Construction of Optimal Portfolio Using Efficient Portfolio and Zero Opportunity Cost

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Abstract In this manuscript, we introduce asset allocation and portfolio selection techniques based on efficiency condition, Sharpe ratio error condition and order three zero opportunity condition. Investors expect the same level of risk and return from alternative investment options unless they want the advantage of diversification for risk. There are two fundamental investment portfolios. The first one is risk free fundamental portfolio, and the second one is risky fundamental portfolio. Investors use zero opportunity cost to select portfolio objective. In this research, mathematical derivation of portfolio construction approach is described in advance. Historical data of this research show that there is a positive linear relationship between natural logarithm of standard deviation of securities' return and square of co-variance between securities and market return. Furthermore, it shows that there is a positive linear relationship between Treynor ratio of securities' return and Sharpe ratio of securities' return. We use global search optimization tool in MatLab and R software to solve empirical portfolio selection and asset allocation problem. Moreover, we apply direct and indirect mathematical proof methods to prove mathematical facts of this study.

Keywords Asset, Efficient Frontier, Opportunity Cost, Return Neutral, Sharpe Ratio

1 Introduction

Investors optimize investment asset allocation to reduce unnecessary losses based on their investment objectives and constraints. Traditionally, investors use 40-60 asset allocation strategies. This 40-60 classical asset allocation strategy need to be optimal. Thus, investors should set their investment objectives and constraints to optimize asset allocation. Practitioners use mean variance model to select portfolio even if the model assumes weights of securities are independent of time parameter. Another problem of mean variance model is that the model does not consider zero opportunity cost between fundamental portfolios. Single index model is developed because mean variance model needs estimations of expected return, risk and co-variance of securities. This indicates that mean variance model is not easy for portfolio with large number of securities. In this research, we propose new portfolio construction method using order three zero opportunity condition, efficiency condition and Sharpe ratio error condition. Clearly, one can express variance of portfolio in terms of expected return by using efficiency condition. Similarly, the variance of portfolio can be expressed in terms of expected return due to Sharpe ratio error condition. Let w_f^* , w_p^* and w_m^* be weight of risk free security, weight of risky portfolio and weight of market portfolio, respectively. Suppose that r_f , r_{tp} and r_{tm} represents risk free rate of return, risky portfolio rate of return and market portfolio rate of return at time t , respectively. Then we define rate of return of complete portfolio r_{tcp} by $r_{tcp} = w_f^* r_f + w_p^* r_{tp} + w_m^* r_{tm}$. In this research, we determine r_{tcp} , w_f^* , w_p^* and w_m^* using order three zero opportunity condition, efficiency condition and Sharpe ratio error condition. Moreover, we use weight-variance regres-

sion model and Treynor-Sharpe regression model to build General order three opportunity model (GOOM). Furthermore, we solve Order three asset allocation model (OAAM) to determine complete portfolio return. We use USA *s&p500* companies one year daily returns for empirical study.

1.1 Motivation

Since investors have different objectives, their portfolio selection need not be the same using mean variance model. So which investor is right? This question motivates researchers to study portfolio selection and asset allocation techniques using zero opportunity cost.

1.2 Statement of the problem

The statement of the problem is constructing optimal complete portfolio using order three zero opportunity condition, efficiency condition and Sharpe ratio error condition. Mathematically, we obtain the optimal risk free security weight w_f^* , market portfolio weight w_m^* and risky portfolio weight w_p^* to construct complete portfolio P_c such that $r_{tcp} = w_f^*r_f + w_p^*r_{tp} + w_m^*r_{tm}$ using order three zero opportunity condition, efficiency condition and Sharpe ratio error condition.

1.3 Objectives of the study

The general objective of this study is to construct complete portfolio using order three zero opportunity condition, efficiency condition and Sharpe ratio error condition. Specific objectives are:

1. to show that there is positive linear relation ship between Treynor ratio and Sharpe ratio of securities,
2. to estimate parameters of weight-variance regression model.
3. to build General order three opportunity model (GOOM),
4. to construct Order three asset allocation model (OAAM).

1.4 Research gap

Since efficient frontier contains many efficient portfolios, investors should choose efficient portfolio using logical constraints. Mean variance model does not explain how to choose efficient portfolio using logical constraints. In this research, we use (GOOM) and (OAAM) to construct optimal complete portfolio.

2 Literature review

Investment is an allocation of resources for medium or long term and the expected effect is to recover the investment costs and have a high profit, [1]. Risk is the possibility that desired objective would not be attained, [2]. One cannot definitely determine whether the risk is only good or bad, [3].

2.1 Financial risk preference and tolerance

Risk preference is the individual's orientation towards risk-taking, [4]. Financial risk tolerance is the level of risk that clients believe they are willing to accept, [5]. Financial risk tolerance is the maximum amount of volatility that investor is willing to accept when making a financial decision, [5]. Risk aversion means an attitude of reluctance to take risky decisions, [6]. Financial Risk Tolerance is the willingness of individual investors to receive negative changes of investment value or the result opposite or different from the expected results, [7]. Willingness to take financial risk depends on portfolio structure, gender, age, educational attainment, income, financial stability, financial literacy, marital status, and family size, [8].

Define standard deviation vector σ , weight vector w , expected return vector μ and vector of ones l as follows. $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)^T$, $w = (w_1, w_2, \dots, w_n)^T$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)^T$ and $l = (1, 1, \dots, 1)^T$. Let Σ be covariance matrix of order n .

2.2 Markowitz' mean variance model

H.M. Markowitz' was the first who introduced Markowitz' mean variance model for portfolio selection.

The Markowitz' model (MM) is given by:

$$\begin{aligned} MM \quad & \max_w \mu^T w \\ & \min_w w^T \Sigma w, \\ \text{subject to} \quad & l^T w = 1. \end{aligned} \quad (1)$$

Note that return maximization and risk minimization objectives are conflicting objectives. If investors attempt to increase return, they will be penalized by risk. The solution of MM depends on investors' risk preference. Consider the following special cases of MM .

2.3 Risk neutral investors

Risk neutral investors are investors who are interested to maximize return without paying attention on risk, [9]. These investors do not care about risk minimization. Risk neutral Markowitz's model ($RINMM$) is given as follows:

$$\begin{aligned} RINMM \quad & \max_w \mu^T w \\ \text{subject to} \quad & l^T w = 1. \end{aligned} \quad (2)$$

2.4 Return neural investors

Some investors prefer risk minimization rather than return maximization. These investors are called return neutral investors. Return neutral Markowitz's model is given as follows:

$$\begin{aligned} RENMM \quad & \min_w w^T \Sigma w, \\ \text{subject to} \quad & l^T w = 1. \end{aligned} \quad (3)$$

Note that return neutral investors gain less return than risk neutral investors. However, they are less exposed to risk.

2.5 Maximum expected return for a given level of risk

Investors who know their risk tolerance potential would like to maximize return for their risk level. These investors would like to solve the following optimization problem:

$$\begin{aligned} & \max_w \mu^T w \\ \text{subject to } & w^T \Sigma w = \sigma_p^*, \end{aligned} \quad (4)$$

$$l^T w = 1. \quad (5)$$

Readers are referred to [9] for further detail about maximum expected return for a given level of risk.

2.6 Minimum risk for a given level of expected return

Those investors who are interested to achieve targeted expected return by minimizing risk use the following model:

$$\begin{aligned} & \min_w w^T \Sigma w \\ \text{subject to } & \mu^T w = \mu_p^*, \\ & l^T w = 1. \end{aligned} \quad (6)$$

The detail of this model is given in [9].

2.7 Risk adjusted model (RAM)

Investors may have different weight levels for risk and return. The investors who think both risk and return are equally important use 50-50 weights for risk and return. In general, risk adjusted portfolio model has the following form as given in [9, 10]:

$$\begin{aligned} RAM \quad & \min_w [\rho w^T \Sigma w - \mu^T w] \\ \text{subject to } & l^T w = 1, \end{aligned} \quad (7)$$

where ρ is a parameter.

2.8 Sharpe ratio maximization model (SRMM)

Portfolio comparison is relative to some reference. For instance, managers may compare two portfolios relative to Sharpe ratio. This does not mean portfolio with high Sharpe ratio is absolutely better than portfolios with low Sharpe ratio. However, portfolio with high Sharpe ratio is more desirable than the one with low Sharpe ratio provided an investor cares about Sharpe ratio rather than other criteria. See reference [11, 12] for more details. Sharpe ratio maximization model is given by:

$$\begin{aligned} SRMM \quad & \max_w \frac{\mu^T w - r_f}{\sqrt{w^T \Sigma w}} \\ \text{subject to } & l^T w = 1, \end{aligned} \quad (8)$$

where r_f risk free rate of return.

2.9 Diversification maximization model (DMM)

Diversification is a key technique to minimize investment risk. Investors who think diversification as a good tool to minimize risk maximize diversification ratio. Diversification ratio maximization model is given in, [13]. The following model is called diversification maximization model.

$$\begin{aligned} DMM \quad & \max_w \frac{\sigma^T w}{\sqrt{w^T \Sigma w}} \\ \text{subject to } & l^T w = 1. \end{aligned} \quad (9)$$

The above investment portfolio construction models presented above are well known fundamental portfolio selection models. However, there are some issues which also affect portfolio selection decision. For instance, managers should consider opportunity cost, variability (instability) of security weight over time, inefficiency, return errors and market factors when they construct portfolio. In this research, portfolio errors such as return error, frontier error, opportunity error, Sharpe error, Treynor error, Sharpe-Treynor error and weight error are considered.

2.10 Global minimum variance and expected return

Suppose that investors want to maximize expected return for a given level of risk, see problem (4)-(5). Apply Lagrange multiplier method to solve expected return maximization problem for a given risk level. Let λ_1 and λ_2 be two scalars. Then define

$$L(w, \lambda_1, \lambda_2) = -\mu^T w - \lambda_2(l^T w - 1) - \lambda_1(w^T \Sigma w - \sigma_p^*). \quad (10)$$

Solve $\nabla L(w, \lambda_1, \lambda_2) = 0$ to find critical points of L . Thus, the following equations follow from $\nabla L(w, \lambda_1, \lambda_2) = 0$.

$$\Sigma w = \frac{-\mu - \lambda_2 l}{2\lambda_1}, \quad (11)$$

$$l^T w = 1, \quad (12)$$

$$w^T \Sigma w = \sigma_p^*. \quad (13)$$

This implies that

$$w^T \Sigma = \frac{-\mu^T - \lambda_2 l^T}{2\lambda_1}, \quad (14)$$

$$l^T w = 1, \quad (15)$$

$$w^T \Sigma w = \sigma_p^*. \quad (16)$$

It follows from equation (14) that

$$w^T = \frac{-\mu^T \Sigma^{-1} - \lambda_2 l^T \Sigma^{-1}}{2\lambda_1}. \quad (17)$$

This implies that

$$(w^T l) 2\lambda_1 = -\mu^T \Sigma^{-1} l - \lambda_2 l^T \Sigma^{-1} l. \quad (18)$$

Since $w^T l = 1$, we get

$$2\lambda_1 = -b - \lambda_2 c. \quad (19)$$

Thus, the following equations are valid.

$$\lambda_2 = \frac{-(b + 2\lambda_1)}{c}, \quad (20)$$

$$\sigma_p^* = \left(\frac{1}{2\lambda_1} \right)^2 \left(\frac{ac - b^2}{c} \right) + \frac{1}{c}. \quad (21)$$

It follows from equation (17) that

$$\mu_p = w^T \mu = \frac{-\mu^T \Sigma^{-1} \mu - \lambda_2 l^T \Sigma^{-1} \mu}{2\lambda_1} = \frac{-a - \lambda_2 b}{2\lambda_1}. \quad (22)$$

Substitute the value of λ_2 in to the equation (22) to get the following equation (23).

$$\frac{1}{2\lambda_1} = \frac{b - c\mu_p}{ac - b^2}. \quad (23)$$

Substitute equation (23) in to the equation (21) to get constant weight efficient frontier of portfolio p . Therefore, constant weight efficient frontier of portfolio p is given by:

$$\sigma_p^2 = \frac{1}{c} \left(1 + \frac{(c\mu_p - b)^2}{ac - b^2} \right). \quad (24)$$

Minimize σ_p^2 with respect to μ_p to derive global minimum variance of portfolio p .

The global minimum variance of portfolio p is given by:

$$(\sigma_p^*)^2 = \frac{1}{c}. \quad (25)$$

Clearly, the global minimum variance of portfolio p is obtained at $\mu_p = \frac{b}{c}$, [14, 15, 16, 17].

2.11 Complete portfolio

A complete portfolio of risk free asset and risky asset portfolio is defined as a combination of a risky asset portfolio and the risk-free asset, [18]. The return r_c of complete portfolio is defined by:

$$r_c = w_p r_p + (1 - w_p) r_f, \quad (26)$$

where r_p is risky asset portfolio return, r_f is risk free asset return and w_p is wight of risky asset portfolio.

2.12 Capital market line

Capital Market Line (CML) contains all optimal portfolios that are combination partners of the risk-free asset and the market portfolio, [19]. All of the portfolios on the CML have the same Sharpe ratio as that of the market portfolio, [19]. The tangency portfolio is the intersection point of the CML and the efficient frontier, [17, 20]. Let r_m be return of market portfolio. Then the return of tangency portfolio r_p is defined by:

$$r_p = w_p r_m + (1 - w_p) r_f, \quad (27)$$

where r_f is risk free asset return and w_p is wight of market portfolio. In this research, we relax the tangency portfolio condition $r_p = w_p r_m + (1 - w_p) r_f$ to order two opportunity cost condition $\rho r_p + (1 - \rho) r_m = w_p r_m + (1 - w_p) r_f$, where

$\rho \in [0, 1]$. Note that tangency portfolio condition is a special case of order two opportunity cost condition, choose $\rho = 1$.

We observe from literature review that portfolio selection models depend on investors' decisions and objectives. Tangency portfolio condition helps investors to choose efficient portfolio when investors would like to construct a complete portfolio of risk free asset and market index. However, there exist order two and three zero opportunity condition which are general conditions of Tangency portfolio condition. In summary, researchers study portfolio selection techniques based on zero opportunity cost condition as investors' decision.

3 Portfolio selection using zero opportunity cost

This research presents the mathematical modeling for portfolio selection based on zero opportunity cost. Hence there are many mathematical expressions used throughout this research. We have used a number symbols in this mathematical expressions which are given in Table 1.

Symbol	Value
a	$\mu^T \Sigma^{-1} \mu$
b	$\mu^T \Sigma^{-1} l$
c	$l^T \Sigma^{-1} l$
z	$\left(\frac{\sigma_m}{\mu_m - r_f} \right)^2$
x	$\frac{-w_p r_f + (w_p - 1) \mu_m}{\mu_m - r_f}$
x^*	$(1 - w_p) + \frac{r_f \mu_m}{(\mu_m - r_f)^2}$
s^*	$\sigma_m^2 (x^* + x)$
q^*	$\frac{-2\sigma_m^2 r_f}{(\mu_m - r_f)^2}$

Table 1. Symbols and their corresponding value

3.1 Frontier error minimization model (FEMM)

The following model is called frontier error minimization model:

$$FEMM : \min_w \left(w^T \Sigma w - \frac{1}{c} \left(1 + \frac{(c\mu^T w - b)^2}{ac - b^2} \right) \right)^2 \quad (28)$$

Subject to : $l^T w = 1$.

A portfolio that corresponds to FEMM is called frontier error (FE) portfolio.

Definition 3.2 (Efficiency condition). *The equation*

$$w^T \Sigma w = \frac{1}{c} \left(1 + \frac{(c\mu^T w - b)^2}{ac - b^2} \right) \quad (29)$$

is called efficiency condition.

3.3 Zero opportunity portfolio

Let r_f be the rate of return of risk free asset. Let r_m be return of market portfolio (index). Let μ_m be expected return of market portfolio. Let r_p be return of risky portfolio. Scenario

one portfolio (risk free fundamental portfolio) SP_1 is a portfolio with return $r_{SP_1} = w_p r_f + (1 - w_p) r_m$, $w_p \in [0, 1]$. Scenario two portfolio (risky fundamental portfolio) SP_2 is a portfolio with return $r_{SP_2} = \rho r_p + (1 - \rho) r_m$, $\rho \in [0, 1]$. Scenario three portfolio (free-risky fundamental portfolio) SP_3 is a portfolio with return $r_{SP_3} = \delta r_f + (1 - \delta) r_p$, $\delta \in [0, 1]$. SP_1 and SP_2 are said to be zero opportunity portfolios if $\mu_m \neq r_f$, $w_p \neq 0$, $\rho = w_p$ and $r_{SP_1} = r_{SP_2}$.

SP_1 and SP_2 are said to be equivalent portfolios of order two if $r_{SP_1} = r_{SP_2}$. SP_1, SP_2 and SP_3 are said to be equivalent portfolios of order three if $r_{SP_1} = r_{SP_2} = r_{SP_3}$. Consider equivalent portfolios of order two. There are two cases for equivalent portfolios of order two. The first case $\rho = w_p$. The second case for equivalent portfolios of order two is $\rho \neq w_p$. Let us begin with $\rho = w_p$ case. The following results are obtained by the help of $\rho = w_p$ case.

Theorem 3.4. Let α_i and β_i be two real numbers. Define

1. $\overline{r_{SP_j}} = E(r_{SP_j})$ for $j = 1, 2, 3$, where E is expectation,
2. $\sigma_{SP_j} = SD(r_{SP_j})$ for $j = 1, 2, 3$, where SD is standard deviation,
3. $\sigma_j = SD(r_j)$, where SD is standard deviation,
4. $\rho_{pm} = \text{corr}(r_p, r_m)$, where corr is correlation,
5. $h(w_p) = \sigma_m + \alpha_p w_p$,
6. $g(w_p) = \beta_p + (2\alpha_p \sigma_m + 2\sigma_m^2) w_p + (\alpha_p^2 - \sigma_m^2) w_p^2$.

If SP_1 and SP_2 are zero opportunity portfolios, then $2\rho_{im} w_p (1 - w_p) \sigma_m \sigma_p = -w_p^2 \sigma_p^2 + g(w_p)$.

Proof

Suppose that SP_1 and SP_2 are zero opportunity portfolios. Then

$$r_{SP_1} = w_p r_f + (1 - w_p) r_m, \quad (30)$$

$$r_{SP_2} = \rho r_p + (1 - \rho) r_m \quad (31)$$

and

$$w_p r_f + (1 - w_p) r_m = \rho r_p + (1 - \rho) r_m. \quad (32)$$

It follows from equation (32) that

$$\overline{r_{SP_1}} = \overline{r_{SP_2}} \quad (33)$$

and

$$\sigma_{SP_1} = \sigma_{SP_2}. \quad (34)$$

It follows from equation (30) that

$$\sigma_{SP_1} = (1 - w_p) \sigma_m. \quad (35)$$

Solve for w_p from equation (35) to get equation (36).

$$w_p = \frac{\sigma_m - \sigma_{SP_1}}{\sigma_m}. \quad (36)$$

Clearly the following equations hold.

$$\overline{r_{SP_1}} = w_p \mu_p + (1 - w_p) \mu_m. \quad (37)$$

$$\overline{r_{SP_2}} = w_p \mu_p + (1 - w_p) \mu_m. \quad (38)$$

Substitute equation (36) in to the equation (37) to get (39).

$$\overline{r_{SP_1}} = r_f + (\mu_m - r_f) \frac{\sigma_{SP_1}}{\sigma_m}, \quad (39)$$

Use equations (33), (38) and (39) to get equation (40).

$$r_f + (\mu_m - r_f) \frac{\sigma_{SP_1}}{\sigma_m} = w_p \mu_p + (1 - w_p) \mu_m. \quad (40)$$

It follows from equation (40) that

$$\mu_p = \frac{r_f + (\mu_m - r_f) \frac{\sigma_{SP_1}}{\sigma_m} + (w_p - 1) \mu_m}{w_p}. \quad (41)$$

It follows from equations (39) and (40) that

$$\overline{r_{SP_1}} = w_p \mu_p + (1 - w_p) \mu_m. \quad (42)$$

Note that market portfolio expected return and standard deviation are constant numbers, whereas weight w_p and σ_p are variables. Clearly,

$$\sigma_{SP_2} = \sqrt{w_p^2 \sigma_p^2 + (w_p - 1)^2 \sigma_m^2 + 2w_p(1 - w_p) \rho_{pm} \sigma_p \sigma_m}. \quad (43)$$

Since $\mu_p = r_f$, $\frac{\partial \mu_p}{\partial w_p} = 0$ and $\frac{\partial \mu_p}{\partial \sigma_p} = 0$. Use $\sigma_{SP_1} = \sigma_{SP_2}$ and equation (43) to find equation (44).

$$\sigma_{SP_1} = \sqrt{w_p^2 \sigma_p^2 + (w_p - 1)^2 \sigma_m^2 + 2w_p(1 - w_p) \rho_{pm} \sigma_p \sigma_m}. \quad (44)$$

Solve for w_p from equation (37) to get equation (45)

$$w_p = \frac{\overline{r_{SP_1}} - \mu_m}{r_f - \mu_m}. \quad (45)$$

Substitute equation (42) in to the equation (45) to find equation (46).

$$w_p = \frac{w_p \mu_p + (1 - w_p) \mu_m - \mu_m}{r_f - \mu_m}. \quad (46)$$

Substitute equation (46) in to the equation (35) to get equation (47).

$$\sigma_{SP_1} = \frac{r_f - w_p \mu_p + (w_p - 1) \mu_m}{r_f - \mu_m} \sigma_m. \quad (47)$$

Not that $\frac{\partial(w_p \mu_p)}{\partial w_p} = \frac{\partial(w_p r_f)}{\partial w_p} = r_f = \mu_p$.

Differentiate both sides of equation (47) with respect to w_p to get equation (48).

$$\mu_p = (\mu_m - r_f) \frac{\frac{\partial \sigma_{SP_1}}{\partial w_p}}{\sigma_m} + \mu_m. \quad (48)$$

$$\frac{\partial \sigma_{SP_1}}{\partial \sigma_p} = 0. \quad (49)$$

It follows from equation (49) that

$$\sigma_{SP_1} = h(w_p). \quad (50)$$

It follows from equations (41), (48) and (50) that

$$\frac{h(w_p)}{\sigma_m} - w_p \frac{\frac{dh(w_p)}{dw_p}}{\sigma_m} = 1. \quad (51)$$

We also note the following equations:

$$2w_p(1 - w_p)\rho_{im}\sigma_p\sigma_m = -w_p^2\sigma_p^2 + g(w_p) \quad (52)$$

$$\frac{dh(w_p)}{dw_p} = \frac{1}{2h(w_p)} (2w_p\sigma_p^2 + 2(w_p - 1)\sigma_m^2 + u), \quad (53)$$

where $u = \frac{\partial(2w_p(1-w_p)\rho_{pm}\sigma_p\sigma_m)}{\partial w_p}$. It follows from equations (52) and (53) that

$$\frac{dg(w_p)}{dw_p} = 2h(w_p) \frac{dh(w_p)}{dw_p} + 2(1 - w_p)\sigma_m^2. \quad (54)$$

Solve for h from equation (51). Substitute h in to equation (54) and solve for g from equation (54). Hence proved.

Theorem 3.5. *If SP_1 and SP_2 are zero opportunity portfolios, then*

1. $\alpha_p = \frac{\mu_p - \mu_m}{\mu_m - r_f} \sigma_m$,
2. $g(w_p) = (2\alpha_p\sigma_m + 2\sigma_m^2)w_p + (\alpha_p^2 - \sigma_m^2)w_p^2$,
3. $2\rho_{pm}w_p(1 - w_p)\sigma_m\sigma_p = -w_p^2\sigma_p^2 + g(w_p)$.

Proof. Clearly, α_p result follows from equation (48). Now solve for σ_{SP_1} from equation (41). Thus, the following result follows:

$$\sigma_{SP_1} = \sigma_m + w_p\alpha_p. \quad (55)$$

It follows from theorem 3.4 that

$$\sigma_{SP_1}^2 = (1 - w_p)^2\sigma_m^2 + g(w_p). \quad (56)$$

The value of β is zero from equations (55) and (56). Hence the proof followed. $\square$

Lemma 3.6. *If risky and risk free fundamental portfolios have equal Sharpe ratio, then*

$$\sigma_{SP_2} = \frac{\overline{r_{SP_2}} - r_f}{\mu_m - r_f} \sigma_m. \quad (57)$$

Proof. Suppose that risky and risk free fundamental portfolios have equal Sharpe ratio. Clearly,

$$w_p = \frac{\overline{r_{SP_2}} - \mu_m}{\mu_p - \mu_m}. \quad (58)$$

Since both risky and risk free fundamental portfolios have equal Sharpe ratio, we get

$$\frac{\mu_m - r_f}{\sigma_m} \sigma_{SP_2} = w_p(\mu_p - \mu_m) + \mu_m - r_f. \quad (59)$$

Solve for w_p from equation (59). Thus, the following equation follows.

$$w_p = \frac{(\sigma_{SP_2} - \sigma_m)(\mu_m - r_f)}{(\mu_p - \mu_m)\sigma_m}. \quad (60)$$

Use equations (58) and (60) to get equation (61).

$$\frac{(\sigma_{SP_2} - \sigma_m)(\mu_m - r_f)}{(\mu_p - \mu_m)\sigma_m} = \frac{\overline{r_{SP_2}} - \mu_m}{\mu_p - \mu_m}. \quad (61)$$

It follows from equation (61) that

$$\sigma_{SP_2} = \frac{\overline{r_{SP_2}} - r_f}{\mu_m - r_f} \sigma_m. \quad (62)$$

Hence the proof followed. $\square$

Corollary 3.7. *If SP_1 and SP_2 are zero opportunity portfolios, then*

1. $\alpha_p = \frac{\mu_p - \mu_m}{\mu_m - r_f} \sigma_m$,
2. $g(w_p) = (2\alpha_p\sigma_m + 2\sigma_m^2)w_p + (\alpha_p^2 - \sigma_m^2)w_p^2$,
3. $\sigma_{SP_2}^2 = (1 - w_p)^2\sigma_m^2 + g(w_p)$.

Proof. The result of this theorem follows from Theorem 3.5. $\square$

Theorem 3.8. *If SP_1 and SP_2 are zero opportunity portfolios, then*

$$\sigma_{SP_2}^2 = z\overline{r_{SP_2}}^2 + q^*\overline{r_{SP_2}} + s^*. \quad (63)$$

Proof. Suppose that SP_1 and SP_2 are zero opportunity portfolios. Then

$$\overline{r_{SP_2}} = w_p\mu_p + (1 - w_p)\mu_m \quad (64)$$

This implies that

$$\mu_p = \frac{\overline{r_{SP_2}} + (w_p - 1)\mu_m}{w_p}. \quad (65)$$

Clearly,

$$\alpha_p = \frac{\mu_p - \mu_m}{\mu_m - r_f} \sigma_m. \quad (66)$$

It follows from equations (65) and (66) that

$$\alpha_p = \frac{\sigma_m}{w_p} \left(\frac{\overline{r_{SP_2}} - \mu_m}{\mu_m - r_f} \right). \quad (67)$$

Consider the following equation.

$$\begin{aligned} (2\alpha_p\sigma_m + 2\sigma_m^2)w_p &= \left(2\frac{\sigma_m}{w_p} \left(\frac{\overline{r_{SP_2}} - \mu_m}{\mu_m - r_f} \right) \sigma_m + 2\sigma_m^2 \right) w_p \\ &= 2\sigma_m \left(\frac{\overline{r_{SP_2}} - \mu_m}{\mu_m - r_f} \right) \sigma_m + 2\sigma_m^2 w_p \\ &= 2\sigma_m^2 \left(\left(\frac{\overline{r_{SP_2}} - \mu_m}{\mu_m - r_f} \right) + w_p \right) \\ &= 2\sigma_m^2 \left(\frac{\overline{r_{SP_2}} + (w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} \right) \\ &= 2\sigma_m^2 \left(\frac{\overline{r_{SP_2}}}{\mu_m - r_f} + x \right) \\ &= (2\mu_m - 2r_f) \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 \overline{r_{SP_2}} \\ &\quad + 2\sigma_m^2 x. \end{aligned} \quad (68)$$

Let us consider the following equation.

$$\begin{aligned}
 (\alpha_p^2 - \sigma_m^2)w_p^2 &= \left(\frac{\sigma_m^2}{w_p^2} \left(\frac{\overline{rSP_2} - \mu_m}{\mu_m - r_f} \right)^2 - \sigma_m^2 \right) w_p^2 \\
 &= \sigma_m^2 \left(\frac{\overline{rSP_2} - \mu_m}{\mu_m - r_f} \right)^2 - \sigma_m^2 w_p^2 \\
 &= \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 (\overline{rSP_2} - \mu_m)^2 - \sigma_m^2 w_p^2 \\
 &= \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 (\overline{rSP_2}^2 - 2\mu_m \overline{rSP_2} + \mu_m^2) \\
 &\quad - \sigma_m^2 w_p^2. \tag{69}
 \end{aligned}$$

It follows from corollary 3.7 that

$$\sigma_{SP_2}^2 = (1 - w_p)^2 \sigma_m^2 + g(w_p). \tag{70}$$

Substitute equations (68) and (69) in to the equation (70) to get (71).

$$\begin{aligned}
 \sigma_{SP_2}^2 &= (1 - w_p)^2 \sigma_m^2 + -2r_f \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 \overline{rSP_2} + 2\sigma_m^2 x + \\
 &\quad \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 (\overline{rSP_2}^2 + \mu_m^2) - \sigma_m^2 w_p^2. \tag{71}
 \end{aligned}$$

Define $d_u = \sigma_m^2 x + \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 \mu_m^2 - \sigma_m^2 w_p^2$. Consider the following equation.

$$\begin{aligned}
 d_u &= \sigma_m^2 \frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} \\
 &\quad + \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 \mu_m^2 - \sigma_m^2 w_p^2. \tag{72}
 \end{aligned}$$

This implies that

$$\begin{aligned}
 d_u &= \sigma_m^2 \frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} + \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 \mu_m^2 - \sigma_m^2 w_p^2 \\
 &= \sigma_m^2 \left(\frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} \right) + \sigma_m^2 \left(\left(\frac{\mu_m}{\mu_m - r_f} \right)^2 - w_p^2 \right) \\
 &= \sigma_m^2 \left(\frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} + \left(\frac{\mu_m}{\mu_m - r_f} - w_p \right) \left(\frac{\mu_m}{\mu_m - r_f} + w_p \right) \right) \\
 &= \sigma_m^2 \left(\frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} \right) \left(1 - \left(\frac{\mu_m}{\mu_m - r_f} + w_p \right) \right) \\
 &= \sigma_m^2 \left(\frac{(w_p - 1)\mu_m - w_p r_f}{\mu_m - r_f} \right) \left(\frac{-w_p \mu_m - (1 - w_p)r_f}{\mu_m - r_f} \right) \\
 &= \sigma_m^2 \left(\frac{(1 - w_p)\mu_m + w_p r_f}{\mu_m - r_f} \right) \left(\frac{w_p \mu_m + (1 - w_p)r_f}{\mu_m - r_f} \right) \\
 &= \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2 ((1 - w_p)\mu_m + w_p r_f)(w_p \mu_m + (1 - w_p)r_f). \tag{73}
 \end{aligned}$$

Define $d_l = ((1 - w_p)\mu_m + w_p r_f)(w_p \mu_m + (1 - w_p)r_f)$. Note that

$$\begin{aligned}
 d_l &= (w_p - w_p^2)(r_f^2 + \mu_m^2) + (1 - 2w_p + 2w_p^2)r_f \mu_m \\
 &= (w_p - w_p^2)(\mu_m - r_f)^2 + r_f \mu_m. \tag{74}
 \end{aligned}$$

It follows from equations (73) and (74) that

$$d_u = (w_p - w_p^2)\sigma_m^2 + r_f \mu_m \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2. \tag{75}$$

Substitute equation (75) in to the equation (71) to get equation (76).

$$\sigma_{SP_2}^2 = (1 - w_p)\sigma_m^2 + q^* \overline{rSP_2} + \sigma_m^2 x + z \overline{rSP_2}^2 + r_f \mu_m \left(\frac{\sigma_m}{\mu_m - r_f} \right)^2. \tag{76}$$

This implies that

$$\sigma_{SP_2}^2 = z \overline{rSP_2}^2 + q^* \overline{rSP_2} + s^*. \tag{77}$$

Hence the proof followed. $\square$

3.9 Sharpe square error model (SSEM)

We define zero Sharpe square error $ZSSE$ by:

$$ZSSE = \left(\sigma_{SP_2} - \frac{\overline{rSP_2} - r_f}{\mu_m - r_f} \sigma_m \right)^2. \tag{78}$$

Investors who prefer to construct efficient portfolio using zero Sharpe square error would like to solve Sharpe square error model. Sharpe square error model is given by:

$$SSEM : \min_{w_p, w} \left(\sigma_{SP_2} - \frac{\overline{rSP_2} - r_f}{\mu_m - r_f} \sigma_m \right)^2$$

Subject to : $l^T w = 1,$

$$0 \leq w_p, w_i \leq 1 \quad \forall i = 1, 2, \dots, n. \tag{79}$$

A portfolio that corresponds to $SSEM$ is called Sharpe square error (SSE) portfolio.

Definition 3.10 (Sharpe ratio error condition). *The equation*

$$\sigma_{SP_2} = \frac{\overline{rSP_2} - r_f}{\mu_m - r_f} \sigma_m \tag{80}$$

is called Sharpe ratio error condition.

3.11 Return neutral portfolio

Use return neutral model to construct return neutral portfolio. Clearly, $\vec{w} = \frac{\Sigma^{-1}l}{c}$.

3.12 Sharpe ratio efficient opportunity portfolio

Sharpe ratio efficient opportunity model ($SREOM$) is an optimization model with Sharpe ratio objective subject to efficient-opportunity constraint and unit weight sum constraint. Sharpe ratio efficient opportunity model ($SREOM$) is given by:

$$SREOM : \max_w \frac{\mu_p - r_f}{\sigma_p}$$

$$\text{Subject to : } \sigma_p^2 = \frac{1}{c} \left(1 + \frac{(c\mu_p - b)^2}{ac - b^2} \right),$$

$$l^T w = 1,$$

$$0 \leq w_i \leq 1 \quad \forall i = 1, 2, \dots, n. \tag{81}$$

Sharpe ratio efficient opportunity ($SREO$) portfolio is a portfolio with optimal weight vector of $SREOM$.

3.13 Equally weighted portfolio

Investors select equally weighted portfolio to keep diversification. However, equally weight portfolio may or may not be optimal portfolio. Let P_{ew} be equally weighted portfolio. Then expected return and risk of P_{ew} are given by:

$$\mu_{ew} = \frac{l^T \mu}{n} \text{ and } \sigma_{ew} = \frac{\sqrt{l^T \Sigma l}}{n}. \text{ Thus, Sharpe ratio of equally weighted portfolio is defined by:}$$

$$SR_{ew} = \frac{l^T (\mu - r_f)}{\sqrt{l^T \Sigma l}}.$$

3.14 Order three complete portfolio

The return of order three complete portfolio P_{3c} is defined by $r_{3c} = \frac{w_p + \delta}{3} r_f + \frac{\rho + 1 - \delta}{3} r_m + \frac{2 - \rho - w_p}{3} r_p$. Without loss of generality, let us assume that w_p is given. Then one can find δ using weight-variance regression model and ρ using order three weight equation. Therefore, for a given w_p there exist δ and ρ which satisfy order three weight equation.

4 Weight-variance regression model

The regression model $\ln(\sigma_p) = \alpha + \beta \sigma_{pm}^2 + \epsilon_p$ is called weight-variance regression model. We use feasible generalized least square *FGLS* regression method to estimate the parameters α and β if variance of residual square is not constant. Solve for optimal solution $\frac{1-\delta}{1-w_p}$ of $\min \left(\ln(\sigma_m) + \ln\left(\frac{1-\delta}{1-w_p}\right) - \hat{\alpha} - \hat{\beta} \sigma_m^4 \left(\frac{1-\delta}{1-w_p}\right)^2 \right)^2$. Therefore, $\frac{1-\delta}{1-w_p}$ can be estimated by the help of weight-variance regression model. Note that $\frac{1-\delta}{1-w_p} = \frac{1}{\sigma_m^2 \sqrt{2\beta}}$. Suppose that there is no opportunity cost between the two fundamental portfolios. That is, fundamental portfolios have the same level of risk and expected return. This assumption is correct because in finance there is no free lunch or arbitrage opportunity. Define expected return of risky fundamental portfolio by $\mu_c = w_p \mu_p + (1 - w_p) \mu_m$. Similarly, define risk of risky fundamental portfolio by $\sigma_c^2 = w_p^2 \sigma_p^2 + (1 - w_p)^2 \sigma_m^2 + 2w_p(1 - w_p) \sigma_{pm}$. Notation: $\mu_p = \mu^T w$ and $\sigma_p^2 = w^T \Sigma w$.

Now one can build a model based on efficiency condition, Sharpe ratio error condition and weight-variance regression model. Define $\sigma_{pm} = (\sigma_{1m}, \sigma_{2m}, \dots, \sigma_{nm})^T$. Therefore, we have the following Sharpe variance model (*SVM*).

$$\begin{aligned} SVM : \quad & \min_{w, w_p} \sigma_c^2 \\ \text{Subject to: } & \sigma_c = \sqrt{z}(\mu_c - r_f), \\ & \ln(\sigma_p) = \hat{\alpha} + \hat{\beta} \sigma_{pm}^2, \\ & l^T w = 1, \\ & 0 \leq w_p, w_i \leq 1 \quad \forall i = 1, 2, \dots, n. \end{aligned} \quad (82)$$

SV is called Sharpe variance portfolio.

5 Treynor-Sharpe regression model

Define Treynor ratio tr_i of security i by $tr_i = \frac{\mu_i - r_f - \beta_i(\mu_m - r_f)}{\beta_i}$. Similarly, define Sharpe ratio sr_i of security i by $sr_i = \frac{\mu_i - r_f}{\sigma_i}$. Then the model

$$tr_i = a + b sr_i + \epsilon_i \quad (83)$$

is called Treynor-Sharpe regression model. The line $tr_i = \hat{a} + \hat{b} sr_i$ called Treynor-Sharpe regression line. We have seen the case $\rho = w_p$ for equivalent portfolios of order two. Let us consider the opposite case of $\rho = w_p$. Suppose that $w_p r_f + (1 - w_p) r_m = \rho r_p + (1 - \rho) r_m$ and $\rho \neq w_p$.

Let us consider three investment portfolio construction scenarios. Let P_{fr} , P_{mr} and P_{fmr} denote free-risky portfolio, market-risky portfolio and free-market-risky portfolio, respectively. Consider the following return condition of these portfolios. Suppose that $n = w_p r_f + (1 - w_p) r_p = \rho r_m + (1 - \rho) r_p = \delta r_f + (1 - \delta) r_m$. The condition $w_p r_f + (1 - w_p) r_p = \rho r_m + (1 - \rho) r_p = \delta r_f + (1 - \delta) r_m$ is called order three opportunity condition.

Lemma 5.1. *If order three opportunity condition is valid, then $\rho = \frac{(1-\delta)w_p}{w_p - \delta}$. This equation is called order three weight equation.*

Proof. Suppose that order three opportunity condition is valid. Then $r_p = \frac{n - w_p r_f}{1 - w_p}$. This implies that $n = \rho r_m + (1 - \rho) r_p = \rho r_m + (1 - \rho) \frac{n - w_p r_f}{1 - w_p}$. Solve for n . Thus, $\frac{\rho(1 - w_p) r_m + (\rho - 1) w_p r_f}{\rho - w_p} = \delta r_f + (1 - \delta) r_m$. Solve for ρ and express it in terms of w_p and δ . Hence the proof followed. $\square$

Lemma 5.2. *If order three opportunity condition is valid, then $(1 - w_p) \sigma_p^2 - (1 - \delta) \sigma_{pm}^2 = 0$. This equation is called weight-variance equation.*

Proof. Suppose that order three opportunity condition is valid. Then $\rho^2 \sigma_m^2 + (1 - \rho)^2 \sigma_p^2 + 2(1 - \rho) \rho \sigma_{pm} = (1 - \delta)^2 \sigma_m^2$ and $\sigma_p^2 = \left(\frac{1 - \delta}{1 - w_p}\right)^2 \sigma_m^2$. Use $\rho = \frac{(1 - \delta) w_p}{w_p - \delta}$ to solve for σ_{pm} . Thus, $\sigma_{pm} = \frac{1 - \delta}{1 - w_p} \sigma_m^2$. Hence the proof followed. $\square$

Lemma 5.3. *If order three opportunity condition holds, then $(w_p - \rho) \frac{1 - \delta}{1 - w_p} = -\rho$.*

Proof. Suppose that order three opportunity condition holds. Then $(w_p - \rho) \frac{1 - \delta}{1 - w_p} = (w_p - \frac{(1 - \delta) w_p}{w_p - \delta}) \frac{1 - \delta}{1 - w_p}$. This implies that $(w_p - \rho) \frac{1 - \delta}{1 - w_p} = -\frac{(1 - \delta) w_p}{w_p - \delta} = -\rho$. Hence the proof followed. $\square$

Lemma 5.4. *If order three opportunity condition holds, then both risky portfolio p and market portfolio m have equal Sharpe ratio.*

Proof. Suppose that order three opportunity condition holds. Then $\mu_p = \frac{w_p r_f - \rho \mu_m}{w_p - \rho}$. This implies that $\mu_p - r_f = \frac{w_p r_f - \rho \mu_m}{w_p - \rho} - r_f = \rho \frac{r_f - \mu_m}{w_p - \rho}$. Clearly, $\sigma_p = \frac{(1 - \delta)}{1 - w_p} \sigma_m$. This implies that $\frac{\mu_p - r_f}{\sigma_p} = \rho \frac{r_f - \mu_m}{(w_p - \rho) \frac{(1 - \delta)}{1 - w_p} \sigma_m}$. Thus, $\frac{\mu_p - r_f}{\sigma_p} = \frac{\mu_m - r_f}{\sigma_m}$. Hence the proof followed. $\square$

Lemma 5.5. *If order three opportunity condition holds, then both risky portfolio p and market portfolio m have equal Treynor ratio.*

Proof. Suppose that order three opportunity condition holds. Let t_p and t_m be Treynor ratio of portfolio p and market portfolio m , respectively. Clearly, $t_m = 0$. Claim: To show that $t_p = 0$. $t_p = \sigma_m^2 \left(\frac{\mu_p - r_f}{\sigma_{pm}} \right) + (r_f - \mu_m)$. Clearly, $\sigma_{pm} = \frac{(1-\delta)}{1-w_p} \sigma_m^2$. This implies that $t_p = \frac{1-w_p}{1-\delta} (\mu_p - r_f) + (r_f - \mu_m)$. Since $\mu_p - r_f = \rho \frac{r_f - \mu_m}{w_p - \rho}$, $t_p = (-r_f + \mu_m) + (r_f - \mu_m) = 0$. Hence the proof followed. $\square$

Theorem 5.6. *If order three opportunity condition holds, then both order three complete portfolio P_{3c} and market portfolio m have equal Treynor ratio.*

Proof. Suppose that order three opportunity condition holds. Then $r_{3c} = \rho r_m + (1-\rho)r_p$. Define Treynor ratio t_{3c} of order three opportunity complete portfolio by:

$$t_{3c} = \frac{\alpha_{3c}}{\beta_{3c}} = \frac{\mu_{3c} - r_f - \beta_{3c}(\mu_m - r_f)}{\beta_{3c}}. \quad (84)$$

Similarly, Treynor ratio t_p of risky portfolio is defined by:

$$t_p = \frac{\alpha_p}{\beta_p} = \frac{\mu_p - r_f - \beta_p(\mu_m - r_f)}{\beta_p}. \quad (85)$$

It follows from equations (84) and (85) that

$$t_p = \frac{\mu_p - r_f}{\beta_p} + t_{3c} - \left(\frac{(1-\rho)\mu_p + \rho\mu_m - r_f}{(1-\rho)\beta_p + \rho} \right) \quad (86)$$

Consider the following result.

$$\begin{aligned} & \frac{\mu_p - r_f}{\beta_p} - \left(\frac{(1-\rho)\mu_p + \rho\mu_m - r_f}{(1-\rho)\beta_p + \rho} \right) \\ &= \frac{(\mu_p - r_f)((1-\rho)\beta_p + \rho) - \beta_p((1-\rho)\mu_p + \rho\mu_m - r_f)}{\beta_p((1-\rho)\beta_p + \rho)} \\ &= \rho \left(\frac{\mu_p - r_f - \beta_p(\mu_m - r_f)}{\beta_p((1-\rho)\beta_p + \rho)} \right) \\ &= \left(\frac{\rho}{(1-\rho)\beta_p + \rho} \right) t_p. \end{aligned}$$

This implies that

$$\frac{\mu_p - r_f}{\beta_p} - \left(\frac{(1-\rho)\mu_p + \rho\mu_m - r_f}{(1-\rho)\beta_p + \rho} \right) = \left(\frac{\rho}{(1-\rho)\beta_p + \rho} \right) t_p. \quad (87)$$

Substitute the equation (87) in to equation (86) to get the equation

$$t_p = \left(\frac{\rho}{(1-\rho)\beta_p + \rho} \right) t_p + t_{3c}. \quad (88)$$

Clearly,

$$t_{3c} = \left(\frac{1-\rho}{(1-\rho)\beta_p + \rho} \right) t_p. \quad (89)$$

Since $t_p = 0$, we get $t_{3c} = 0$. Hence the proof followed. $\square$

Theorem 5.7. *Let s_{3c} be Sharpe ratio of order three complete portfolio P_{3c} . Suppose that $0 < w_p < 1$. If order three opportunity condition holds, then $s_{3c} = s_p$.*

Suppose that order three opportunity condition holds. Then $\sigma_p^2 = \left(\frac{1-\delta}{1-w_p} \right)^2 \sigma_m^2$ and $\sigma_{pm} = \frac{1-\delta}{1-w_p} \sigma_m^2$. Consider the ratio of complete and risky portfolios Sharpe ratios. Suppose that $\sigma_{3c}^2 \neq 0$. Define g_p and s_p by: $g_p = \left(\frac{(1-\rho)\mu_p + \rho\mu_m - r_f}{\mu_p - r_f} \right)$ and $s_p = \sqrt{\frac{\sigma_p^2}{(1-\rho)^2 \sigma_p^2 + \rho^2 \sigma_m^2 + 2\rho(1-\rho)\sigma_{pm}}} = \frac{1}{1-w_p}$. Clearly, $\frac{s_{3c}}{s_p} = g_p s_p$ and $(1-\rho)\mu_p + \rho\mu_m - r_f = (1-\delta)(\mu_m - r_f)$. Note that $\mu_p - r_f = -\rho \frac{\mu_m - r_f}{w_p - \rho}$. It is obvious that $s_{3c} = s_p$. Hence the proof followed.

Theorem 5.8. *Suppose that $\rho \neq w_p$. $w_p r_f + (1-w_p)r_m = \rho r_p + (1-\rho)r_m$ if and only if $r_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) r_m$.*

Proof. Suppose that $w_p r_f + (1-w_p)r_m = \rho r_p + (1-\rho)r_m$ and $\rho \neq w_p$. Set $r_h = w_p r_f + (1-w_p)r_m$. First solve for r_h in terms of r_h . Thus, $r_m = \frac{r_h - w_p r_f}{1-w_p} = \frac{r_h - \rho r_p}{1-\rho}$. Solve for r_h from $\frac{r_h - w_p r_f}{1-w_p} = \frac{r_h - \rho r_p}{1-\rho}$. Clearly, $r_h = \frac{w_p(1-\rho)r_f + (w_p-1)\rho r_p}{w_p - \rho}$. This implies that $w_p r_f + (1-w_p)r_m = \frac{w_p(1-\rho)r_f + (w_p-1)\rho r_p}{w_p - \rho}$. Now solve for r_p . Hence the result follows immediately. Conversely, suppose that $r_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) r_m$. Then $\rho r_p + (1-\rho)r_m = \rho \left(\frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) r_m \right) + (1-\rho)r_m$. The result follows immediately. Hence the proof followed. $\square$

Theorem 5.9 (Fundamental theorem of order two zero investment opportunity). *Suppose that a point (sr_p, tr_p) lies on Treynor-Sharpe regression line and $\mu_m \neq r_f$. If $w_p r_f + (1-w_p)r_m = \rho r_p + (1-\rho)r_m$, then $w_p = \rho$.*

Proof. Suppose that the a point (sr_p, tr_p) lies on Treynor-Sharpe regression line and $\mu_m \neq r_f$. Then $tr_p = \hat{a} + \hat{b}sr_p$. Apply proof by contradiction method to prove this theorem. Suppose $w_p \neq \rho$. Then $w_p > \rho$ or $w_p < \rho$. Case one: Suppose $w_p > \rho$. It follows from theorem 5.8 that $r_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) r_m$. This implies that $\beta_p = (1 - \frac{w_p}{\rho})$, $\mu_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) \mu_m$ and $\sigma_p = (\frac{w_p}{\rho} - 1) \sigma_m$. Clearly, $tr_p = 0$. Thus, $\mu_m = r_f$. This contradicts with the given fact $\mu_m \neq r_f$. Therefore, $w_p > \rho$ is false. Case two: Suppose $w_p < \rho$. Similarly, it follows from theorem 5.8 that $r_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) r_m$. This implies that $\beta_p = (1 - \frac{w_p}{\rho})$, $\mu_p = \frac{w_p}{\rho} r_f + (1 - \frac{w_p}{\rho}) \mu_m$ and $\sigma_p = (1 - \frac{w_p}{\rho}) \sigma_m$. Again $tr_p = 0$. Thus, $\mu_m = r_f$. This contradicts with the given fact $\mu_m \neq r_f$. Therefore, $w_p < \rho$ is false. This shows that $w_p = \rho$. Hence the proof followed. $\square$

6 General order three opportunity model

Use weight-variance regression model to estimate $\frac{1-\delta}{1-w_p}$.

Thus, $(1-\delta)\sigma_m^2 \sqrt{2 \frac{\text{cov}(\ln(\sigma_p), \sigma_{pm}^2)}{\text{var}(\sigma_{pm}^2)}} = (1-w_p)$. Define $\hat{u} = \sigma_m^2 \sqrt{2 \frac{\text{cov}(\ln(\sigma_p), \sigma_{pm}^2)}{\text{var}(\sigma_{pm}^2)}}$. Clearly, $(1-\delta)\hat{u} = (1-w_p)$.

Theorem 6.1. *The following equations are necessary conditions for order three opportunity condition.*

$$1. \beta_p = \frac{1-\delta}{1-w_p},$$

2. $\mu_p - r_f = \beta_p(\mu_m - r_f)$,
3. $\ln(\sigma_p) \approx \hat{\alpha} + \hat{\beta}\sigma_{pm}^2$.

The following model is called general order three opportunity model (*GOOM*).

$$\begin{aligned} \text{GOOM : } & \min_w \sigma_p^2 \\ \text{Subject to : } & \\ & \beta_p = \frac{1}{\sigma_m^2 \sqrt{2\hat{\beta}}}, \\ & \mu_p - r_f = \beta_p(\mu_m - r_f), \\ & \ln(\sigma_p) = \hat{\alpha} + \hat{\beta}\sigma_{pm}^2, \\ & l^T w = 1, \\ & 0 \leq w_i \leq 1 \quad \forall i = 1, 2, \dots, n. \end{aligned} \quad (90)$$

Order three (*OTH*) portfolio is a portfolio with optimal weight vector of *GOOM*.

7 Order three asset allocation model

$$\text{Define } d_b = \frac{1}{c} \left(1 + \frac{\left(c \left(\frac{\delta(\mu_m - r_f) - \mu_m + w_p r_f}{w_p - 1} \right) - b \right)^2}{ac - b^2} \right).$$

Definition 7.1 (Order three efficiency condition). *The equation*

$$\left(\frac{1 - \delta}{1 - w_p} \right)^2 \sigma_m^2 = d_b \quad (91)$$

is called order three efficiency condition.

$$\text{Define } d_k = \left(w_p^2 \left(\frac{1 - \delta}{1 - w_p} \right)^2 + (1 - w_p)^2 + 2w_p(1 - \delta) \right).$$

Definition 7.2 (Order three Sharpe ratio error condition). *The equation*

$$d_k = \left(w_p^2 \left(\frac{1 - \delta}{1 - w_p} \right)^2 + (1 - w_p)^2 + 2w_p(1 - \delta) \right) \quad (92)$$

is called order three Sharpe ratio error condition.

Theorem 7.3. *If efficiency condition, Sharpe ratio error condition and order three zero opportunity condition hold, then order three efficiency condition and order three Sharpe ratio error condition hold.*

Proof. Suppose that efficiency condition, Sharpe ratio error condition and order three zero opportunity condition hold. Then

$$w^T \Sigma w = \frac{1}{c} \left(1 + \frac{(c\mu^T w - b)^2}{ac - b^2} \right) \quad (93)$$

and

$$\sigma_{SP_2} = \frac{\overline{r_{SP_2}} - r_f}{\mu_m - r_f} \sigma_m. \quad (94)$$

Clearly, the following equation follows from order three zero opportunity condition.

$$\delta = \left(\frac{\mu_p - r_f}{\mu_m - r_f} \right) w_p - \left(\frac{\mu_p - \mu_m}{\mu_m - r_f} \right). \quad (95)$$

Solve for μ_p from equation (95) to find equation (96).

$$\mu_p = \frac{\delta(\mu_m - r_f) - \mu_m + w_p r_f}{w_p - 1}. \quad (96)$$

Clearly,

$$\sigma_{SP_2}^2 = w_p^2 \sigma_p^2 + (1 - w_p)^2 \sigma_m^2 + 2w_p(1 - w_p) \sigma_{pm}. \quad (97)$$

It follows from order three zero opportunity condition that

$$\sigma_p^2 = \left(\frac{1 - \delta}{1 - w_p} \right)^2 \sigma_m^2 \quad (98)$$

and

$$\sigma_{pm} = \frac{1 - \delta}{1 - w_p} \sigma_m^2. \quad (99)$$

Substitute equations (98) and (99) in to the equation (97) to get equation (100).

$$\sigma_{SP_2}^2 = \sigma_m^2 \left(w_p^2 \left(\frac{1 - \delta}{1 - w_p} \right)^2 + (1 - w_p)^2 + 2w_p(1 - \delta) \right). \quad (100)$$

Consider the following equation.

$$\begin{aligned} \overline{r_{SP_2}} - r_f &= w_p \mu_p + (1 - w_p) \mu_m - r_f \\ &= w_p \left(\frac{\delta(\mu_m - r_f) - \mu_m + w_p r_f}{w_p - 1} \right) + (1 - w_p) \mu_m \\ &\quad - r_f \\ &= \left(\frac{(1 + \delta)w_p - 1 - w_p^2}{w_p - 1} \right) (\mu_m - r_f). \end{aligned} \quad (101)$$

Substitute equation (101) in to the equation (94) to get equation (102).

$$\sigma_{SP_2}^2 = \sigma_m^2 \left(\frac{(1 + \delta)w_p - 1 - w_p^2}{w_p - 1} \right)^2. \quad (102)$$

It follows from equations (100) and (102) that

$$\left(\frac{(1 + \delta)w_p - 1 - w_p^2}{w_p - 1} \right)^2 = d_k. \quad (103)$$

Substitute equations (96) and (98) in to the equation (93) to get equation (104)

$$\left(\frac{1 - \delta}{1 - w_p} \right)^2 \sigma_m^2 = \frac{1}{c} \left(1 + \frac{\left(c \left(\frac{\delta(\mu_m - r_f) - \mu_m + w_p r_f}{w_p - 1} \right) - b \right)^2}{ac - b^2} \right) \quad (104)$$

Hence the proof followed. $\square$

Order three asset allocation model (*OAAM*) is given by:

$$\text{OAAM : } \min_{w_p, \delta, \rho} (\rho(w_p - \delta) + (\delta - 1)w_p)^2$$

$$\text{Subject to : } \left(\frac{1 - \delta}{1 - w_p} \right)^2 \sigma_m^2 = d_b,$$

$$\left(\frac{(1 + \delta)w_p - 1 - w_p^2}{w_p - 1} \right)^2 = d_k,$$

$$0 \leq w_p, \delta, \rho \leq 1.$$

Figure 1. Scatter plot of logarithm of standard deviation against square of co-variance

As shown in figure 1, there is linear relation ship between natural logarithm of risky portfolio standard deviation and square of co-variance between risky and market portfolio. Both white and alternative white test hetroscedasticity problem testing methods show that there is no statistically significant hetroscedasticity problem. Therefore, we can apply *OLS* to estimate weight-variance model parameters. The required weight-variance model equation is given by:

$$\ln(\sigma_p) = -4.031 + 2.1206e + 06\sigma_{pm}^2. \quad (105)$$

The next step is estimating the value of $\frac{1-\delta}{1-w} = \frac{1}{\sigma_m^2 \sqrt{2\hat{\beta}}}$. Use $\hat{\beta} = 2.121(10^6)$ and $\sigma_m = 0.02145974$ to find $\frac{1-\delta}{1-w}$. Hence $\frac{1-\delta}{1-w} = 1.054398$. Let us consider basic portfolios and their optimal weight vector. In practice, investors use long selling assuming that short selling is not allowed. Thus, we assume weights of securities as non-negative. Treynor ratio maximization model (*TRMM*) is given by:

$$\begin{aligned} TRMM \quad & \max \frac{\alpha_p}{\beta_p} \\ \text{subject to : } & l^T w = 1. \end{aligned} \quad (106)$$

Now we want to test the assumption $\rho = w_P$ for second order zero opportunity portfolio. It is proved that $tr_p = \hat{a} + \hat{b}sr_p$ is sufficient condition for $\rho = w_p$. Thus, we should test the linear relation between Treynor ratio and Sharpe ratio using linear regression. One can observe from figure 2 that there is positive linear relation ship between Treynor ratio and Sharpe ratio.

Figure 2. Plot of Treynor ratio against Sharpe ratio

Figure 2 shows that there is positive linear relation ship between Sharpe ratio of companies and Treynor ratio of companies. Thus, we assume portfolio follows Treynor-Sharpe regression line. As a result, we assume $\rho = w_P$.

Figure 3. Plot of Sharpe ratio process against time

Let consider the performance of *OTH* and *CP*. Note that *CP* is more diversified portfolio than *OTH*. We observe from Figure 3 that *CP* outperforms *OTH* based on Sharpe ratio process.

9 Discussion

Mean variance model is applicable only either expected return of portfolio or variance of portfolio return is given. But in reality both expected return and variance of portfolio are not known. Therefore, mean variance model has no supportive evidence unless either expected return of portfolio or variance of portfolio return is given. Another problem of mean variance model is it does not consider some logical constraints like opportunity cost and arbitrage opportunity constraints. By market efficiency hypothesis, information about securities prices is available in public. Thus, there is no arbitrage opportunity and opportunity cost. In this research, we have investigated two portfolio selection and asset allocation strategies based on the evidence that there is positive linear relationship between Treynor ratio of securities and Sharpe ratio of securities and positive linear relationship between natural logarithm of standard deviation of securities' return and square of co-variance between securities' return and market return. These two strategies are portfolio selection and asset allocation strategies using order three asset allocation model and general order three opportunity model.

We would like to test the quality or (validity) of order two zero opportunity cost condition using historical data. We observe from results of this study that securities follow Treynor-Sharpe regression line. Thus, it is reasonable to construct a portfolio that follows Treynor-Sharpe regression line. It follows from fundamental theorem of order two investment opportunity that risk free fundamental portfolio and risky fundamental portfolio are zero opportunity portfolios.

Order three zero opportunity condition suggests that there is linear relationship between variance of security return and co-variance of security return with market return. However, we realized that there is positive linear relationship between natural logarithm of standard deviation of securities' return and square of co-variance between securities' return and market return using historical data of securities' return. We have introduced portfolio weight-variance regression line constraint in *GOOM*. We have included portfolio market line constraint and portfolio beta estimation constraint in *GOOM*. Note that *GOOM* needs optimization tool for its optimal solution.

The tangency portfolio is constructed by the help of efficiency condition and tangency portfolio condition. Let consider efficiency condition and order three zero opportunity condition to construct *OAAM*. Order three Sharpe error condition and order three efficiency condition follow from efficiency condition and order three zero opportunity condition. We use *OAAM* to determine weight of risky portfolio, risk free portfolio and market portfolio in order three complete portfolio.

The two models *OAAM* and *GOOM* are favorable for investors who do not want opportunity gain or loss from the three investment scenarios. Tangency portfolio does not con-

sider portfolio beta estimation constraint, portfolio market line constraint and weight-variance regression line constraint. Let us discuss on mean variance portfolio, tangency portfolio and order three complete portfolio. Tangency portfolio is a special case of mean variance portfolio with capital market line constraint. Thus, mean variance portfolio may outperform tangency portfolio because tangency portfolio is constrained by capital market line constraint. However, mean variance portfolio has opportunity cost risk. Since tangency portfolio is a special portfolio with special case of order three opportunity cost condition, it has order three opportunity cost risk. Order three complete portfolio has no order three opportunity cost risk. Therefore, investors who take zero order three opportunity cost would choose order three complete portfolio for their investment portfolio selection and asset allocation decision. In summary, we have relaxed tangency portfolio condition to order three zero opportunity cost condition to construct order three complete portfolio. In summary, the two models *OAAM* and *GOOM* are realistic models than mean variance portfolio and tangency portfolio for investors who takes order three zero opportunity cost risk.

10 Conclusion and recommendation

In this research, we have introduced basic portfolio conditions, namely, efficiency condition, Sharpe ratio error condition and order three zero opportunity condition. We have proved some mathematical facts related efficiency condition, Sharpe ratio error condition and order three zero opportunity condition. We have shown that there is positive linear correlation between Treynor ratio and Sharpe ratio of companies using empirical study. Furthermore, we observed that logarithm of standard deviation of companies and square of co-variance of companies with market index are positively correlated. We constructed *GOOM* using portfolio Treynor-Sharpe line and order three opportunity condition. Moreover, we constructed *OAAM* using efficiency condition, Sharpe ratio error condition and order three zero opportunity condition. Finally, we applied *GOOM* and *OAAM* to construct complete portfolio *CP*.

Let us consider practical application of *GOOM* and *OAAM*. First, we checked Treynor-Sharpe regression model using historical data of securities' return. Second, we checked the validity of weight-variance regression model using historical data of securities' return. Third, we solved for optimal solution of *GOOM*. We constructed *OTH* from *GOOM*. Fourth, we applied *OAAM* to determine weights of assets in order three complete portfolio. Finally, we evaluated the performance of *OTH* and *CP* using Sharpe ratio process. The result of this study shows that *CP* is outperforming *OTH* with respect to Sharpe ratio process. Moreover, we constructed *CP* using proposed models and historical data. The result of this study suggests that investors who want to take order three zero opportunity cost risk should allocate $\frac{2}{3}$ of the investment capital for risky portfolio.

In mean variance model, weight of each company is assumed as independent of time. In this research, we assumed

that weight of each company is assumed as independent of time. However, in reality weight may be dynamic with respect to time. Therefore, we recommend researchers to study the impact of dynamic (time dependent) weight on portfolio selection. Another research recommendation for researchers is that the study of impact of market index on portfolio selection and asset allocation. This study does not include the impact of economic and fundamental factors on portfolio selection. Moreover, we will study the impact of market factor and dynamic weight in our future study.

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