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ALTERNATIVE HAMILTONIANS OF CLASSICAL MECHANICS AND NONCANONICAL QUANTIZATIONS

Andrzej Horzela
H.Niewodniczański Institute of Nuclear Physics,
Polish Academy of Sciences,
ul. Eliasza-Radzikowskiego 152, PL 31342 Krakow, Poland
e-mail: Andrzej.Horzela@ifj.edu.pl

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Abstract

Noncanonical quantization schemes have been proposed in quantum physics long time ago. Recalling some of them I associate these investigations to the quantum mechanical implications of the canonically inequivalent Hamilton formulations of classical Newtonian dynamics. As the simplest illustration of such *alternative* Hamiltonian descriptions I consider one dimensional system of two particles interacting harmonically. Description of the motion of their center of mass is standard and guarantees the Galilean covariance but the internal motion is described as a noncanonical oscillator for which the Hamiltonian and the total energy are not related by a simple proportionality. Alternative Hamiltonians lead to alternative Poisson structures which may be considered as classical analogues of the noncanonical quantum commutation rules. Gen-

erated *alternative* quantizations are consistent with the same Heisenberg equations of motion, *i.e.*, they satisfy the Wigner principle of quantization. For the one dimensional harmonic oscillator the simplest noncanonical algebraic description is given in terms of the Lie algebra $so(p,q), p+q=3$. In such a case the Hamiltonian is still a linear function while the energy is a quadratic polynomial of the main quantum number. This leads to nonstandard termodynamical properties obeyed by such a noncanonical oscillator. The 3-dimensional harmonic oscillator also has a noncanonical description given in terms of the simple Lie algebra, $so(p,q), p+q=5$ in the case under consideration. The most important physical consequence of such a description is the noncommutativity of the coordinate operators, interpreted to be an implication of the noncommutative geometry of the underlying spacetime.

1 Introduction

It is well known that any quantization, *i.e.,* a prescription how to construct a quantum theory having known its classical analogue, cannot be given unambigously. Obviously, we can ask whether such a prescription is necessary because this is the classical physics which should emerge from the quantum theory in its physically well-defined limits and not the opposite. But constructions of quantum models without references to their classical analogues are very difficult, if possible at all with our present knowledge. This is known from the early days of quantum mechanics and because of this fact the *correspondence principle* still remains an epistemological basis of the quantum theory. The correspondence means that the classical description is a shadow of the quantum one - any quantum theory has to have a classical limit, there always exists a quantum observable corresponding to a classical one and the structure of noncommutative multiplication of quantum observables is mirrored in the structure of the Poisson brackets of the classical phase-space description.

The rules of the *correspondence principle* are general enough to provide us quantization recipes leading to widely applicable and effective calculation schemes which give us surprisingly good descriptions of the vast majority of the microworld physical phenomena. Nevertheless all the quantizations procedures need further assumptions, sometimes quite arbitrary both from physical, as well as from mathematical point of view, to be added in order to achieve unique formulation. Rules of the canonical quantization, taken as a straightforward analogy to the Poisson brackets, are the prescription valid in the Cartesian coordinates only. The quantization of the system in non-Cartesian coordinates replaced by operators satisfying the canonical rules does not reproduce results got in the Cartesian coordinates and each time a quantization of the classical expression written down in the non-Cartesian coordinates has to be done carefully. Another frequently quoted illustration of the quantization ambiguities is the "ordering problem". When one asks how to order noncommuting operators $\hat{q}$ and $\hat{p}$ in an operator product which should be an analogue of a classical monomial $q^n p^m$ there one obviously can not give an unigue answer. Various orderings lead to various quantizations and, in general, to different physical implications [1].

But the "ordering problem" is neither the only one nor the most important reason why such ambiguities do appear. In the following I am going to convince the reader that asking for a *correspondence* we should ask which classical phase-space description, or even which formulation of the classical mechanics, should be addressed as a reference. I shall show that the problem is neither an artificial question if considered in its physical context nor a mere mathematical curiosity. Its studies and efforts to find a solution are related to questions which touch fundamental concepts of the classical physics and find implications in the nature of the spacetime geometry, its commutativity or noncommutativity, reflected in its continuous or possibly discrete structure.

2 Towards noncanonical quantizations - a brief recollection

2.1 The Wigner idea

It was E. Wigner who noticed that the problem how to choose a quantization has its source in the fundamentals of quantum physics. More, he showed that the canonical quantization is a quantization scheme which arises as an arbitrarily chosen possibility. It is not, as often claimed, justified by fundamental reasons but only it is suggested by its analogy to the standard Hamilton's formalism of classical mechanics. Wigner's seminal paper on this topic, [2], published in 1950, is entitled "Do the equations of motion determine the canonical commutation relations?". The paper is shorter than two printed pages but for the author this was enough room to prove that the title question has to be answered negatively. Taking as fundamental assumptions that *...equations of motion in Ehrenfest picture have a more immediate physical significance than Heisenberg - Born - Jordan canonical commutation relations ...* ,[3], and *...purpose of the above - mentioned considerations is to avoid using Hamiltonian theory...* Wigner followed ideas emerging from the Ehrenfest theorem. He abandoned the usual rules of the canonical quantization - encapsulated by P.A.M. Dirac in the recipe *...replace Poisson brackets by commutator brackets...*,[4] - as an *a priori* given and the only one unquestionable scheme which admits to quantize physical systems. Instead, Wigner investigated a physical justification of commutation rules, which, if existed, would result from equations of motion written down in terms of the Newtonian variables. Wigner's test model was the one dimensional harmonic oscillator of mass m and frequency ω studied in terms of the energy, coordinate and velocity operators in order to avoid notions coming from Hamilton's formalism. Here we emphasize that Wigner identified the time evolution generator, *i.e.*, the Hamiltonian, with the total energy given as a sum of the kinetic and potential energies. Assuming that the Heisenberg equations of motion have the form of Hamilton's equations

$$\begin{aligned} v &= \dot{x} = -\frac{i}{\hbar}\left[\; x, \frac{m}{2}\left(v^2+\omega^2x^2\right)\;\right], \\ -\omega^2 x &= \dot{v} = -\frac{i}{\hbar}\left[\; v, \frac{m}{2}\left(v^2+\omega^2x^2\right)\;\right], \end{aligned} \tag{1}$$

Wigner solved them as operator equations and showed that besides of the canonical commutation relation satisfied by the coordinate and velocity operators there exists also also a more general solution for which the commutator satisfies the relation

$$\left(i+\frac{m}{\hbar}\left[\; v, x\;\right]\right)^2 = -\left(\frac{2E_0}{\hbar\omega}-1\right)^2, \tag{2}$$

with an arbitrary parameter E_0. Obviously (2) includes the canonical case ($E_0 = \hbar\omega/2$) but in general it provides us an example of a noncanonical commutation relation or, as we are used to say now, it gives an alternative quantization scheme, [5].

The Wigner generalization of the commutation relations, as well as those which have been proposed later, becomes easier to see if we introduce ladder operators

$$a^- = \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{i}{\omega}v\right), \qquad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{i}{\omega}v\right), \tag{3}$$

and rewrite the oscillator equations of motion (1) as triple relations

$$[a^-, a^+a^- + a^-a^+] = 2a^-, \qquad [a^+, a^+a^- + a^-a^+] = -2a^+. \tag{4}$$

These lead to the identity

$$a^+\left(1+[\; a^+, a^-\;]\right) = -\tfrac{1}{2}\,[[\; a^+, a^-\;], a^+] \tag{5}$$

which the canonical a^- and a^+ obviously satisfy. But this is not the only one solution. In fact, representing the ladder operators as infinite matrices with matrix elements

$$a^-_{kl} = \alpha_{k+1}\delta_{k,l-1}, \qquad\qquad a^+_{kl} = \alpha^*_k\delta_{k-1,l} \tag{6}$$

one finds the general solution of (5)

$$\begin{array}{ll} \alpha_{2k} = \sqrt{2k} & k = 1, 2, 3, ..., \\ \alpha_{2k+1} = \sqrt{2k+1+\theta} & k = 0, 1, 2, 3, ..., \end{array} \tag{7}$$

depending on an arbitrary parameter θ. Any choice $\theta \neq 0$ means that we deal with a noncanonical scheme of quantization.

Relations (4) may be completed to

$$\left[\{a^\xi, a^\eta\}, a^\epsilon\right] = (\epsilon - \xi)\, a^\eta + (\epsilon - \eta)\, a^\xi, \tag{8}$$

with $\xi, \eta, \epsilon = -, +$ or ± 1. These, if generalized to the multimode case and to the field theory, are known as the Green ansatz and form the algebraic basis of the parastatistics approach, [6], [7]. The latter is a noncanonical quantization scheme within which generators of the spacetime translations have their canonical shape but ladder operators satisfy multiplication rules more general than usual creation and annihilation operators do. As a consequence the standard algebraic structures with their dichotomous choice between commutation or anticommutation relations may be generalized to the structures described in terms of orthosymplectic Lie superalgebras, [8]. Representations of such algebras were found to describe nonrelativistic quantum systems, called the *Wigner quantum systems* and defined, [9], as quantum systems for which

w1) the Hamiltonian is identified with the total energy,

w2) the state spaces W are Hilbert spaces and observables are selfadjoint operators in W,

w3) the Heisenberg equations coincide with the Hamilton equations (as operator equations in W),

w4) the description is Galilean covariant .

The only standard requirement which is dropped in the definition above is the canonical form of the fundamental commutation relations. This changes a lot. One can show that the *Wigner quantum* *systems* exhibit unusual physical properties even for the simple harmonic interaction, [10] - [12]. The list of such unusual properties includes:

i) finite and discrete spectra of the coordinate and energy operators,

ii) noncommutative geometry of the internal configuration space,

iii) appearance of nonstandard uncertainty relations,

iv) behavior with respect to noncanonical, so called exclusion, statistics characterized by a number p, called the order of the statistics, defined by $aa^\dagger|0\rangle = p|0\rangle$ or $p = \sqrt{1+\theta}$ in the notation of (7).
Recently, the concepts of generalized statistics emerging from the parastatistics have found an interesting and physically promising application. This is the theory of particle-like, fractional spin and statistics objects called anyons, [13], investigated nowadays as relevant to the physics of quantum Hall effect and the high temperature superconductivity, [14]. Both these phenomena are still not satisfactorily understood and are supposed to be described by effectively planar models which algebraic structure is noncanonical, [15], and the underlying geometry is noncommutative, [16].

2.2 Deformed commutators

The commutation relation (2), or the commutation relations got for the ladder operators with (7) used, provide us the examples of, as called today, *deformed commutation relations*. Such relations do not share simple analogies characteristic for the canonical formalisms of classical and quantum physics. Moreover, besides of the Planck constant $\hbar$, they depend on other parameters which physical interpretation is to be specified for each case under consideration. It is clear that such modified description of the physical systems influences both the mathematical, as well as the physical content of the noncanonically quantized models. Even for the harmonic oscillator the only modification of the canonical rule $\{x, v\} = 1/m$ allowed by the Jacobi identities within an associative dynamical algebra built of x, v and H, namely $\{x, v\} \propto f(H)$ with f being an arbitrary function, contradicts the equations of motion if one puts into it the standard expression $H = \frac{m}{2}\left(v^2 + \omega^2 x^2\right)$. So, when we have decided to study *deformed commutation relations* we must agree to abandon some properties of the standard approach, among them either basic properties of the algebraic structures under consideration or an interpretation of the physical quantities chosen in order to describe the system.

The best known example of the *deformed commutation relations* are those related to the harmonic oscillator dynamics, [17] - [20], named also *deformed bosons*, [21], [22], if the Fock space description is used as a notion of the primary meaning. For the latter we define the ladder operators a and $a^\dagger$ (analogous to a^- and a^+ of (3), respectively) through their commutation relations with a self-adjoint, semibounded operator $\hat{N}$ which spectrum consists of nonnegative integers, $\hat{N}|n\rangle = n|n\rangle$,

$$\left[a, \hat{N}\right] = a, \qquad \left[a^\dagger, \hat{N}\right] = -a^\dagger. \tag{9}$$

The general form how a and $a^\dagger$ act on the eigenvectors of $\hat{N}$ reads

$$a|n\rangle = \sqrt{[n]}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle, \tag{10}$$

where a conventionally used symbol $[n]$ denotes an arbitrary function of n. This implies

$$\left[a, a^\dagger\right]|n\rangle = \big([n+1]-[n]\big)|n\rangle = \big([\hat{N}+1]-[\hat{N}]\big)|n\rangle. \tag{11}$$

As $[\hat{N}]$ is not restricted to be a linear function of $\hat{N}$ then (11) can be noncanonical and (9) and (11) together may form an algebraic structure which, in contradistinction to the canonical case, is not a Lie algebra at all. Such examples are provided by the so-called q-deformations of the basic commutation relations for which

$$\begin{aligned} [n] &= \frac{1-q^n}{1-q} && \text{'math' } \quad q-\text{case}, \\ [n] &= \frac{q^n - q^{-n}}{q-q^{-1}} && \text{'physics' } \quad q-\text{case}, \end{aligned} \tag{12}$$

associated to the simplest realizations of the Hopf algebras. Another choice,

$$[n] = n + \frac{p}{2}\left(1+(-1)^{n+1}\right), \tag{13}$$

transforms (9) and (11) into the single mode paraboson algebra with a parameter p giving the order of the (generalized) statistics, or, equivalently, to the R-deformed Heisenberg algebra. The latter is known to describe the two particle Calogero model, [16], which, however, is not the only one interpretation of (9) - (11) in terms of the nonlinear systems. Equally, the problem may be formulated as the problem of the oscillator which frequency depends on the energy. Such oscillators have been named nonlinear f-oscillators, [23] - [26]. Keeping the notation of these references the basic properties of the f-oscillators are seen when one notices that the equations (9) remain invariant when the canonical a and $a^\dagger$ are transformed according to

$$a \to A = af(\hat{N}), \qquad a^\dagger \to A^\dagger = f^*(\hat{N})a^\dagger, \tag{14}$$

with an arbitrary function f. According to (9) the operators A and $A^\dagger$ act in the usual Fock space as lowering and raising operators but they are noncanonical because

$$[A, A^\dagger] = |f(\hat{N}+1)|^2(\hat{N}+1) - |f(\hat{N})|^2\hat{N}, \tag{15}$$

as requested by (11). Because of (9) it is also that both $AA^\dagger$ and $A^\dagger A$ commute with $\hat{N}$. This means that if we interpret $\hat{N}$ as a number (or as an excitation number) operator then operators of such physically meaningful quantities like the energy and the time evolution generator (which are to be commeasurable with $\hat{N}$ in the nonrelativistic physics) should be expressable in terms of $AA^\dagger$ and $A^\dagger A$. Asking for a dynamics of the A and $A^\dagger$ we may begin with the classical expression for the energy given in terms of the kinematical variables. Such an analogy suggests that the energy operator E is given by

$$\begin{aligned} E &= \frac{\hbar\omega}{2}(AA^\dagger + A^\dagger A) \\ &= \frac{\hbar\omega}{2}\left(|f(\hat{N}+1)|^2(\hat{N}+1) + |f(\hat{N})|^2\hat{N}\right). \end{aligned} \tag{16}$$

Next, if we assume that the generator of the time evolution H (which defines the dynamics) is identical with the energy then the equations of motion for A and $A^\dagger$ become

$$[A, H] = \omega(\hat{N})A, \qquad \left[A^\dagger, H\right] = -\omega(\hat{N})A^\dagger, \tag{17}$$

where

$$\omega(\hat{N}) = H(\hat{N}+1) - H(\hat{N}). \tag{18}$$

The above allows us to state that the system under consideration can be treated as an oscillator which frequency depends on the energy.

This is an example of the general dynamical systems characterized by constants which are not external parameters but which are functions of the integrals of motion. Obviously in our case the functional shape of such a dependence is a consequence of the definition which we have taken for the energy, not obligatory to be given by (16). As an alternative one can choose $E = \hbar\omega A^\dagger A$, a possibility exaustively discussed in the Ref. [27].

Here I would like to emphasize that the above illustrates aspects of the Wigner approach which are different from the original one. Beginning with the same ladder structure used to describe the system in the Fock space we arrive at different rules of the time evolution, postulated independently from the ladder structure. This shows that the Wigner approach - to begin with operator rules having clear physical interpretation and to investigate their consistency with other operator rules - must not be restricted to the appearance of the noncanonical commutation relations. The identification of the number, the energy, and the time evolution operators as mutually proportional for the canonical oscillator does not emerge from the equations of motion as well. Having (10) and (14) we can ask which pair of the operators $a, a^\dagger$ or $A, A^\dagger$ we will use to get kinematical quantities x and v. In the following I shall show that formulating the problem in terms of the equations of motion we will conclude that the equations of motion coexist not only with inequivalent fundamental quantum commutators but also with inequivalent descriptions in the sense of the formalism of classical mechanics. I will show that the latter are related to the existence of the, so-called, alternative Hamiltonian and alternative phase - space descriptions. Moreover, I will show that in the framework of the alternative descriptions the energy and the time evolution generator should be treated as two independent quantities which identification is an additional, quite restrictive, requirement.

2.3 An almost forgotten result of classical mechanics

As said, the original Wigner approach is based on the identification of the total energy and the Hamiltonian. But this assumption can not be considered as universally true even in the case of the simplest dynamics driven by a potential force. It is known that the same Newtonian dynamical systems may be described in terms of

Lagrangians or Hamiltonians which lead to the same equations of motion but which are inequivalent in the sense of canonical transformations. In classical mechanics this problem was noticed longer than a century ago, [28]. Its extensive investigations in 70's and 80's of the XX-th century resulted in the statement that the existence of classical alternative descriptions is related to a (complete) integrability of the system, [29]-[32], easy to understand because if two quantities may be treated as independent Hamiltonians of the same conservative system then both of them have to be constant in time, *i.e.*, they should be functions of the integral(s) of motion. According to my best knowledge the meaning of the alternative classical descriptions for quantum physics was for the first time noted and investigated in [33]. The study [34] reviews the topic in the context of the Feynman idea of late 40's, recalled by Dyson, [35], and extensively discussed, under the name of the Feynman problem, in early 90's, [36]. Feynman's aim was to find a dynamics consistent with the Newton equations of motion and the canonical commutation relations but beyond the Lorentz force. Feynman, considering such a hypothesis, hoped that if such a dynamics existed it would open possibility to get rid of difficulties arising in just formulated quantum electrodynamics. The program failed as it appeared that the only one dynamics simultaneously consistent with these assumptions is driven by the Lorentz-type forces, the condition which guarantees the existence of the Lagrangean formulation of the equations of motion. Because of that the rigid conjecture *No Lagrangian? No quantization!* has even been formulated, [37]. Nevertheless this no-go result breaks down if the assumption that the coordinates commute is dropped, even under a weak condition that the noncommutativity is realized by a proportionality to a constant operator, [38]. More complicated realizations of the coordinates noncommutativity appear when the quantum systems are studied in noneuclidean spaces, which origin comes from both the general relativity, [39], as well as from noncanonical algebraic descriptions, [40].

It must be emphasized here that if we drop the canonical shape of some fundamental commutators then in general we have to deal with noncanonical commutators for all kinematical variables. We have shown in [41] - [44] that the noncanonical commutators which mix the coordinates and the velocities are to be correlated with the noncanonical commutators which contain the coordinates and the velocities separately. To see this let us consider a single particle dynamics, *i.e.,* a motion of a material point of mass m put in a field of a given force f. Assume that (1) holds in general, *i.e.,* the operators of coordinates x_i, velocities $\dot{x}_j$, accelerations $\ddot{x}_k$ and a generator of the time evolution H obey the Heisenberg evolution equations which may be written down in the Newtonian form

$$\dot{x}_k = -\frac{i}{\hbar}\,[\,x_k, H\,]\,, \qquad \ddot{x}_k = \frac{1}{m} f_k = -\frac{i}{\hbar}\,[\,\dot{x}_k, H\,]\,, \tag{19}$$

with f_k have been used to denote an operator of the acting force. Evolution equations (19) imply that

$$\begin{aligned}
&[[x_i,\ \dot{x}_k]_{\mathcal{A}}\,,\ H] = i\hbar\left([\dot{x}_i,\ \dot{x}_k] + \frac{1}{m}\,[x_i,\ f_k]_{\mathcal{A}}\right),\\
&[[x_i,\ \dot{x}_k]_{\mathcal{S}}\,,\ H] = \frac{i\hbar}{m}\,[x_i,\ f_k]_{\mathcal{S}}\,,\\
&[[x_i,\ x_k]\,,\ H] = 2i\hbar\,[x_i,\ \dot{x}_k]_{\mathcal{A}}\,,\\
&[[\dot{x}_i,\ \dot{x}_k]\,,\ H] = \frac{2i\hbar}{m}\,[f_i,\ \dot{x}_k]_{\mathcal{A}}\,,
\end{aligned} \tag{20}$$

where subscripts $\mathcal{A}$ and $\mathcal{S}$ denote antisymmetric and symmetric parts of the tensor operators $[\cdot,\cdot]$, respectively.

The canonical formalism of quantum mechanics, *i.e.*, the canonical quantization exploits operators of generalized coordinates x_i and momenta p_k with commutation relations

$$[x_i,\ p_k] = i\hbar\delta_{ik}, \qquad [x_i,\ x_k] = 0, \qquad [p_i,\ p_k] = 0. \tag{21}$$

Operators of velocities $\dot{x}_l$, defined as secondary quantities according to the minimal coupling rule, $\dot{x}_l = \frac{1}{m}\,(p_l - A_l(x,t))$, obey commutation relations coming from (21)

$$\begin{aligned}
&[x_i,\ \dot{x}_k] = \frac{i\hbar}{m}\delta_{ik},\\
&[p_i,\ \dot{x}_k] = -\frac{1}{m}\,[p_i,\ A_k(x,t)]\,,\\
&[\dot{x}_i,\ \dot{x}_k] = -\frac{1}{m^2}\,([p_i,\ A_k(x,t)] - [p_k,\ A_l(x,t)])\,.
\end{aligned} \tag{22}$$

One can check that in the canonical quantization (21) the equalities (20) are satisfied by a Lorentz type force provided the minimal coupling rule is valid. Nevertheless another relations, with the harmonic oscillator force (the first column) and the simplest magnetic force, *i.e.*, the Lorentz force in a constant, c-number magnetic field $\vec{B}$ (the second column)

$$
\begin{array}{ll}
f_i = -kx_i, & f_i = \mu\varepsilon_{ijk}\dot{x}_j B_k, \\
[x_i,\ \dot{x}_k] = i\hbar\delta_{ik}\mathcal{E}, & [x_i,\ \dot{x}_k] = i\hbar\delta_{ik}\mathcal{E}, \\
\frac{k}{m}[x_i,\ x_k] = i\hbar\varepsilon_{ikl}V_l, & [x_i,\ x_k] = i\hbar\varepsilon_{ikl}V_l, \\
[\dot{x}_i,\ \dot{x}_k] = i\hbar\varepsilon_{ikl}V_l, & [\dot{x}_i,\ \dot{x}_k] = i\hbar\varepsilon_{ikl}B_l\mathcal{E}, \\
[H,\ \mathcal{E}] = [H,\ V_l] = 0, & [H,\ \mathcal{E}] = [H,\ V_l] = 0.
\end{array}
\tag{23}
$$

fulfill (20) as well. There is no argument seen in (20) to single out the canonical case $\mathcal{E} = 1$ and $V_l = 0$, or, more generally, to put these operators into the center of the algebra. If we consider the Jacobi identity for a triple $x_i, x_j, \dot{x}_k$

$$
\begin{array}{l}
[[x_i,\ x_j], \dot{x}_k] + [[x_j,\ \dot{x}_k], x_i] + [[\dot{x}_k,\ x_i], x_j] = \\
i\hbar\left(\varepsilon_{ijl}\left[\tilde{V}_l,\ \dot{x}_k\right] + \delta_{jk}[\mathcal{E},\ x_i] - \delta_{ki}[\mathcal{E},\ x_j]\right) = 0,
\end{array}
\tag{24}
$$

then we see that a nontrivial $\mathcal{E}$ requires nontrivial V_l's, and *vice versa.* It will be shown in the Sect.4 that for model systems $\mathcal{E}$ is associated with the time evolution generator, while V_l's are related to the generators of the space rotations. Relations in the last line in (23) show that possible modifications of the canonical commutation rules indeed are connected to the existence of integrals of motion and to the symmetries. The quantities which nontrivially change fundamental commutators do commute with the Hamiltonian, so they obey properties of the integrals of motion, either constructed from kinematical variables or the time independent generators of symmetries of the dynamical system under consideration. Besides of that comment, I would like to stress that because the commutator of the coordinate and the velocity is meaningful also for one dimensional problems we should consider namely it as the most fundamental quantity.

2.4 Alternative quantizations

During last few years the alternative quantizations, initially understood and discussed as solutions to the Wigner problem, [5], [24], [41] - [45], have gained a new interest. Quantum descriptions related to the alternative Hamiltonians have been found explicitly and their properties studied for oscillator-like systems, [46] - [51]. The problem has also been linked to the multihamiltonian systems of classical mechanics, well known and investigated by mathematicians but almost completely discarded in the physical literature and recalled only recently, also in relation to the Wigner problem, [52], [53]. It has been noticed that having more than one, alternative and canonically inequivalent, Hamiltonian descriptions of the same Newtonian dynamics we cannot give any universal relation connecting various Hamiltonians (all being the time evolution generators) with the total energy of the system. The reason is simple - if the energy is a constant of motion then it has to obey this property for any of the alternative Hamiltonians admitted by the system under consideration. Moreover, the energy is physically measurable quantity which has to take the same value independently from any auxiliary notion used to define it. This implies directly that the relation between the energy and different Hamiltonians must be given by different functions. If the total energy which is a sum of the kinetic and potential energies (for systems driven by potential forces) is to be identified with a Hamiltonian this may be true for one Hamiltonian only and not for the other admitted by a system. This confirms Wigner's remark that basic physical properties of the system should be described in a way independent from the Hamiltonian formalism - in particular we should express the energy in terms of the coordinates and velocities. Also generalized velocities are not simply proportional to the space translation generators - the canonical momenta. Phase-space descriptions based on such alternative Hamiltonians lead to nonstandard Poisson brackets, which, if considered as classical analogues of the quantum mechanical commutators, mirror in a noncanonical shape of the algebras of quantum mechanical observables. Their representations, obeying all fundamental requirements, provide us the quantum descriptions of the system. Nevertheless, our lack of knowledge of the alternative Hamiltonians, and the "ordering problem" which occurs for the examples known, [46]-[50], makes Dirac's recipe of quantization difficult to follow, mathematically complicated and giving results which depend on the order of the operators been adopted.

In our generalization of the Wigner approach, [41] - [45], we have developed the idea to study the Wigner problem without specifying the form of the Hamiltonian. Considering the Hamiltonian, and the other generators of the spacetime symmetries, as independent algebraic elements (*i.e.*, following the concept suggested by R. F. Streater more than 40 years ago, [54]) we have constructed new algebras of quantum mechanical observables under the requirement that they belong to the class of the Lie deformations of the Heisenberg algebra. In the next sections I am going to explain how such alternative quantum algebras are related to alternative classical structures and to show that they do provide alternative quantum descriptions, being different for different dynamics and leading to nonstandard properties of the system. In the framework of our approach the alternative descriptions are constructed from the very beginning in a way consistent with the Galilean symmetry and they generalize the notion of the *Wigner quantum systems* because we do not restrict the Hamiltonian to be the total energy. I am convinced that such a freedom is necessary when one wants to generalize ideas of the Wigner quantization to the relativistic physics where, as the *Zitterbewegung* phenomenon shows, [55], neither the time translation nor the space translations generators can be simply related to the total energy and the velocities, respectively [56].

3 One dimensional systems

3.1 Alternative classical description

Consider a system of point masses m_1 and m_2 moving on a line. Assume also that their mutual interaction is given by a distance dependent potential force, $f_{\rm int}(x_1 - x_2) = -{\rm grad}\ V(x_1 - x_2)$. The Newton equations of motion are

$$m_1\ddot{x}_1 = f_{\rm int}(x_1 - x_2), \qquad m_2\ddot{x}_2 = -f_{\rm int}(x_1 - x_2). \tag{25}$$

In order to solve the above the center of mass and the relative motion coordinates are introduced

$$R = (m_1 + m_2)^{-1}(m_1x_1 + m_2x_2), \qquad r = x_1 - x_2, \tag{26}$$

and the equations (25) rewritten as

$$\ddot{R}(t) = 0, \qquad \ddot{r}(t) = \frac{m_1 + m_2}{m_1m_2} f_{\rm int}(r). \tag{27}$$

In the phase-space description of the system the transformation (26), together with the transformation of momenta

$$P = p_1 + p_2, \qquad p = \frac{m_2}{m_1+m_2}p_1 - \frac{m_1}{m_1+m_2}p_2, \tag{28}$$

is a canonical transformation which links two sets of variables: x_1, p_1, x_2, p_2 and R, P, r, p, where P denotes the center of mass, and p the relative, momenta, respectively. In terms of R, P, r, p the Hamilton equations are

$$\begin{aligned} \dot{R} &= \frac{\partial H(R,P,r,p)}{\partial P}, & \dot{P} &= -\frac{\partial H(R,P,r,p)}{\partial R}, \\ \dot{r} &= \frac{\partial H(R,P,r,p)}{\partial p}, & \dot{p} &= -\frac{\partial H(R,P,r,p)}{\partial r}. \end{aligned} \tag{29}$$

Taking the time derivatives of equations in the first column of (29) and using equations in the second column one arrives at consistency conditions, [33], of (29) with the Newton equations (27). They may be expressed in terms of the Poisson brackets and read

$$\begin{aligned} &\frac{\partial^2 H}{\partial R \partial P}\frac{\partial H}{\partial P} - \frac{\partial^2 H}{\partial P^2}\frac{\partial H}{\partial R} + \frac{\partial^2 H}{\partial r \partial P}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p \partial P}\frac{\partial H}{\partial r} = \\ &= \left\{\frac{\partial H}{\partial P}, H\right\} = \ddot{R} = 0, \\ &\frac{\partial^2 H}{\partial r \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial r} + \frac{\partial^2 H}{\partial R \partial p}\frac{\partial H}{\partial P} - \frac{\partial^2 H}{\partial P \partial p}\frac{\partial H}{\partial R} = \\ &= \left\{\frac{\partial H}{\partial p}, H\right\} = \ddot{r} = \frac{m_1+m_2}{m_1 m_2} f_{\text{int}}(r). \end{aligned} \tag{30}$$

Obviously the standard Hamiltonian corresponding to (25)

$$H_{\text{TOT}}(R,P,r,p) \;=\; \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(r), \tag{31}$$

where $M = m_1 + m_2$ and $\mu = m_1 m_2 / M$, is the solution of (30). Such a Hamiltonian provides us the usual relations between canonical momenta and velocities

$$P = M\dot{R}, \qquad p = \mu \dot{r}, \tag{32}$$

and is identified with the total energy of the system

$$\begin{aligned} H_{\mathrm{TOT}} = E_{\mathrm{TOT}} &= \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} + V(x_1 - x_2) = \\ &= \frac{M\dot{R}^2}{2} + \frac{\mu \dot{r}^2}{2} + V(r). \end{aligned} \tag{33}$$

But (31) is only a particular solution of (30), singled out by the requirement that the solution is a sum of monomials in R, P, r and p, [46],[47]. Equally well we can take

$$H_{\mathrm{TOT}}(R, P, r, p) = \frac{P^2}{2M} + H_{\mathrm{alt,int}}(r, p), \tag{34}$$

where an alternative internal Hamiltonian $H_{\mathrm{alt,int}}(r, p)$ is any solution to the equation

$$\frac{\partial^2 H}{\partial r \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial r} = -\frac{1}{\mu}\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{\mu} f_{\mathrm{int}}(r), \tag{35}$$

by no means restricted to the form of (31). An example of such a solution is

$$H(r, p) = \alpha(r)\exp(\lambda p) + \beta(r)\exp(-\lambda p) \tag{36}$$

with

$$\alpha(r)\beta(r) = \frac{1}{2\mu\lambda^2}\left(V(r) + C\right), \tag{37}$$

where λ and C are dimensionful constants. Because Eq.(35) is invariant with respect to $p \to -p$ we require H to be the even function of p. This leads to $\alpha(r) = \beta(r)$ and

$$\begin{aligned} H_{\mathrm{prod}}(r, p) &= \sqrt{\frac{1}{2\mu\lambda^2}\left(V(r) + C\right)}\left(\exp(\lambda p) + \exp(-\lambda p)\right) = \\ &= \sqrt{\frac{2}{\mu\lambda^2}\left(V(r) + C\right)}\cosh(\lambda p), \end{aligned} \tag{38}$$

which is the only one solution of (35) consistent with the requirement that it is a product of factors which separately depend on r and p, [46], [47].

Any alternative internal Hamiltonian gives, through the Hamilton equation, a relation between the generalized velocity $\dot{r}$ and the alternative canonical momentum $p_{\rm alt}$. For the Hamiltonian of the form (38) we get

$$\begin{aligned} \dot{r} &= \sqrt{\frac{2}{\mu}\left(V(r)+C\right)}\sinh(\lambda p_{\rm alt}), \\ p_{\rm alt} &= \frac{1}{\lambda}\ln\frac{\dot{r}\pm\sqrt{\frac{2}{\mu}\left(E_{\rm int}+C\right)}}{\sqrt{\frac{2}{\mu}\left(V(r)+C\right)}}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} E_{\rm int} &= \frac{\mu}{2}\dot{r}^2+V(r)=\frac{\mu\lambda^2}{2}H^2_{\rm int,alt}-C= \\ &= \left(V(r)+C\right)\cosh^2(\lambda p_{\rm alt})-C={\rm const}, \end{aligned} \tag{40}$$

i.e., the internal energy and the alternative internal Hamiltonian are related in a nonstandard way. The relation $\dot{r}=p_{\rm standard}/\mu$ allows us to use (39) in order to define the mutual relation between the alternative and the standard canonical momenta

$$p_{\rm alt}=\frac{1}{\lambda}\ln\frac{p_{\rm standard}\pm\sqrt{2\mu\left(E_{\rm int}+C\right)}}{\sqrt{2\mu\left(V(r)+C\right)}}, \tag{41}$$

and to compare phase-space descriptions of the same Newtonian system given by two sets of canonical variables: $r, p_{\rm standard}$ and $r, p_{\rm alt}$. We find that the transformation leading from the standard to the alternative variables do not preserve the form of the Poisson brackets so this transformation can not be a canonical one. Namely, the fundamental brackets with respect to alternative variables are

$$\begin{aligned} \{r, p_{\rm standard}(r,p_{\rm alt})\}_{r,p_{\rm alt}} &= \mu\,\{r,\dot{r}(r,p_{\rm alt})\}_{r,p_{\rm alt}}= \\ &= \mu\lambda^2 H_{\rm int,alt}(r,p_{\rm alt}). \end{aligned} \tag{42}$$

This proves that for the system (25) we have constructed canonically inequivalent Hamilton descriptions given by

$$\{R, r, P, p_{\rm standard}, H_{\rm TOT,standard}=H_{\rm CM}+H_{\rm int,standard}\}$$

and

$$\{R, r, P, p_{\rm alt}, H_{\rm TOT,alt}=H_{\rm CM}+H_{\rm int,alt}\}\,.$$

The Poisson brackets calculated for the alternative description are

$$\{R, r\} = 0, \qquad \{P, r\} = 0, \qquad \{R, p\} = 0, \qquad \{p, P\} = 0, \tag{43}$$

$$\begin{gathered}\{R, H_{\mathrm{TOT}}\} = \dot{R}, \qquad \left\{\dot{R}, H_{\mathrm{TOT}}\right\} = 0, \qquad \{P, H_{\mathrm{TOT}}\} = 0, \\ \{r, H_{\mathrm{TOT}}\} = \dot{r}, \quad \{p, H_{\mathrm{TOT}}\} = -\frac{\partial H_{\mathrm{int}}}{\partial r}, \quad \{\dot{r}, H_{\mathrm{TOT}}\} = -\frac{1}{\mu}\frac{\mathrm{d}V}{\mathrm{d}r},\end{gathered} \tag{44}$$

$$\{R, P\} = 1, \qquad \left\{R, \dot{R}\right\} = \frac{1}{M}, \qquad \left\{\dot{R}, P\right\} = 0, \tag{45}$$

$$\begin{aligned}\{r, p\} &= 1, \qquad\qquad \{r, \dot{r}\} = \lambda^2 H_{int}, \\ \{\dot{r}, p\} &= \left(\frac{\partial H_{\mathrm{int}}}{\partial p}\right)^{-1} \frac{\partial}{\partial r}\left(\frac{1}{2}\lambda^2 H_{\mathrm{int}}^2 - \frac{1}{\mu}V\right).\end{aligned} \tag{46}$$

The total Hamiltonian and the center of mass variables R and P, *i.e.,* the variables which describe the motion of the system as a whole, can be identified with generators of the infinitesimal transformations of the (1 + 1) extended Galilei group. For such a symmetry of the underlying spacetime the Poisson brackets of the generators are

$$\{K, \mathcal{H}\} = \Pi, \qquad \{K, \Pi\} = \mathcal{M}, \qquad \{\Pi, \mathcal{H}\} = 0, \tag{47}$$

where $\mathcal{H}$, Π and K denote generators of the time and the space translations, and the generator of the Galilean boost, respectively, and $\mathcal{M}$ is a central charge. The Poisson brackets of the Galilei generators with R, P, r and p read

$$\begin{aligned}&\{R, \mathcal{H}\} = \dot{R}, && \left\{\dot{R}, \mathcal{H}\right\} = 0, && \{r, \mathcal{H}\} = \dot{r}, && \{\dot{r}, \mathcal{H}\} = -\frac{1}{\mu}\frac{\mathrm{d}V}{\mathrm{d}r}, \\ &\{K, R\} = t, && \left\{K, \dot{R}\right\} = 1, && \{K, r\} = 0, && \{K, \dot{r}\} = 0, \\ &\{R, \Pi\} = 1, && \left\{\dot{R}, \Pi\right\} = 0, && \{r, \Pi\} = 0, && \{\dot{r}, \Pi\} = 0,\end{aligned} \tag{48}$$

which, if compared with (43) - (45), enables us to represent the Galilean generators in terms of the center of mass variables

$$\mathcal{H} = H_{\mathrm{TOT}}, \qquad K = M\left(R - t\dot{R}\right), \qquad \Pi = M\dot{R}, \qquad \mathcal{M} = M\mathbf{1}. \tag{49}$$

This means that the alternative description of internal degrees of freedom coexists with the Galilean symmetry exactly like it is within the standard approach. For the internal motion we are left with the fundamental Poisson brackets

$$\begin{aligned}
\{r, p_{\rm alt}\}_{r,p_{\rm alt}} &= 1, \\
\{r, \dot{r}(r, p_{\rm alt})\}_{r,p_{\rm alt}} &= \lambda^2 H_{\rm alt}(r, p_{\rm alt}), \\
\{\dot{r}(r, p_{\rm alt}), p_{\rm alt}\}_{r,p_{\rm alt}} &= \frac{V'(r)\dot{r}}{2\left(V(r)+C\right)} = \frac{\rm d}{{\rm d}t}\left(\log\sqrt{V(r)+C}\right).
\end{aligned} \tag{50}$$

From the above we see once more that the total energy (33) has to be related to the alternative Hamiltonian (38) squared. The Poisson brackets of the total energy and r (or $\dot{r}$)

$$\begin{aligned}
\{r, E\}_{r,p_{\rm alt}} &= \mu\lambda^2 H_{\rm alt}(r, p_{\rm alt})\dot{r}, \\
\{\dot{r}, E\}_{r,p_{\rm alt}} &= -\lambda^2 H_{\rm alt}(r, p_{\rm alt})\frac{\rm dV(r)}{\rm dr},
\end{aligned} \tag{51}$$

are to be identified as the Poisson brackets of r (or $\dot{r}$) and $\frac{\mu\lambda^2}{2}H^2_{\rm alt}$. This is not, however, an universal relation. In general, as we have shown in [49] - [51], the fundamental Poisson bracket $\{r, \dot{r}\}$ is

$$\{r, \dot{r}(r, p_{\rm alt})\}_{r,p_{\rm alt}} = \frac{\delta F(H_{\rm alt}(r, p_{\rm alt}))}{\delta H_{\rm alt}(r, p_{\rm alt})} \tag{52}$$

if $E_{\rm int} = F(H_{\rm alt}(r, p_{\rm alt}))$, *i.e.,* we may have an arbitrary function of H on the right hand side of (52). Nevertheless, in what follows I am going to consider only minimal Lie structures and because of that I restrict my further investigations to (40).

3.2 Wigner quantization

In order to quantize the alternative structure, like the Eqs. (50) are, we can follow a two-fold way. The first method is to apply the straightforward analogy to the standard case. In such a case one quantizes the canonical pair q and p, then the earlier found classical Hamiltonian as an element of the linear envelope of the Heisenberg algebra, and at last solves the Schroedinger equation. But, as seen from (38), the alternative Hamiltonians do contain products of q and p so doing this we unavoidably face the ordering problem which different solutions lead to different results, [46], [47].

As a challenger idea I am going to exploit the approach in which all elements of the algebra of observables are independent and their mutual relations are determined only if we know representations. Following this way I expect to find a connection between such an approach and various solutions of (30) and to relate alternative classical descriptions to the studies of noncanonical quantizations which are based on the Wigner idea: *to find and to investigate these noncanonical fundamental commutation relations which coexist with the equations of motion.* In spite of that let us assume that the quantum Hamiltonian is not *a priori* given by its classical analogue. Under this requirement we can search for new algebras of quantum mechanical observables which join symmetries and dynamical equations without any reference to the form of symmetry generators brought to quantum physics from the classical canonical formalism. Following the second Wigner's suggestion *to avoid using Hamiltonian theory* we can also describe the quantum system in terms of the coordinate and velocity operators. In fact, having in mind that the alternative Hamiltonians exist, one feels that the description in terms of the coordinate and the velocity is fundamental. Inequivalent Lagrangians and Hamiltonians lead to inequivalent canonical momenta while the definition of the velocity is free from ambiguities, at least if it is treated as a vector field tangent to the trajectory of the Newtonian dynamics. Here we will consider it as an operator given as the time derivative of the coordinate operator. Because of that we choose, as a classical counterpart of the algebra of observables, only those Poisson brackets which contain the coordinates, the velocities and the Hamiltonian, the latter introduced as the generator of the time evolution and independent from r and $\dot{r}$. Such a structure closes if we know the form of the interaction and we can construct the algebra of observables according to the Dirac procedure using (43) - (46) as its classical analogue. If, in addition, we assume that the masses of (25) interact harmonically that for the case in question the set of commutation relations form a Lie algebra which splits into direct sum of algebras of the center of mass and the relative motions. The first of them is standard while the algebra of the relative coordinate, $x = \hat{r}$, and the relative velocity, $v = \hat{\dot{r}}$, operators consists of

$$[\ x,\ H\ ] = i\hbar v, \qquad [\ v,\ H\ ] = -i\hbar\omega^2 x, \tag{53}$$

i.e., the usual Heisenberg equations of motion and

$$[\ x, v\ ] = -\frac{i\hbar}{\mu}\left(\ B + \frac{1}{\hbar\omega}A\ H\right) \tag{54}$$

where $\lambda^2 \to -\frac{A}{\hbar\mu\omega}$ has been used. The above is the most general Lie algebra which closes the oscillator equations of motion in a minimal way, *i.e.,* without introducing operators besides those which describe the motion, [41] - [45].

Eqn. (54) differs from the Born-Jordan-Heisenberg rule and determines the nonstandard part of the algebra of observables. For $A \neq 0$ we deal with the structural relations of the Lie algebra $so(p, q)$ with $p + q = 3$. The choice $\text{sign}\, A = 1$ points out $so(3)$ while $\text{sign}\, A = -1$ leads to $so(2, 1)$. Obviously $A = 0$ and $B = -1$ reduces the algebra (53) and (54) to the Heisenberg algebra. Properties of $so(p, q)$, $p + q = 3$ algebras and their possible applicability to physical problems are known from a long time, [57]. Physically meaningful representations must be constructed in terms of self-adjoint operators so we will deal with finite dimensional ones for $so(3)$, [58], and with infinite dimensional for $so(2, 1)$. A particular case of the latter I shall present as an illustration of the general scheme just constructed.

3.3 The Barut - Girardello representation of $so(2, 1)$ - a soluble example of noncanonical oscillator

In order to quantize (53) and (54) we use the representation of $so(2, 1)$ which was found in [57] and studied in details in [59]. Namely such a quantization I consider to be the closest to the canonical case. Another examples of the quantization of (53) and (54), like the Jordan-Schwinger representation for $so(2, 1)$ or the standard angular momentum-like quantization of $so(3)$, have been discussed in [50].

Following [57] and [59] I will use the discrete series of representations of $so(2, 1)$ given by self-adjoint operators which act in separable Hilbert spaces, labeled by real numbers Φ, $-2\Phi = 1, 2, 3, ...$, and spanned by the basis vectors $\{|\Phi, m\rangle\}_{m=0}^{\infty}$. In the notation

$\tilde{H} = H + \frac{\hbar\omega B}{A}$ we have

$$\begin{aligned}
\tilde{H}|\Phi, m\rangle &= \hbar\omega\,(m-\Phi)\,|\Phi, m\rangle, \\
x|\Phi, m\rangle &= \frac{1}{2}\sqrt{\frac{\hbar|A|}{\mu\omega}}\left(\sqrt{(m+1)\,(-2\Phi+m)}|\Phi, m+1\rangle\right. \\
&\left. + \sqrt{m\,(-2\Phi+m-1)}|\Phi, m-1\rangle\right), \\
v|\Phi, m\rangle &= \frac{i}{2}\sqrt{\frac{\hbar\omega|A|}{\mu}}\left(\sqrt{(m+1)\,(-2\Phi+m)}|\Phi, m+1\rangle\right. \\
&\left. - \sqrt{m\,(2\Phi+m-1)}|\Phi, m-1\rangle\right).
\end{aligned} \tag{55}$$

In this representation the $so(2,1)$ Casimir operator acts

$$\left(\frac{\tilde{H}^2}{(\hbar\omega)^2} - \frac{\mu}{\hbar\omega|A|}\left(v^2+\omega^2x^2\right)\right)|\Phi, m\rangle = \Phi\,(\Phi+1)\,|\Phi, m\rangle \tag{56}$$

from which one finds the spectrum of the internal energy operator

$$\begin{aligned}
E|\Phi, m\rangle &= \frac{1}{2}\left(\mu v^2+\mu\omega^2x^2\right)|\Phi, m\rangle = \\
&= \frac{\hbar\omega|A|}{2}\left((m-\Phi)^2-\Phi\,(\Phi+1)\right)|\Phi, m\rangle.
\end{aligned} \tag{57}$$

In contradistinction to the linear dependence which characterizes the canonical oscillator we deal with a quadratic function of the quantum number m. The noncanonical oscillator ground state energy, equals to $-\frac{\hbar\omega\Phi|A|}{2}$ and is smaller from the canonical one if $\Phi|A| > -1$. In the adopted units A is dimensionless and modulus of any its nonzero value may be normalized to 1 as a result of a choice of the unit of lenght. This means that the most interesting noncanonical case is that with $\Phi = -1/2$. The condition $\Phi|A| = -1$, under which canonical and noncanonical ground states energies are the same, is interesting as well - under it the representation (55) contracts to the canonical representation in the Fock space in the double limit $\Phi \to \infty$ and $A \to 0$, [59].

The energy spectrum of the type $E \propto m^2$ characterizes not only the model under investigation but also a free particle trapped in an infinite square-well. Boundary conditions in the configuration space put on the motion of the latter lead both to the quantization of the energy levels as well as restrict the space region available for the motion. I am convinced that the concidence of the spectra is not incidental but suggests a deeper connection between noncanonical descriptions of the dynamics and descriptions of the nondynamically confined motions. An example is the energy spectrum of the particle moving in the shifted Pöschl - Teller (PT) potential of the trigonometric type

$$V(x) = \frac{V_0}{2}\left(\frac{\xi_1(\xi_1-1)}{\cos^2(x/2a)} + \frac{\xi_2(\xi_2-1)}{\sin^2(x/2a)}\right) - \frac{\hbar^2(\xi_1+\xi_2)^2}{8\mu a^2}, \qquad (58)$$

where V_0, a and $\xi_1, \xi_2 > 1$ are parameters which fix the particular form of the potential and the shift is given by the last term. The energy levels of the PT potentials are

$$E_m = \frac{\hbar^2 m(m+\xi_1+\xi_2)}{2\mu a^2}. \qquad (59)$$

The Pöschl-Teller type potentials (as well as their modifications, [60]) are interpreted as an interpolation between the infinite square-well and the harmonic oscillator potentials considered as an analytic regularization of the infinite well potential [61]. Their algebraic description is given by the underlying dynamical algebra $su(1,1)$) and this fact clarifies and supports, at least mathematically, their close relation to the noncanonical oscillator just considered.

4 Is noncanonicality hidden? Remarks on the termodynamics of the noncanonical oscillator

Having known the energy spectrum (57) one can ask for the physical properties of the noncanonical oscillators which make them different from the canonical ones. I shall show that such differences do exist and are seen on thermodynamical properties of the system, *i.e.*, I will follow ideas proposed in order to investigate the physical properties of the q-deformed oscillators, [23], [24].

The partition function for the noncanonical oscillator (55) - (57) can be expressed in terms of the Jacobi theta functions, [62], [63], namely by $\vartheta_2(z,q)$ if Φ's are negative half-integers and by $\vartheta_3(z,q)$ if Φ's are negative integers. For the case $\Phi = -\frac{1}{2}$ we have

$$\begin{aligned} Z_{\Phi=-1/2}(T) &= \sum_{m=0}^{\infty} \exp\left(-\frac{E_{\Phi=-1/2}(m)}{k_B T}\right) = \\ &= \frac{1}{2}\exp\left(-\frac{\hbar\omega|A|}{8k_B T}\right)\vartheta_2(0,q)\Big|_{q=e^{-\frac{\hbar\omega|A|}{2k_B T}}}, \end{aligned} \qquad (60)$$

where k_B and T denote the Boltzmann constant and the (absolute) temperature, respectively. Knowing the partition function we may find physically meaningful quantities. For example the mean value of the excitation energy in the simplest case $\Phi = -\frac{1}{2}$ reads

$$\langle E_{\Phi=-1/2}(T) - E^{\mathrm{vac}}_{\Phi=-1/2}\rangle = \\ = -\frac{\hbar\omega|A|}{8}\left(1 + \frac{\vartheta_2''(0,q)}{\vartheta_2(0,q)}\right)\bigg|_{q=e^{-\frac{\hbar\omega|A|}{2k_BT}}} \tag{61}$$

with the standard abbreviation $\vartheta_2''(0,q) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\vartheta_2(z,q)|_{z=0}$ used. Properties of the theta functions allow to rewrite (61) as

$$\langle E_{\Phi=-1/2}(T) - E^{\mathrm{vac}}_{\Phi=-1/2}\rangle = \\ = \hbar\omega|A|\frac{\exp\left(-\frac{\hbar\omega|A|}{k_BT}\right)}{1-\exp\left(-\frac{\hbar\omega|A|}{k_BT}\right)}\times \\ \times\left[\sum_{n=1}^{\infty}\frac{n(-1)^{n-1}\kappa^{n-1}}{1+\kappa+\ldots+\kappa^{n-1}}\right]_{\kappa=e^{-\frac{\hbar\omega|A|}{k_BT}}}, \tag{62}$$

in which one recognizes the Planck distribution of $\omega' = \omega|A|$ multiplicatively deformed by a factor increasing monotonically from 1/2 to 1 for $\omega' \in (0,\infty)$. Eq. (62) provides us the noncanonical black-body radiation spectrum. As mentioned before any finite $|A|$ may be normalized to 1 and we can compare the canonical and noncanonical cases as functions of the same variable $\hbar\omega/k_BT$. The Fig.1 shows that the shape of the noncanonical curve differs from the canonical one by the low frequencies slope, its maximum is smaller and slightly shifted towards higher frequencies. For high frequencies both curves are indistinguishable.

In the classical limit, $\hbar \to 0$, we find

$$\lim_{\hbar\to 0}\langle E_{\Phi=-1/2}(T) - E^{\mathrm{vac}}_{\Phi=-1/2}\rangle = \frac{k_BT}{2}, \tag{63}$$

twice smaller than the result got for the canonical oscillator but characteristic for the canonical free case. This cannot be considered as completely unexpected, at least as a heuristics. I recall that the energy spectrum $E \propto m^2$ characterizes a free particle trapped in

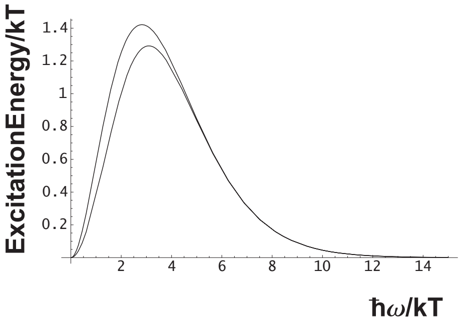

Figure 1: Canonical (left) *versus* noncanonical (right) Planck distributions

an infinite square-well. Quantization of the energy levels disappears however in the classical limit and looking at the classical macroscopic properties of the system we recognize only the free motion. Our example shows that the same limiting properties characterize another motion but described in a different way - what we see classically as a free motion may be related to the nontrivial interaction at the quantum level. I would also like to emphasize that (63) agrees with the excitation energy value got from the classical partition function calculated for the energy (40) with $C = 0$

$$\begin{aligned} Z_{\text{classical}}(T) &= \frac{1}{V_{qp}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}p \exp\left(-\frac{\mu\omega^2 x^2}{k_B T}\cosh^2\lambda p\right) \\ &= \frac{1}{V_{qp}} \sqrt{\frac{2\pi k_B T}{\mu\omega^2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\cosh\lambda p} \\ &= \frac{1}{V_{qp}} \sqrt{\frac{2\pi k_B T}{\mu\omega^2}} \, \frac{\pi}{\lambda}, \end{aligned} \tag{64}$$

where V_{qp} provides us a normalization of the phase-space volume and in order to evaluate the second integral I have used Ref. [64].

Here I would like to pay the readers attention to the fact that the calculation of the partition function defined as an object over the phase - space and with the Hamiltonian put into the exponential gives the same result for both canonical and noncanonical cases, proved in [65] to be a general property. Nevertheless, we have seen that defining the partition function as a functional of the energy we get different results. Having this in mind we must judge which entity enters the definition of the partition function - is it either the energy, or the Hamiltonian? Both these quantities, interchanged freely in the framework of the canonical approach, should not be identified any longer within our scheme. I advocate here the point of view that it is the energy which, through the Maxwell - Boltzmann probability distribution, determines the partition function. Namely the energy and its metamorphoses are fundamental concepts used in order to describe thermodynamical processes. Termodynamics does not consider the time evolution and it is the statistical mechanics which introduces it and the Hamiltonian in order to give a justification to the thermodynamical considerations.

The most important quantity of the equilibrium macroscopic termodynamics is the entropy. In terms of the partition function it is given by

$$S = k_B \frac{\partial}{\partial T} \left(T \log Z(T)\right) = k_B \log Z(T) + \frac{1}{T}\langle E \rangle. \tag{65}$$

Taking (60), (62) and the product representation of the $\vartheta_2(z,q)$ we get for the noncanonical case

$$\begin{aligned} &S^{\text{noncanonical}}_{\text{quantum}}(T) = \\ &= k_B \left\{ \log \left[\prod_{n=1}^{\infty} \left[1 - \exp\left(-n\frac{\hbar\omega|A|}{k_B T}\right)\right] \left[1 + \exp\left(-n\frac{\hbar\omega|A|}{k_B T}\right)\right]^2 \right] + \right. \\ &\left. + \frac{\hbar\omega|A|}{k_B T} \frac{\exp\left(-\frac{\hbar\omega|A|}{k_B T}\right)}{1 - \exp\left(-\frac{\hbar\omega|A|}{k_B T}\right)} \left[\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}\kappa^{n-1}}{1 + \kappa + \ldots + \kappa^{n-1}}\right]_{\kappa = e^{-\frac{\hbar\omega|A|}{k_B T}}} \right\}, \end{aligned} \tag{66}$$

which we can compare with the canonical expression

$$S^{\text{canonical}}_{\text{quantum}}(T) = \\ = k_B \left\{ \log \frac{1}{1 - \exp - \dfrac{\hbar\omega}{k_B T}} + \frac{\hbar\omega}{k_B T} \frac{\exp\left(-\dfrac{\hbar\omega}{k_B T}\right)}{1 - \exp\left(-\dfrac{\hbar\omega}{k_B T}\right)} \right\}. \tag{67}$$

As should be expected both (66) and (67) satisfy the Third Principle of Thermodynamics and vanish in the limit $T \to 0$. Nevertheless their asymptotic behavior is different. Setting $A = -1$ we may investigate their mutual relation in a more detailed way. The difference

$$\Delta S_{\text{quantum}} = S^{\text{noncanonical}}_{\text{quantum}}(T) - S^{\text{canonical}}_{\text{quantum}}(T) = \\ = k_B \left\{ 2\log\left[1 - \exp\left(-\frac{2\hbar\omega}{k_B T}\right)\right] + \right. \\ + \log\left[\prod_{n=2}^{\infty}\left[1 - \exp\left(-2n\frac{\hbar\omega}{k_B T}\right)\right]\left[1 + \exp\left(-n\frac{\hbar\omega}{k_B T}\right)\right]\right] + \\ \left. + \frac{\hbar\omega}{k_B T}\frac{\exp\left(-\dfrac{\hbar\omega}{k_B T}\right)}{1 - \exp\left(-\dfrac{\hbar\omega}{k_B T}\right)}\left[\sum_{n=2}^{\infty}\frac{n(-1)^{n-1}\kappa^{n-1}}{1 + \kappa + \ldots + \kappa^{n-1}}\right]_{\kappa = e^{-\frac{\hbar\omega}{k_B T}}} \right\}, \tag{68}$$

is negative for $T \to 0_+$ as a sum of three negative terms. So, for low T the canonical entropy is greater than the noncanonical one. In order to find the asymptotic behaviour of the noncanonical entropy (and next of (68)) for $T \to \infty$ we will use the relation

$$\frac{\partial \log Z}{\partial T} = \frac{\langle E \rangle}{k_B T^2} \tag{69}$$

which, because of (62) and (63), leads asymptotically to

$$\frac{\partial \log Z^{\text{noncanonical}}_{\infty}}{\partial T} = \frac{1}{2T}. \tag{70}$$

The solution to (70) is

$$\log Z^{\text{noncanonical}}_{\infty}(T) = \log e^{c}\sqrt{\frac{k_B T}{\hbar\omega}}. \tag{71}$$

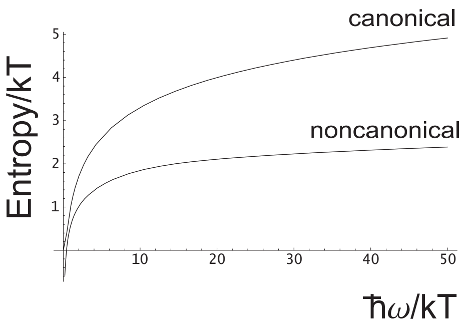

Figure 2: Entropies of canonical and noncanonical harmonic oscillators

Similarly, we have

$$\begin{aligned} S_\infty^{\text{noncanonical}}(T) &= k_B \log e^{c+1/2}\sqrt{\frac{k_BT}{\hbar\omega}}, \\ S_\infty^{\text{canonical}}(T) &= k_B \log e\frac{k_BT}{\hbar\omega}, \end{aligned} \tag{72}$$

and

$$S_\infty^{\text{noncanonical}}(T) - S_\infty^{\text{canonical}}(T) = -k_B \log e^{c-1/2}\sqrt{\frac{k_BT}{\hbar\omega}}, \tag{73}$$

i.e., also for high temperatures the canonical entropy is greater than the noncanonical. As checked numerically and plotted on the Fig.2. this property holds for all temperatures. I consider this as an important result which allows to formulate the following heuristics:
If the Newtonian system admits alternative Hamiltonian descriptions, i.e., if it can be described in canonically inequivalent phase-spaces, then the entropy of the canonical system is greater than the entropy of its noncanonical analogue. Because of that, equilibrium states prefer to be canonical and systems which approach to the equilibrium simultaneously become canonical.

If such a statement is true then it explains why we do not observe noncanonical effects looking at systems being at the equilibrium - but obviously it does not exclude noncanonical effects influencing systems being far from the equilibrium.

5 Wigner quantum systems - a multidimensional case

In Sect.2.1 I listed unusual physical properties obeyed by multidimesional Wigner systems described within the parastatistics approach, among them the noncommutativity of the underlying configuration space. Here, I will show that similar results we may get also for Wigner systems studied in a way suggested by alternative classical description.

Considering multidimensional systems in the same way as the one dimensional systems of the previous section we find that equations analogous to (30) do have alternative solutions - a sum of (38) over three coordinates provides us a nontrivial example. The Poisson brackets may be calculated, $\{q_i, \dot{q}_j\} \propto \delta_{ij} H$, but, as shown in (19) - (24), it requires $\{q_i, q_j\} \neq 0$. This cannot be satisfied by the standard Poisson bracket and it means either a contradiction, or a necessity to change the definition of the Poisson brackets. However, in our approach we do not refer to the shape of the classical rules and we construct alternative quantum algebras following the scheme of their construction as relations which close the equations of motion and the symmetries to an algebraic structure with well-defined properties.

5.1 3d harmonic oscillator

Let us assume that two point masses m_1 and m_2 interact harmonically. Denote their position and velocity operators as $\vec{x}^{\,1}$, $\vec{x}^{\,2}$, $\vec{v}^{\,1}$ and $\vec{v}^{\,2}$, and the generators of the extended Galilei group by: H - the time evolution, Π_i - the space translations , S_i - the three dimensional rotations, K_i - the Galilean boosts , all with $i = 1, 2, 3$, and M - the central charge. Algebraic description of the system, for $a, b = 1, 2$ numerating particles, reads

$$[x^a{}_k,\ H] = i\hbar v^a{}_k, \qquad [v^a{}_k,\ H] = i\hbar \frac{F^{ab}}{m_a} = i\hbar \frac{k(x^b{}_k - x^a{}_k)}{m_a}, \tag{74}$$

$$
\begin{aligned}
&[S_i,\ x^a{}_j] = i\hbar\varepsilon_{ijk}x^a{}_k, \quad \left[\Pi_i,\ x_j^a\right] = -i\hbar\delta_{ij}, \quad \left[K_i,\ x_j^a\right] = i\hbar t\delta_{ij}, \\
&[S_i,\ v^a{}_j] = i\hbar\varepsilon_{ijk}v^a{}_k, \quad \left[\Pi_i,\ v_j^a\right] = 0, \qquad \left[\ K_i,\ v_j^a\right] = i\hbar\delta_{ij},
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
&[S_i,\ S_j] = i\hbar\varepsilon_{ijk}S_k, \quad [K_i,\ K_j] = 0, \quad [K_i,\ H] = i\hbar\Pi_i, \\
&[S_i,\ \Pi_j] = i\hbar\varepsilon_{ijk}\Pi_k, \quad [\Pi_i,\ \Pi_j] = 0, \quad [\Pi_i,\ H] = 0, \\
&[S_i,\ K_j] = i\hbar\varepsilon_{ijk}K_k, \quad [K_i,\ \Pi_j] = i\hbar M\delta_{ij}, \quad [S_j,\ H] = 0.
\end{aligned}
\tag{76}
$$

For the center of mass and relative motion coordinates the above become

$$
[R_i,\ H] = i\hbar\dot{R}_i, \qquad \left[\dot{R}_i,\ H\right] = 0, \tag{77}
$$

$$
\begin{aligned}
&[S_i,\ R_j] = i\hbar\varepsilon_{ijk}R_k, \quad [K_i,\ R_j] = i\hbar t\delta_{ij}, \quad [\Pi_i,\ R_j] = -i\hbar\delta_{ij}, \\
&\left[S_i,\ \dot{R}_j\right] = i\hbar\varepsilon_{ijk}\dot{R}_k, \quad \left[K_i,\ \dot{R}_j\right] = i\hbar\delta_{ij}, \quad \left[\Pi_i,\ \dot{R}_j\right] = 0,
\end{aligned}
\tag{78}
$$

and

$$
[r_i,\ H] = i\hbar\dot{r}_i, \qquad [\dot{r}_i,\ H] = -i\hbar\omega^2 r_i \tag{79}
$$

$$
\begin{aligned}
&[S_i,\ r_j] = i\hbar\varepsilon_{ijk}r_k, \qquad [K_i,\ r_j] = 0, \qquad [\Pi_i,\ r_j] = 0, \\
&[S_i,\ \dot{r}_j] = i\hbar\varepsilon_{ijk}\dot{r}_k, \qquad [\ K_i,\ \dot{r}_j] = 0, \qquad [\Pi_i,\ \dot{r}_j] = 0.
\end{aligned}
\tag{80}
$$

Now the question for a closure of (77) - (80) to an oscillator algebra $\mathcal{HO}_{3\mathrm{d}}$, being a Lie one, is a little bit more complicated. In a minimal version, like it was for 1-d system, it exists for a canonical case only. Nevertheless if we repeat a separation of the center of mass and the relative motion variables

$$
[r_i, R_j] = \left[\dot{r}_i, \dot{R}_j\right] = \left[r_i, \dot{R}_j\right] = [\dot{r}_i, R_j] = 0 \tag{81}
$$

and require the canonical relations for the center of mass operators

$$
[R_i, R_j] = 0, \quad \left[\dot{R}_i, \dot{R}_j\right] = 0, \quad \left[R_i, \dot{R}_j\right] = i\hbar\frac{1}{m_1 + m_2}\delta_{ij}, \tag{82}
$$

then a subalgebra spanned by $R_i, \dot{R}_i, H, S_i, \Pi_i$ and K_i is a canonical ideal $\mathcal{HO}_{3\mathrm{d,CM}}$ and we may look for remaining commutators within a Lie algebra $\mathcal{HO}_3/\mathcal{HO}_{3\mathrm{d,CM}}$. Commutators which complete (77) - (80), (81) and (82) to such a quotient algebra are

$$
\begin{aligned}
&[r_i, r_j] = \omega^{-2}\left[\dot{r}_i, \dot{r}_j\right] = \frac{i\omega}{\mu}A\varepsilon_{ijk}S_k, \\
&[r_i, \dot{r}_j] = -\frac{i\hbar}{\mu}\left(B + \frac{1}{\hbar\omega}AH\right)\delta_{ij},
\end{aligned}
\tag{83}
$$

where A and B are arbitrary. For $A \neq 0$ the relations (79) - (83) describe in a Galilean covariant way a noncanonical harmonic oscillator whose 10 dimensional algebra of internal motion, spanned by r_i, $\dot{r}_i$, S_i and H, is one of the $so(p,q), p+q=5$ algebras, depending on the $\operatorname{sign}\omega^2$ and $\operatorname{sign}A$. Rewriting (81), (82) and (83) in terms of $\vec{x}^{\,a,b}$ and $\vec{v}^{\,a,b}$, $(a,b=1,2)$ we get

$$\left[{x^a}_i, {x^b}_j\right] = \omega^{-2}\left[{v^a}_i, {v^b}_j\right] = \frac{i\omega m_a m_b}{m_1 m_2 (m_1+m_2)} A\varepsilon_{ijk} S_k, \tag{84}$$

$$\left[{x^a}_i, {v^a}_j\right] = -\frac{i\hbar}{m_a+m_b}\left(m_a - m_b B + \frac{m_b}{\hbar\omega} AH\right)\delta_{ij}, \tag{85}$$

which implies that noncanonical coordinates and velocities of both particles are noncommutative if the interaction is present and if the mass ratio of particles and the system total mass are finite.

5.2 Representations

The algebraic structure which we have got distinguishes the center of mass motion from the relative motion. If the center of mass motion is canonical then its algebra has a structure of a semidirect product of the Galilei and the Heisenberg algebras. If, like we have done previously, we consider the carrier space of representations as the space of functions $\Psi(t;\rho_i)\otimes\phi(t;\eta)$ (with ρ_i having the meaning of the spacetime variables and η to be specified) then all the generators of the extended Galilei algebra may be represented by differential operators

$$R_i = \rho_i,\ \dot{R}_i = -i\hbar\frac{1}{M}\frac{\partial}{\partial\rho_i},\ K_i = M\ \left(R_i - t\dot{R}_i\right),\ \Pi_i = M\dot{R}_i, \tag{86}$$

$$S_i = \varepsilon_{ijk}R_j\dot{R}_k + S_i^{\mathrm{int}}, \qquad H = \frac{1}{2}M\sum_{k=1}^{3}\dot{R}_k^2 + H_{\mathrm{int}}, \tag{87}$$

where $M = m_1 + m_2$. The generators of rotations and of the time evolution have now, besides of the standard terms connected with the center of mass motion, also additional terms related to the internal properties of the system. Both S_i^{int} and H_{int} may have noncanonical forms, they both commute with the center of mass variables and act on the internal degrees of freedom. They, together with noncanonical coordinates and velocities, should be fixed by a choice of representation of the internal motion algebra. We should also expect that it will be the choice of representation which will provide us a physical interpretation of the alternative, noncanonical description.

It is not new that studying physical systems we are enforced to distinguish between the center of mass and the internal degrees of freedom. This is the situation which we meet in the Dirac equation theory - almost hidden when we look at the equation only but explicitly emerging when we study its algebraic background, [56]. The analysis of the *Zitterbewegung* phenomenon confirms such a statement and warns us that identification of the physical observables and the generators of the spacetime symmetries which works in the canonical formalism of the classical mechanics is not so simple in general.

6 Conclusions

The existence of the alternative Hamiltonians justifies, even enforces, investigations of noncanonical quantization schemes. It confirms more than 50 years old Wigner's idea that the equations of motion and the Ehrenfest theorem are crucial to understand the meaning of the classical analogy in quantum physics. The latter statement can be formulated even stronger - the alternative descriptions provide us a classsical background of the noncanonical quantizations and may be a shadow of the properties of the quantum world which are unusual from the canonical point of view. Solutions to the simplest solvable models, the oscillator-like dynamics, show that noncanonical quantum systems, *i.e.*, representations of noncanonical algebras of quantum mechanical observables exhibit physical properties which distinguish them from canonical ones. Among them the most fundamental and the most important is the noncommutative geometry of the underlying spacetime. Here one has to notice that this concept dates back to the beginning of quantum physics and quantum mechanical description of elementary physical phenomena - in fact it is hidden in the, so called, Landau problem of motion of a charged particle driven by a constant magnetic field, [66], [67]. When noncommutative geometry was for the first time proposed explicitly to be used in the quantum physics, [68], it was considered as curious, artificial and leading nowhere. But now this new mathematics found a rigorous and well established background, [69]. Its enthusiastic supporters promise that the noncommutative field theory will help to solve *unsolvable* problems of the quantum physics, [70]. It is also claimed that "noncommutative methods" open possibilities to investigate some mysterious quantum phenomena - like the quarks confinement or the fractional Hall effect - as nondynamical correlations, much more similar to the Pauli exclusion principle than to effects caused by interactions understood in an elementary field-theoretical sense. I am convinced in favour of all these arguments. But considering their details we should remember that the same dynamical systems may be described both in commutative and in noncommutative geometrical background and there is no fundamental reason to favour any of them. It may be that ideas emerging from the toy-model based considerations given in Sect.4 will appear useful in order to understand why observable physical world is so strongly dominated by canonical, commutative behaviour and leave room for noncanonical descriptions only in extremal, but physically not excluded, situations.

The approach which I have presented suggests that noncommutativity of the quantum spacetime has something common with alternative Hamiltonian descriptions. But the alternative Hamiltonians are not of the quantum origin - in fact they are beyond the quantum physics at all and belong to the foundations of the classical mechanics. Their almost forgotten existence and up-to-date rediscovery warns us that our knowledge gained through studies of the simple solutions to classical problems can not be treated as an universal and ultimate principle when we build more fundamental theories. The canonical Hamiltonian formalism provides us methods which allow to solve mechanical problems easier and much more effectively than the Newtonian approach does. It allows also to describe classical mechanics in an elegant geometrical language and to get rid of many ambiguities which quantizations of classical theories born. Nevertheless it is a beautiful, but may be going too far, simplification, by no means singled out to be an unique one. In view of that I believe that the canonical approach cannot be considered as an absolutely universal ontological background which determines the shape of the quantum physics. I am convinced that in numerous cases the difficulties which we face solving problems of the quantum physics are not of the technical origin only and we should look for a way out modifying fundamental assumptions of quantum physics. But such modifications may not be new and revolutionary concepts. Many of them may be long-time known, forgotten and waiting for revival.

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