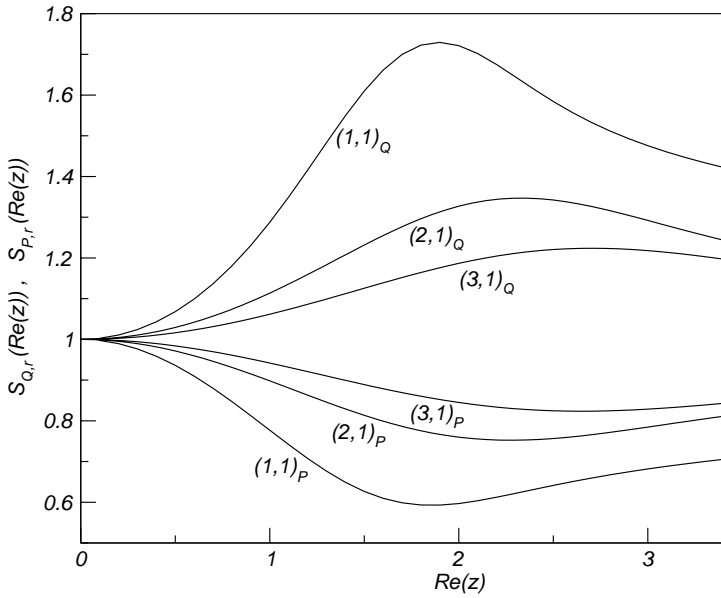


Figure 4: The squeezing parameters of Eqs.(201) and (202) for the coordinate Q (three upper curves) and for the momentum P (three lower curves) for different r , as a function of $Re(z)$, for $r = 1, 2, 3$.

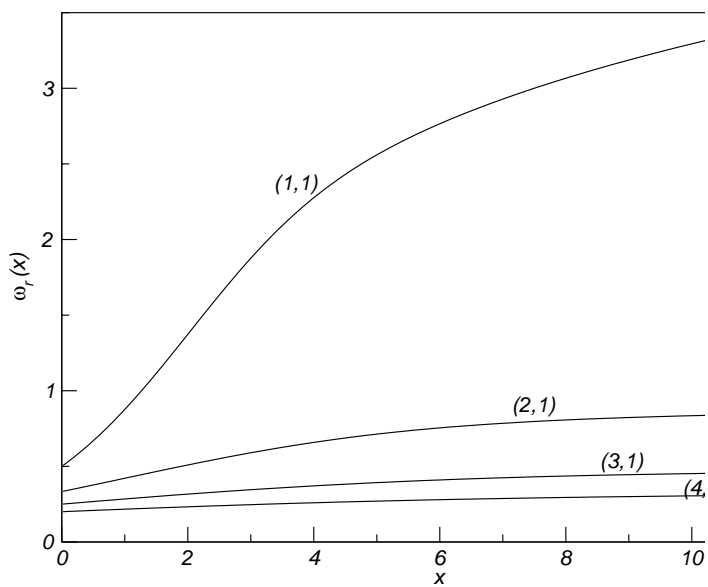


Figure 6: Metric factor $\omega_r(x)$, calculated with Eq.(204) as a function of $x = |z|^2$, for $r = 1 \dots 4$.

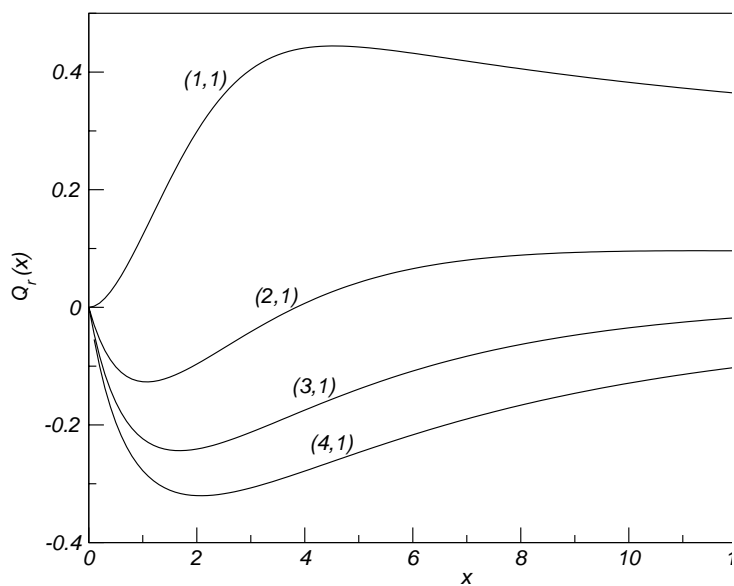


Figure 7: Mandel parameters $Q_r(x)$ for $r = 1 \dots 4$, as a function of $x = |z|^2$, for states with $\rho(n) = B_{r,1}(n)$.

**Comment on
COMBINATORICS OF BOSON NORMAL
ORDERING
AND SOME APPLICATIONS**

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The normal ordering of an operator function $F(a, a^\dagger)$, where a and a^\dagger are the usual creation and annihilation operators, consists in explicit moving all creation operators to the left by applying the commutation relation $[a, a^\dagger] = 1$. The resulting expression is called the normally ordered form of $F(a, a^\dagger)$ and denoted by $\mathcal{N}[F(a, a^\dagger)]$. Of course, the operators $F(a, a^\dagger)$ and $\mathcal{N}[F(a, a^\dagger)]$ are equal to each other, but in general they have different forms. It is clear that the normal ordering is a purely algebraic operation. For this reason, topics such as domains of operators in Hilbert space, or definitions of products of unbounded operators, *etc.*, lie outside the scope of this problem.

It is a trivial task to normally order simple operators F . On the other hand, such a task becomes extremely difficult when F is a polynomial of high order or an infinite series. In such cases even computer algebra systems do not help much. This problem is encountered in

many branches of theoretical physics, in particular in quantum field theory and atomic optics. P. Błasiak has solved analytically the problem of normal ordering for wide classes of the functions F .

The main results are presented in Chapters 4 and 5. Chapter 4 is devoted to the normal ordering of powers of so called homogeneous boson polynomials. Also exponential of such boson expressions is considered. The method used by P. Błasiak consists in deriving and solving recurrence relations for coefficients in $\mathcal{N}[F(a, a^\dagger)]$, which are called the generalized Stirling numbers (ordinary Stirling numbers were originally introduced in combinatorial analysis). This is done with the help of an algebraic realization of the operators a, a^\dagger in a space of polynomials. It is also shown that expectation values of such boson expressions in a coherent state $|z\rangle$ are given by generalized Bell polynomials in $|z|^2$. Furthermore, generating functions for the generalized Bell polynomials and the Stirling numbers are found. Chapter 4 is closed by several examples. Among them exponential of a generalized Kerr Hamiltonian is considered.

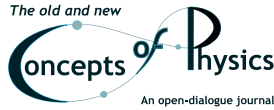
Another class of operators is considered in Chapter 5: boson operators of the form C^m and $\exp C$, where $C = q(a^\dagger)a + v(a^\dagger)$. Here q and v are arbitrary functions. Prior to P. Błasiak's work only the case $C = a^\dagger a + a^\dagger$ was considered in literature. He has tackled the normal ordering problem for such operators with the help of theory of Sheffer-type polynomials. I am impressed by the elegance as well as the effectiveness of the considerations presented in this Chapter, as well as in the preceding one.

Chapter 6 contains three very interesting applications of the results on the normal ordering problem. The first one shows that the methods worked out in Chapter 4 can be applied also to the operator $(A^\dagger A)^n$, where A and A^\dagger are deformed annihilation and creation operators, respectively. In particular, A and A^\dagger can be elements of the 3-dimensional Lie algebras $so(3)$ or $so(2, 1)$. The second application is a construction of new, interesting families of so called 'nonlinear' coherent states. Here the generalized Stirling and Bell numbers are utilized in order to produce explicit solutions of an ensuing moment problem. As the third application, the action of a generalized translation operator on a function is calculated. Hence, a direct proof of so called substitution theorem is obtained. The three applications illustrate very well the usefulness of P. Błasiak's results on the normal

Comment

ordering problems.

Even though the topic of the paper - the algebraic approach to the normal ordering of nontrivial operators - is a purely theoretical one, it has strong links with physical problems. This further increases the value of the results obtained by P. Błasiak, which are very interesting already from the purely theoretical point of view.



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Comment on COMBINATORICS OF BOSON NORMAL ORDERING AND SOME APPLICATIONS

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The paper by Paweł Błasiak concentrates around the "old" problem of quantum field theory, or rather the second - quantization scheme, i. e. ordering of bosonic creation and annihilation operators. The "classical" problem of normal ordering of the n -th power of the number operator $N = a^\dagger a$ serves as a guiding example to the more complicated tasks of ordering monomials containing arbitrary powers of creation and annihilation operators, powers of such monomials, powers of linear combinations of normally ordered monomials, and finally, powers of a linear function of the annihilation operator with coefficients being arbitrary functions of the creation operator.

In principle, investing enough time and patience should allow to reorder an arbitrary monomial in creation and annihilation operators to the normal form (*i.e.* with an appropriate power of the creation operator preceding the annihilation operator raised to a power), using

only the canonical commutation relation between a and a^\dagger . A practical way of reaching the goal is provided by the Wick theorem. Alas, even for rather moderately high powers and lengths of the monomials the task becomes too complicated even for computer algebraic manipulation programs. It is thus highly desirable to find manageable formulas or, at least some guiding principles leading to the solutions of normal ordering problems.

In order to attack the problem effectively, the author employs a representation of the bosonic creation and annihilation operators as, respectively, the multiplication and differentiation over a single variable fulfilling the same commutation relation as a^\dagger and a . If we want to map in this way real quantum mechanical problems we should probably be more specific in the space in which both multiplication and differentiation operator act (like in the so called Bargman representation). In the considered situation, however, analytic properties of the underlying function space are irrelevant - it is thus enough to consider the action of the operators on the formal power series in one variable.

Already the simplest problem, the above mentioned exemplary guiding task of normally ordering an arbitrary power of the number operator leads to some nontrivial and elegant purely combinatorial problems concerning Bell polynomials, Stirling numbers and their sums *i.e.* the Bell numbers, known from the combinatorial analysis. In his thesis Błasiak generalizes the above mentioned notions, derives relations among new constructed in this way mathematical objects - the most interesting are general Dobiński-type formulas expressing the generalized Bell polynomials in terms of the exponential function and an infinite series.

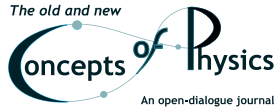
Of special interest are normal orderings of powers of monomials. Amenable formulas for arbitrary powers may be thus used to derive the normal ordering for the exponential function generalizing "classical" formula (36). Such results can find immediate application in obtaining explicit forms of (semi-) group operations for which generators are given in terms of creation and annihilation operators. Normal ordering allows immediate calculations of matrix elements in coherent states representations, reducing in this way operator problems to C -number ones. Interesting application of the methods employed in the thesis is presented in Section 6.3 in the form of the *substitution*

theorem evaluating explicitly the action of the exponential of the general linear differential operator with variable coefficients. The results can be easily obtained by considering a first order partial differential equation equating the derivative of a function over the parameter λ with the action on this function of the operator to be exponentiated. Solving the resulting equation *via* the method of characteristics gives the result equivalent to Eqs.(208) and (209) of the manuscript. It is however remarkable how easily and nicely the appropriate formulae are derived by employing methods developed in this approach.

The above discussed treatment of the normal ordering problem constitute the core of the thesis, comprising its Chapters 4 and 5 (Chapter 1 is an introduction, Chapter 2 contains preliminary definition and notational convention, Chapter 3 is devoted to the analysis of the guiding example of powers of the number operators). Additional interesting results are offered in Chapter 6, where previous considerations are extended to deformed boson algebras and coherent states associated with them. Some physical applications are also provided in Chapter 4, where the author discusses higher order generalizations of the fourth order Kerr Hamiltonian describing *e.g.* light propagation in a nonlinear medium. Indeed, quantum optics is the area where various results obtained in the thesis could find concrete applications, although this topic deserves definitely more discussion than it is offered in the approach.

The manuscript is clearly written, its construction is logical. It contains in the introductory chapters and appendices enough of a background material to be accessible even to non-specialist in the topic. (I could only complain that Appendix B, proudly entitled "Formal power series. Umbral calculus" does not really explain what the latter notion means). I did not find any important or worth mentioning faults or shortcomings.

As it should be clear from my discussion I found the results of it new, nontrivial and interesting. As such their provide a well founded basis for a very positive assessment Błasiak's work.



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**Comment on
COMBINATORICS OF BOSON NORMAL
ORDERING
AND SOME APPLICATIONS**

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This paper is devoted to certain mathematical problems of quantum mechanics, in particular to combinatorial problems arising in the theory of coherent states.

The main technical point is to find closed forms for normal ordered expressions of arbitrary products of bosonic creation and annihilation operators. The simplest cases of this problem were known to be related to classical combinatorial numbers, such as Bell and Stirling numbers. The present memoir introduces and studies in depth several infinite families of generalizations of these numbers. Among the physical applications, one finds explicit power series solutions of the Schrödinger equation for various (one dimensional) non linear problems.

The first part is very detailed, well written and accessible to non

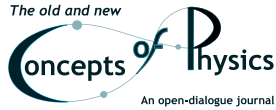
specialists. Chapters 1-3 contain a complete and useful discussion of the generic case of normal order of powers and exponential of the number operator, which illustrates how the combinatorial structures naturally appear in this context.

The presentation of original results begins in Chapter 4. Here, generalizations of Bell and Stirling numbers are obtained from the investigation of the normal ordering problem for arbitrary strings of one mode bosonic creation and annihilation operators a^\dagger and a . Recurrence relations, generating functions, and Dobiński-type formulas are obtained, and closed forms in terms of special functions are obtained for various particular cases. The results are then applied to generalizations of the Kerr hamiltonian. In Chapter 5, the normal ordering problem is solved for expressions of the form $(q(a^\dagger)a + v(a^\dagger))^n$, where q and v are arbitrary functions in one variable. This leads to an explicit solution of the evolution equation associated to the operator $H = q(a^\dagger)a + v(a^\dagger)$. This is achieved through the introduction of what is called the monomiality principle, according to the terminology introduced by the Frascati group (Dattoli *et al.*), which is investigated by means of the theory of Sheffer-type polynomials. The results are well illustrated by interesting examples.

Chapter 6 describes various applications of the previous results and of the techniques introduced so far. Several identities involving various kinds of deformed bosons are obtained, and families of coherent states built from the new combinatorial sequences previously obtained are presented.

The results presented above demonstrate that Blasiak has an excellent command of the combinatorial methods, which he applies with real ease to a number of well chosen physical examples. I should also stress that it is extremely enjoyable for a combinatorialist like myself to see such a development, an example of cross-fertilisation between two disciplines. I strongly urge the author of this work (and/or any other person) to pursue his investigations which I seriously consider to be very important.

This is a very interesting work containing many novel results with clear interdisciplinary flavor. It is pointing out unexpected connections between classical combinatorics and important computational problems of quantum mechanics.



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Comment on COMBINATORICS OF BOSON NORMAL ORDERING AND SOME APPLICATIONS

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This is a very interesting paper that applies techniques from combinatorics to quantum mechanics. There are deep connections between combinatorics and some of the quantum mechanical formulas which are related to the non-commutativity of the creation and annihilation operators. For example Bell and Stirling numbers are related to normal ordering of powers of the number operator. There is only a limited amount of work in this exciting and interdisciplinary area of research; and these results make an important, novel and timely contribution to it.

The first three chapters present a comprehensive review of existing work. Chapter 4 considers strings of powers of creation and annihilation operators and through normal ordering it introduces generalizations of the Bell and Stirling numbers. Chapter 5 studies the normal ordering of expressions of the type $(q(a^\dagger)a + v(a^\dagger))^n$. The results

Comment

are used to study exponentials of these expressions which are generalizations of the shift operators. Deformed bosons are considered in Chapter 6 and they lead to deformed versions of the usual Stirling numbers. The formalism is also used to introduce generalized coherent states. The author concludes in Chapter 7 with a discussion of his results. There are two appendices on coherent states and the umbral calculus.

The results are presented clearly and in good detail. The candidate demonstrates a deep knowledge of both combinatorics and quantum mechanical methods. He displays considerable physical insight and maturity in a difficult interdisciplinary area of research. I was very impressed by the quality and novelty of the work presented in the manuscript.