

OBSERVABLES, QUANTUM FLUCTUATIONS, AND “QUANTUM REALITY”

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Abstract

We discuss difference between the classical and quantum methods of description of Nature in terms of fundamental observables related to the dynamical symmetry group of the Hilbert space and quantum fluctuations, specifying their measurement. We show that the total amount of quantum fluctuations of fundamental observables can be used to measure the “remoteness” of quantum states from “classical reality” provided by the choice of observables in terms of c -numbers. The proposed picture is more general than Bell’s approach to definition of “classical reality”.

1 Introduction

In 1935, Einstein, Podolsky, and Rosen (EPR) proposed a thought experiment that appeared to demonstrate quantum mechanics to be an incomplete theory [1]. EPR believed the predictions of quantum mechanics to be correct, but only as the result of statistical distributions of *hidden* but real properties of the particles. Almost thirty years latter, Bell [2] subsequently formulated Bell's inequalities, which seemed to be a physically reasonable prove of nonexistence of hidden variables in quantum physics (also see Ref. [3]). In 1982, this fact was successfully demonstrated experimentally [4].

Immediately after the EPR paper [1], Schrödinger [5] recognized the informational aspect of EPR experiment connected with the non-locality of the EPR states. He also introduced the term *entanglement* to specify the peculiar quantum correlation between the parts of a quantum mechanical system in the EPR state. Recent discovery of quantum key distribution free of eavesdropping [6] and of quantum teleportation [7] have putted entanglement at the very heart of *quantum information*. At present, entanglement is considered to be a *physical resource*, like energy, for quantum communication, information processing, and computing. In particular, quantum systems in an entangled state can be used as a *quantum communication channel* to perform computational and cryptographic tasks that are impossible with classical systems [8].

Because of the simple mathematical structure provided by the Schmidt decomposition [9] (for recent review on Schmidt decomposition, see [10]), practically everything is known about the entanglement of two particles. In particular, the violation of Bell's conditions of "classical reality" can be used as a definition of entanglement for bipartite states. It is not the case for multipartite states, for which Bell's conditions can be violated by unentangled states as well.

An example is provided by the so-called *W*-state of three *qubits* [11] (three spin- $\frac{1}{2}$ particles)

$$|W\rangle = 3D \frac{1}{\sqrt{3}} (|1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle), \quad (1)$$

where $|\ell, \ell', \ell''\rangle = 3D|\ell\rangle \otimes |\ell'\rangle \otimes |\ell''\rangle$ and $\ell = 3D0, 1$ are the two possible sates of a single spin. The point is that any measure of

entanglement should be an *entanglement monotone* [12] and that the only entanglement monotone of three qubits is provided by *Cayley’s hyperdeterminant* [13], which is similar to *3-tangle* [14]. For the W -state (1), Cayley’s hyperdeterminant has zero value, so that this state does not manifest entanglement [13]. At the same time, W -state (1) violates Bell’s inequality [15].

The matter of Bell’s definition of “classical reality” can be expressed in the following way. Let S be a quantum mechanical system defined in the Hilbert space \mathbb{H}_S . Let \mathcal{O}_i ($i \in \mathbf{I}$) be an *observable* associated with the system S (the Hermitian operator acting in \mathbb{H}_S). According to the principles of quantum mechanics [16], an actual measurements of observable \mathcal{O}_i in the state $\psi \in \mathbb{H}_S$ results in a random quantity m_i characterized by a certain probability distribution. The subset of *commuting* observables \mathcal{O}_j ($j \in \mathbf{J} \subset \mathbf{I}$) has a joint probability distribution. Bell’s interpretation of “classical reality” consists in the assumption that all observables independent of whether they are commuting or not are specified by a *hidden* joint probability distribution.

Bell’s picture reflects Einstein’s idea of existence of hidden variables in quantum mechanics. Thus, violation of Bell’s “classical reality” means nothing but the absence of hidden variables in quantum mechanics. From the mathematical point of view, Bell’s notion of “classical reality” belongs to the *marginal problem*, which examines conditions to have probability density in a coordinate space with given projections onto the coordinate subspaces [17]. For more detailed discussion of connection between the marginal problem and Bell’s conditions, see Ref. [18].

Let us stress that any attempt to distinguish between the quantum and classical levels of description of physical “reality” should be based on the consideration of observables and their measurement. In fact, the principle difference between the quantum mechanics and classical physics consists in the definition of observables in terms of Hermitian operators in the former case and in terms of c -numbers in the latter case. The characteristic trait of quantum mechanics, following from this difference, is the existence of *quantum fluctuations* of observables. The quantum fluctuations can be measured directly (at least, in quantum optics [19]). They cause a number of important physical phenomena such as Lamb shift [20], quantum beats [21],

coherence of light [22], and squeezing [19, 22].

It has been shown recently that quantum fluctuations play important role in the understanding of quantum entanglement [23, 24, 25, 26, 27, 28, 29, 30, 31]. In particular, it has been justified that the *complete* (perfect) entangled states are defined to be *the manifestation of quantum fluctuations at their extreme* [28, 30, 31].

The aim of this paper is to discuss the difference between the quantum and classical descriptions of Nature in terms of quantum fluctuations. In particular, we show that the picture based on the consideration of quantum fluctuations gives a natural quantitative measure of *remoteness* of quantum states from “classical reality”.

The paper is organized as follows. In the second section, we define the *fundamental quantum observables*, depending on the symmetry properties of the Hilbert space. Then, in the third section, we discuss classification of states, depending on the amount of quantum fluctuations manifested by the fundamental observables. In the fourth section, we illustrate the concept of quantum fluctuations through the consideration of the single-particle entangled states. The summary is represented in the fifth section.

2 Fundamental quantum observables

Examining a given quantum mechanical system S , we should not think of all possible observables, but only of the *fundamental set* of observables, which can be specified by the symmetry properties of the corresponding Hilbert space \mathbb{H}_S [27]. The definition of fundamental observables, that we are going to discuss, traces back to famous Wigner’s approach to quantum mechanics [32] that has demonstrated an “unexpected efficiency” in quantum field theory and other brunches of quantum physics [33].

Let the Hilbert space \mathbb{H}_S of a given quantum system S be specified by the dynamical symmetry group G . Then, the set of fundamental observables $\{\mathcal{O}\}$ is determined by the basis of Lie algebra \mathcal{L} associated with the dynamical symmetry group G : $G = \exp(\mathcal{L})$ [27]. Since the observables \mathcal{O} should be represented by the Hermitian operators, sometimes the *complexified* Lie algebra $\mathcal{L}^c = \mathcal{L} \otimes \mathbb{C}$ should be used instead of \mathcal{L} .

For example, if the system S consists of a single spin- $\frac{1}{2}$ (qubit), the two-dimensional Hilbert space $\mathbb{H}_{\frac{1}{2}}$ has the dynamical symmetry

$SU(2)$. At the same time, the observables in this case are provided by the three Pauli operators, forming an infinitesimal representation of the $sl(2, \mathbb{C})$ algebra, which is known to be the complexification of the $su(2)$ algebra. In turn, the system of N qubits defined in the Hilbert space

$$\mathbb{H}_{N, \frac{1}{2}} = \bigotimes_{k=1}^N \mathbb{H}_{\frac{1}{2}}$$

has the dynamical symmetry

$$G = \prod_{j=1}^N SU(2)$$

and hence is specified by the $3N$ local observables, provided by the Pauli operators, acting in the subspaces $\mathbb{H}_{\frac{1}{2}}$ of $\mathbb{H}_{N, \frac{1}{2}}$.

Another example is provided by the Weil-Heisenberg algebra \mathcal{WH} represented by the canonical variables (observables) q and p

$$[q, p] = i.$$

In this case, the complexification \mathcal{WH}^c is provided by the annihilation and creation operators

$$a = \frac{q + ip}{\sqrt{2}}, \quad a^+ = \frac{q - ip}{\sqrt{2}}, \quad [a, a^+] = 1. \quad (2)$$

The importance of the use of the complexified algebra \mathcal{WH}^c is caused by the fact that the maximum symmetry (degeneracy) of the vacuum state is given just by the complexified group $G^c = \exp \mathcal{WH}^c$, while $G = \exp \mathcal{WH}$ is incapable of this [34].

In many cases, observables acting in the same Hilbert space, can be determined differently. For example, the creation and annihilation operators (2) can be used to describe photons. At the same time, the polarization properties of photons are described through the use of the symplectic subalgebras in \mathcal{WH}^c [35, 36, 37].

Another example is provided by the case of the three-dimensional Hilbert space \mathbb{H}_3 , where we can consider either the dynamical symmetry group $G = SU(3)$ or the spin group $G = SU(2)$. In the former

case, the set of fundamental observables is provided by the eight independent operators out of nine generators of the $su(3)$ algebra. In the latter case, the observables are given by the three spin-1 operators.

From the physical point of view, the choice of fundamental observables is determined by the measurements that we are going to perform over the system, or, what is the same, by the Hamiltonians, which are accessible for manipulation with quantum states.

3 Quantum fluctuations

The quantum fluctuation of an observable \mathcal{O}_i in a state $\psi \in \mathbb{H}_S$ is represented by the *variance*

$$V_i(\psi) = \langle \psi | \mathcal{O}_i^2 | \psi \rangle - \langle \psi | \mathcal{O}_i | \psi \rangle^2. \quad (3)$$

Physically (3) defines the uncertainty of measurement of \mathcal{O}_i in the state ψ . We choose the *total amount of quantum fluctuations*

$$\mathbb{V}(\psi) = \sum V_i(\psi), \quad (4)$$

where summation is taken over the whole set $\{\mathcal{O}\}$, to specify the *remoteness* of the state ψ from what is called the “classical reality” [28, 30]. In fact, for the classical observables represented by the c-numbers, the variance (3) is equal to zero for any state $\psi \in \mathbb{H}_S$.

The quantity (4) has a more deep physical meaning than just the total amount of quantum fluctuations. According to Wigner [39, 40] (also see Ref. [41]), (4) can be interpreted as the *skew information*, that is the measure of our knowledge with respect to the quantities that need *macroscopic* systems for their measurement.

All states of a quantum mechanical system have nonzero remoteness (4) with respect to the measurement of fundamental quantum observables. The two opposite extrema of remoteness define the important classes of quantum states.

Definition 1. The *coherent* states $\psi_{coh} \in \mathbb{H}_S$ are defined by the condition

$$\mathbb{V}(\psi_{coh}) = \min_{\psi \in \mathbb{H}_S} \mathbb{V}(\psi). \quad (5)$$

This condition, in fact, represents a *variational principle*, defining the coherent states as minimally remote states from the “classical

reality”. This coincides with the common opinion that coherent states provide almost classical description of physical systems [34, 42].

The definition (5) is compatible with the conventional definition of quantum coherence. Consider as an example of interest a system of N qubits (N spin- $\frac{1}{2}$ particles). If the base vectors of the two-dimensional spin- $\frac{1}{2}$ space of states $\mathbb{H}_{\frac{1}{2}}$ are denoted as $|\ell\rangle$, $\ell = 0, 1$, then an arbitrary pure state in $\mathbb{H}_{(N, \frac{1}{2})}$ takes the form

$$|\psi\rangle = \sum \psi_{\ell\ell'\dots\ell''} |\ell, \ell', \dots, \ell''\rangle, \quad (6)$$

where the coefficients $\psi_{\ell\ell'\dots\ell''}$ form the *multidimensional* matrix $[\psi]$. (Concerning the multidimensional matrices, see Ref. [43]). Here the normalization

$$\sum |\psi_{\ell\ell'\dots\ell''}|^2 = 1$$

is assumed.

The local measurements in each subspace $\mathbb{H}_{\frac{1}{2}}$ are represented by the Pauli operators, which in the given basis have the form

$$\begin{cases} \sigma_1 = & |0\rangle\langle 1| + |1\rangle\langle 0| \\ \sigma_2 = & i(|1\rangle\langle 0| - |0\rangle\langle 1|) \\ \sigma_3 = & |0\rangle\langle 0| - |1\rangle\langle 1| \end{cases} \quad (7)$$

In the case of a qubit, the $SU(2)$ coherent state in $\mathbb{H}_{\frac{1}{2}}$ is defined to be [44]

$$|\alpha\rangle = e^{\alpha J_+ - \alpha^* J_-} |1\rangle = \cos |\alpha| \times |1\rangle + e^{i\phi} \sin |\alpha| \times |0\rangle, \quad (8)$$

where $\alpha \in \mathbb{C}$, $J_{\pm} \equiv (\sigma_1 \pm i\sigma_2)/2$ and $\phi \equiv \arg \alpha$. Thus, the remoteness of this state in $\mathbb{H}_{\frac{1}{2}}$ is

$$\mathbb{V}(\alpha) = \sum_{k=1}^3 V_k(\alpha) = 2.$$

It is easily seen that this is the minimum value of \mathbb{V} for a single qubit.

The coherent state in $\mathbb{H}_{(N, \frac{1}{2})}$ can now be defined as the product of states (8). The corresponding remoteness then is

$$\mathbb{V}(\alpha_1, \alpha_2, \dots, \alpha_N) = N \times 2, \quad (9)$$

which gives the minimal remoteness of states of N qubits. Let us stress that $|\alpha_1, \dots, \alpha_N\rangle$ is the separable state in $\mathbb{H}_{N, \frac{1}{2}}$ by construction.

Definition 2. The completely entangled states $\psi_{ME} \in \mathbb{H}_S$ are defined by the condition

$$\mathbb{V}(\psi_{ME}) = \max_{\psi \in \mathbb{H}_S} \mathbb{V}(\psi). \quad (10)$$

This definition again represents a variational principle, defining the completely entangled states as the manifestation of the highest level of quantum fluctuations of fundamental observables.

It should be stressed that it is enough to define the complete entangled states. The point is that all entangled states of a given system are equivalent to within certain local transformations such as the Lorentz transformations [45, 46, 47] and SLOCC (stochastic local transformations assisted by classical communications) [11, 48, 49].

Corollary. If fundamental observables are represented by the generators of *compact* Lie algebra, the maximal remoteness is determined by the Casimir operator of this algebra.

By definition, the Casimir operator has the form

$$\hat{C} = \sum \mathcal{O}_i^2 = C \times \mathbf{1}, \quad (11)$$

where summation is again taken over the whole set of fundamental observables. Bearing in mind the definitions of variance (3) and remoteness (4), it is easily seen that the maximal remoteness takes the value

$$\mathbb{V}(\psi_{ME}) = \max_{\psi \in \mathbb{H}_S} \mathbb{V}(\psi) = C \quad (12)$$

under the following condition

$$\forall i \in \mathbf{I} \quad \langle \psi_{ME} | \mathcal{O}_i | \psi_{ME} \rangle = 0. \quad (13)$$

This condition (13) is equivalent to the variational principle (10) in the case of fundamental observables forming a representation of compact Lie algebra. This condition was proposed in [23, 24] as an *operational definition* (definition in terms of what can be observed) of maximum entanglement in the case of qubit systems.

Using the same example as above, we notice that the algebra of Pauli operators (7) is specified by the Casimir operator

$$\hat{C} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3 \times \mathbf{1}.$$

Therefore, the completely entangled state of N qubits of the form of (6) has the remoteness $\mathbb{V}_{max} = 3N$.

The remoteness (4) has different values for different classes of states. The complete classification of states of a given quantum system S is provided by the method of *geometric invariants* [13, 18]. Namely, different classes of states belong to the different invariant orbits in complex space. The latter are defined through the use of the G -invariant functions $\mathcal{I}(g\psi) = \mathcal{I}(\psi)$, where $g \in G$. From the physical point of view, the invariants correspond to the integrals of motion. Concerning the theory of geometric invariants and its applications in physics, see Ref. [50].

For example, the states of three qubits (state (6) with $N = 3$) form the following four classes:

Class 1: entangled states. The generic state is represented by the Greenberger-Horne-Zeilinger (GHZ) state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 1\rangle). \quad (14)$$

This class is specified by the maximal remoteness $\mathbb{V}(GZH) = 9$.

Class 2: nonseparable unentangled states. Typical example is provided by the W state (1). The remoteness has the value $\mathbb{V}(W) = 8 + 2/3$.

Class 3: biseparable states of the type of

$$|\psi_{BS}\rangle = \frac{1}{\sqrt{2}} \begin{cases} (|0, 0, 1\rangle + |0, 1, 0\rangle) \\ (|0, 0, 1\rangle + |1, 0, 0\rangle) \\ (|0, 1, 0\rangle + |1, 0, 0\rangle) \end{cases} \quad (15)$$

The remoteness is $\mathbb{V} = 8$.

Class 4: coherent (completely separated) states of the type of

$$|\psi_{coh}\rangle = |0, 0, 0\rangle. \quad (16)$$

In this case, the remoteness takes the minimal value $\mathbb{V}(\psi_{coh}) = 6$.

It should be noted that beginning with GHZ state (14) one can easily construct the basis of completely entangled states in $\mathbb{H}_{(3, \frac{1}{2})}$. The procedure has been described in [30].

4 States of a single “spin-1” particle

One more illustration of importance of consideration of the total amount of quantum fluctuations (4) is provided by a single “spin-1” object (a single qutrit in the $SU(2)$ sector). This means that the system is defined in the three-dimensional space of states \mathbb{H}_1 with the fundamental observables provided by the spin operators

$$\begin{cases} S_x &= \frac{1}{\sqrt{2}}(|+1\rangle\langle 0| + |0\rangle\langle -1| + H.c.) \\ S_y &= \frac{i}{\sqrt{2}}(|+1\rangle\langle 0| + |0\rangle\langle -1| - H.c.) \\ S_z &= |+1\rangle\langle +1| - |-1\rangle\langle -1| \end{cases} \quad (17)$$

such that $S_x^2 + S_y^2 + S_z^2 = 2 \times \mathbf{1}$. Here $|m\rangle$, $m = +1, 0, -1$, are the base vectors in \mathbb{H}_1 , specifying three projections of spin 1 on the quantization axis. A general pure state $\psi \in \mathbb{H}_1$ has the form

$$\begin{aligned} |\Psi\rangle &= x|+1\rangle + y|0\rangle + z|-1\rangle, \\ |x|^2 + |y|^2 + |z|^2 &= 1. \end{aligned} \quad (18)$$

Using the definition of coherent states (Definition 1 in Sec. III), it is a straightforward matter to show that the states $|\pm 1\rangle$ belong to the class of coherent states. Unexpectedly, the state $|0\rangle$ is not a coherent one. In fact, the remoteness (4) achieves the maximum value for $|0\rangle$, so that this is the completely entangled state.

To clarify this fact, consider a system of two qubits (two spin- $\frac{1}{2}$ particles). The Hilbert state of such a system can be decomposed as follows

$$\mathbb{H}_{\frac{1}{2}} \otimes \mathbb{H}_{\frac{1}{2}} = \mathbb{H}_1 \oplus \mathbb{H}_0, \quad (19)$$

where \mathbb{H}_1 and \mathbb{H}_0 are the Hilbert spaces of spin 1 and spin 0, respectively. If we denote the two spin- $\frac{1}{2}$ states by $|\uparrow\rangle$ and $|\downarrow\rangle$, it can be seen that the spin-1 states from \mathbb{H}_1 formally correspond to the symmetric combinations

$$\begin{cases} |+1\rangle &\sim |\uparrow\uparrow\rangle \\ |0\rangle &\sim (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \\ |-1\rangle &\sim |\downarrow\downarrow\rangle \end{cases} \quad (20)$$

In turn, the space \mathbb{H}_0 is equivalent to the antisymmetric state

$$|A\rangle \sim \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (21)$$

According to Eq. (19), the space of states of a single “spin”-1 particle formally corresponds to the space of states of two qubits with discarded antisymmetric base state (21). In particular, this means that the state $|0\rangle$ in (20) is really the completely entangled state. The qubits in this picture can correspond to the intrinsic degrees of freedom of the particle.

An example is provided by a single π^0 meson. According to the modern standard model of particle physics [51], pions are composed of the first generation quarks, namely, up u , anti-up \bar{u} , down d , and anti-down \bar{d} quarks. Using the language of quantum information, it is possible to say that the isodoublets of quarks form two qubits. The states of π^\pm mesons

$$\pi^+ = u\bar{d}, \quad \pi^- = \bar{u}d$$

correspond to unentangled (coherent) combinations, while the state of π^0 meson

$$\pi^0 = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}$$

corresponds to the maximum entangled state of the type of $|0\rangle$ in (20).

Taking into account that the completely entangled state corresponds to the maximum amount of quantum fluctuations, one can conclude that π^0 meson should be less stable than π^\pm mesons. In fact, the lifetime of π^0 is 10^{-9} of the lifetime of π^\pm .

There is a strong similarity between the triplet of pions and a certain phase of quantum Fermi liquid, namely, the so-called A_1 -phase in superfluid ${}^3\text{He}$, where the Cooper pairs are represented by the spin-triplet configuration ($s = 1$) of two particles [52]. It should be stressed that the notion of nonlocality is practically meaningless in the case of quantum liquids because of the strong overlap of the wave functions of individual atoms [53].

Thus, the state of a “spin-1” object with projection $m = 0$ is maximum entangled, i.e. manifests the maximal remoteness from “classical reality”. A simplest example of a single “spin-1” object that can be prepared in the state $|0\rangle$ is provided by the orbital angular momentum of a photon, belonging to the paraxial beam, which is

described by means of Laguerre-Gaussian modes in cylindrical geometry [54]. The orbital angular momentum of photons can be measured and is widely discussed in the context of quantum information (see [55] and references therein). It should be emphasized that, in the usual treatment, entanglement of two or more photons with respect to the orbital angular momentum is considered [55].

It is also possible to consider the single-photon completely entangled state with respect to the total angular momentum in spherical geometry (photon is emitted by a point-like source like atom) [56]. An electric dipole photon with the total angular momentum $J = 1$ and parity $P = 1$, emitted by the atomic electric-type transition

$$|J = 1, m = 0\rangle \rightarrow |J' = 0, m' = 0\rangle$$

between the two atomic states with given angular momentum and its projection [57], is prepared in the single-particle maximum entangled state.

In this case, it is easy to trace the similarity with the two-qubit structure, because $\vec{J} = \vec{S} + \vec{L}$, where \vec{S} and \vec{L} represent the spin and orbital parts of the total angular momentum. Since photon is a massless particle, it has only two spin states (helicities) associated with the polarization. Another qubit corresponds to the orbital angular momentum, which for an electric dipole photon has only two allowed values $L = J \pm 1 = 2, 0$ [58].

This fact that the state $m = |0\rangle$ of the electric dipole photon is completely entanglement can be clarified in the following way. The decay of such a photon with creation of electron-positron pair should lead to the EPR state of charged particles with respect to their spin. It can be verified in the presence of strong electric field, which separates the charged particles but, unlike the magnetic field, does not influence their spin states [59].

Since the fundamental observables (17) correspond to the compact Lie algebra, the condition (13) can be used to determine all completely entangled states of the “spin-1” object under consideration. It can be easily seen that there are infinitely many different maximum entangled states in this case (the state $|0\rangle$ among them).

5 Summary and conclusions

In this paper we have discussed an approach to definition of “quantum reality”. According to this approach, the key point in the understanding of difference between the classical and quantum levels of description of “reality” consists in the specification of observables and their measurements. In the former case, the observables are represented by c-numbers, and their measurement is free from quantum fluctuations. In the latter case, the observables are chosen to be the Hermitian operators, and the specific trait of quantum mechanical measurements is the existence of quantum fluctuations.

To specify quantitatively the remoteness of a given quantum state $\psi \in \mathbb{H}_S$, we need the fundamental set of observables, acting in \mathbb{H}_S and defined by the dynamical symmetry properties of \mathbb{H}_S . Namely, if G is the dynamical symmetry group of the Hilbert space \mathbb{H}_S , then the fundamental observables are associated either with the Lie algebra \mathcal{L} such that $G = \exp \mathcal{L}$ or with the complexified Lie algebra \mathcal{L}^c : $\exp \mathcal{L}^c = G^c$, where G^c denotes the complexification of G .

The total amount of quantum fluctuations $\mathbb{V}(\psi)$ (4) of fundamental observables in a given quantum mechanical state $\psi \in \mathbb{H}_S$ can be used to quantify the remoteness of this state from “classical reality”. All quantum states have nonzero remoteness. The minimum value of remoteness defines the coherent states (5), while the maximal remoteness corresponds to the maximum entangled states (10). The classification of states in terms of the geometrical invariants can also be interpreted in terms of different values of remoteness.

This physical quantity (4) is similar to the so-called skew information has been introduced by Wigner to distinguish between the knowledge coming from the microscopic and macroscopic measurements [39, 40]. The point is that the observables that commute with the additive conservative quantities like energy can be measured with the microscopic apparatuses, while the observables that do not commute with conservative quantities need for their measurement macroscopic systems. The quantum mechanical definition of information has the form [16]

$$I = Tr(\rho \ln \rho), \quad (22)$$

where ρ is the density matrix. The information (22) is always negative, except the case of pure states $\rho = |\psi\rangle\langle\psi|$. In the latter case,

$I = 0$. Thus, from the informational point of view, we cannot distinguish between different pure states. In the case of *composite systems*, the picture can be improved through the use of the reduced entropy [2, 3], which sometimes is called the entropy of formation [14]. This way does not work in the case when we deal with the states of a single or local system. At the same time, the remoteness (4) (or skew information) permits us to distinguish between the information carried by different quantum mechanical states.

The definition of “quantum reality” based on the consideration of quantum fluctuations of fundamental observables is more general than Bell’s conditions. In particular, the former definition permits to distinguish between the entangled and unentangled states of multipartite systems, while the latter one is capable of this only for the bipartite systems.

An undoubted advantage of the above discussed approach is that it can be applied to the systems of different physical nature independent of whether they are local or not.

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