On Tensor Product and Colorability of Graphs

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Abstract The idea of graph coloring problem (GCP) plays a vital role in allotment of resources resulting in its proper utilization in saving labor, space, time and cost effective, etc. The concept of GCP for graph G is assigning minimum number of colors to its nodes such that adjacent nodes are allotted a different color, the smallest of which is known as its chromatic number $\chi(G)$. This work considers the approach of taking the tensor product between two graphs which emerges as a complex graph and it drives the idea of dealing with complexity. The load balancing on such complex networks is a hefty task. Amidst the various methods in graph theory the coloring is a quite simpler tool to unveil the intricate challenging networks. Further the node coloring helps in classifying the nodes with least number of classes in any network. So coloring is applied to balance the allocations in such complex network. We construct the tensor product between two graphs like path with wheel and helm, cycle with sunlet and closed helm graphs then structured their nature. The coloring is then applied for the nodes of the extended new graph to determine their optimal bounds. Hence we obtain the chromatic number for the tensor product of $P_m \otimes W_n$, $P_m \otimes H_n$, $C_m \otimes S_n$ and $C_m \otimes CH_n$.

Keywords Vertex Coloring, Tensor Product, Wheel, Helm, Sunlet

1 Introduction

A graph represents a set of objects called nodes or vertices in which some pairs of nodes are connected by edges. GCP is one of the earlier and interesting concepts in graph theory initially started with coloring the regions of every map. Many researches were done for the past few decades on various coloring concepts such as vertex coloring, edge coloring, total

coloring, equitable coloring and star coloring, etc. It has been investigated in the form of optimal assignment by finding the chromatic number and developing coloring algorithms. Murat et al. [6] studied the performance of various graph coloring algorithms based on their solution and execution time. Graph coloring is widely applied in the process of frequency assignment, task scheduling and load balancing etc. Jong et al. [4] designed and implemented a load balancing approach in edge cloud computing environments with GCP based on a genetic algorithm.

Very few research works has been carried out in the graph theoretical concepts like product of graphs and node coloring due to its complexity. The tensor product (TP) of graphs with node coloring is a recent approach. We considered the combination of TP and its chromatic number of graphs. For our study we considered the TP of path with wheel and helm graphs, cycle with sunlet and closed helm graph and applied node coloring to obtain $\chi(G)$. The coloring is done by the method of mapping the node set V in the TP of two different graphs to the color set $\mathcal{C}$. This mapping transforms into a proper node coloring G . Reed [8] established the relation among the bounds of chromatic number χ , maximum degree Δ and maximum clique number ω of any graph G . Further this Reed's conjecture is studied for the TP of graphs $P_m \otimes W_n$, $P_m \otimes H_n$, $C_m \otimes S_n$ and $C_m \otimes CH_n$. These minimum bounds will resolve the allocation problems in larger and complex networks and enhance the performance with minimum resources.

The **Wheel graph** $n \geq 4$ is obtained by connecting all the nodes $\{v_1, v_2, \dots, v_n\}$ of the cycle C_n to the hub node v_0 .

The **Helm graph** [3] H_n is acquired from a Wheel graph W_n in which each node on the outer cycle C_n is joined with a pendant edge.

A **Closed Helm** [3] CH_n is obtained from a helm H_n after joining each pendant node with an edge between them.

The **Sunlet graph** S_n [10] is the graph with $2n$ nodes con-

sisting of a cycle C_n and n pendant edges each connected to a node $\{v_1, v_2, \dots, v_n\}$ of the cycle.

The **Proper Node Coloring** [5] of a graph $G(V, E)$ is defined as

$$f : V \rightarrow \{1, 2, \dots, |V|\}, \exists f(i) \neq f(j), \forall (i, j) \in E$$

i.e., no two neighboring nodes receives the identical color. The least number of colors to which the proper node coloring exists for a graph G is known as the chromatic number $\chi(G)$.

The **Tensor product**(TP) [11] of graphs $\mathcal{G}$ and $\mathcal{H}$ is represented as $\mathcal{G} \otimes \mathcal{H}$, whose elements are

$$V(\mathcal{G} \otimes \mathcal{H}) = \{(g, h) / g \in V(\mathcal{G}), h \in V(\mathcal{H})\}$$

$$E(\mathcal{G} \otimes \mathcal{H}) = \{(g, h)(g', h') / gg' \in E(\mathcal{G}), hh' \in E(\mathcal{H})\}$$

where each node (g, h) and (g', h') are adjacent precisely if $gg' \in E(\mathcal{G})$ and $hh' \in E(\mathcal{H})$.

Theorem[2] [**Brooks Theorem**] Any graph G which is connected, apart from a complete graph K_n or an odd cycle $C_n, n \equiv 1 \pmod{2}$ then $\chi(G) \leq \Delta(G)$.

Conjecture[8] [**Reed's Conjecture**] Any graph G , having chromatic number χ , maximum degree Δ and clique number ω satisfies the inequality $\chi(G) \leq \left\lceil \frac{1+\Delta+\omega}{2} \right\rceil$.

Theorem[7] For any graph G which does not contains a complete sub graph with r nodes then

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G) + 1}{r} \right\rfloor,$$

where $4 \leq r \leq \Delta(G) + 1$.

Theorem [1] Let $G := C_n(a)$ be a circulant graph. Then $\chi(G) = \begin{cases} 2, & \text{when } n = \text{even} \\ 3, & \text{when } n = \text{odd} \end{cases}$.

Theorem[9] For any two graphs G_1 and G_2 which are connected, then its TP $G_1 \otimes G_2$ is also a connected graph if and only if either G_1 or G_2 consists of an odd cycle.

Corollary[9] For two graphs G_1 and G_2 which are connected without having any odd cycles, then its TP $G_1 \otimes G_2$ has absolutely two connected components.

2 Node coloring for tensor product of graphs

Theorem 1 For any positive integer values of $m \geq 2$ and $n \geq 3$, the chromatic number for TP of path P_m and wheel graph W_n is $\chi(P_m \otimes W_n) = 2$.

Proof

The path P_m has m nodes and $m - 1$ edges and is represented as

$$V(P_m) = \{u_i : 1 \leq i \leq m\}$$

and $E(P_m) = \{u_i u_{i+1} : 1 \leq i \leq m - 1\}$ respectively.

The wheel W_n contains of $n + 1$ nodes and $2n$ edges which are defined as

$$V(W_n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\}$$

and

$$E(W_n) = \{e_i \cup e'_i : 1 \leq i \leq n\},$$

where the edges e_i are between the nodes $v_0 v_i (1 \leq i \leq n)$ and the edge e'_i is between the nodes $v_i v_{i+1} (1 \leq i \leq n - 1)$.

By considering the TP of these two graphs, it generates a product graph with $m(n + 1)$ nodes and $4(m - 1)n$ edges such that

$$V(P_m \otimes W_n) = \{u_i v_j : 1 \leq i \leq m, 0 \leq j \leq n\}$$

$$E(P_m \otimes W_n) = \{(u_i v_0, u_{i+1} v_j) \cup (u_{i+1} v_0, u_i v_j) : 1 \leq i \leq m - 1, 1 \leq j \leq n\}$$

$$\cup \{(u_i v_j, u_{i+1} v_{j+1}) : 1 \leq i \leq m - 1, 1 \leq j \leq n - 1\}$$

$$\cup \{(u_i v_j, u_{i+1} v_{j-1}) \cup (u_i v_1, u_{i+1} v_n) : 1 \leq i \leq m - 1, 2 \leq j \leq n\}$$

The construction of TP with $P_m, m \geq 2$ and wheel graph $W_n, n \geq 3$ is shown in Figure 1.

The mapping of the function f is from $V(P_m \otimes W_n)$ to $\mathcal{C} = \{1, 2\}$.

$$f : V(P_m \otimes W_n) \rightarrow \mathcal{C}.$$

The node coloring are made as follows

$$f(u_i, v_j) = 1, \text{ where } i = 2k - 1, k = 1, 2, \dots, \left\lceil \frac{m}{2} \right\rceil, j = 0, 1, 2, \dots, n$$

$$f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, y = 0, 1, 2, \dots, n$$

Figure 1. Tensor Product of P_m and W_n .

From the above process of coloring as shown in Figure 1, the generalized structure of TP between path $P_m, m \geq 2$ and

wheel graph W_n , $n \geq 3$. It is observed that it requires only 2 colors to color the entire graph. Therefore the chromatic number of this tensor product of path and wheel graphs is 2. Hence $\chi(P_m \otimes W_n) = 2$.

Remark 1 For any $m \geq 2$, $n \geq 3$ the tensor product between the two graphs path P_m and W_n with the maximum degree Δ and the clique number ω the Reed's conjecture $\chi(P_m \otimes W_n) \leq \lceil \frac{1+\Delta+\omega}{2} \rceil$ holds. For $P_m \otimes W_n$ ($m \geq 2$, $n \geq 3$), it is observed that $\Delta = 6$ and $\omega = 2$ which satisfies the Reed's conjecture $\chi(P_m \otimes W_n) \leq 5$.

Theorem 2 For any positive integer $m \geq 2$ and $n \geq 3$, the chromatic number for TP of path P_m and helm graph H_n is $\chi(P_m \otimes H_n) = 2$.

Proof

The elements of the path is formulated as $V(P_m) = \{u_i : 1 \leq i \leq m\}$ and $E(P_m) = \{u_i u_{i+1} : 1 \leq i \leq m-1\}$ respectively.

The helm graph H_n contains $2n+1$ nodes and $3n$ edges which is defined as

$$V(H_n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v_i : n+1 \leq i \leq 2n\} \text{ and}$$

$$E(H_n) = \{e_i \cup e'_i \cup e''_i : 1 \leq i \leq n\},$$

where the edges e_i are between the nodes $v_0 v_i$ ($1 \leq i \leq n$), the edge e'_i is between the nodes $v_i v_{i+1}$ ($1 \leq i \leq n-1$) and the edges e''_i are between the nodes $v_i v_{n+i}$ ($1 \leq i \leq n$).

By considering the TP of these two graphs, it results in a bigger graph with $m(2n+1)$ nodes and $6(m-1)n$ edges such that

$$V(P_m \otimes H_n) = \{u_i v_j : 1 \leq i \leq m, 0 \leq j \leq 2n\}$$

$$\begin{aligned} E(P_m \otimes H_n) = & \{(u_i v_0, u_{i+1} v_j) \cup (u_{i+1} v_0, u_i v_j) \\ & \cup (u_i v_j, u_{i+1} v_{j+n}) \\ & \cup (u_{i+1} v_j, u_i v_{j+n}) : 1 \leq i \leq m-1, \\ & 1 \leq j \leq n\} \\ & \cup \{(u_i v_j, u_{i+1} v_{j+1}) : 1 \leq i \leq m-1, \\ & 1 \leq j \leq n-1\} \\ & \cup \{(u_i v_j, u_{i+1} v_{j-1}) : 1 \leq i \leq m-1, \\ & 2 \leq j \leq n\} \\ & \cup \{(u_i v_1, u_{i+1} v_n) : 1 \leq i \leq m-1\} \end{aligned}$$

Consider f from $V(P_m \otimes H_n)$ to $\mathcal{C} = \{1, 2\}$, such that

$$f : V(P_m \otimes H_n) \rightarrow \mathcal{C}.$$

The nodes are allotted with colors as follows

$$\begin{aligned} f(u_i, v_j) = 1, \text{ where } i = 2k-1, k = 1, 2, \dots, \left\lceil \frac{m}{2} \right\rceil, \\ j = 0, 1, 2, \dots, 2n \end{aligned}$$

$$\begin{aligned} f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ y = 0, 1, 2, \dots, 2n \end{aligned}$$

From this method of coloring the TP as graphed in Figure 2, is easy to state that it requires only 2 colors for coloring the entire graph of any order $m \geq 2$ and $n \geq 3$. Therefore $\chi(P_m \otimes H_n) = 2$.

Figure 2. Tensor Product of P_m and H_n .

Remark 2 For the tensor product between path P_m and H_n , ($m \geq 2$, $n \geq 3$) the maximum degree Δ and the clique number ω is observed that $\Delta = 8$ and $\omega = 2$ which satisfies the Reed's conjecture $\chi(P_m \otimes W_n) \leq 6$.

Theorem 3 For any positive integers of $m, n \geq 3$,

$$\chi(C_m \otimes S_n) = \begin{cases} 3, & \text{if } m \text{ is odd} \\ 2, & \text{if } m \text{ is even} \end{cases}.$$

Proof

The cycle C_m consists both nodes and edges of order and size m which is represented as

$$V(C_m) = \{u_i : 1 \leq i \leq m\} \text{ and}$$

$$E(C_m) = \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{u_m u_1\} \text{ respectively.}$$

The sunlet graph S_n consists both nodes and edges of order and size $2n$ which is formulated as

$$V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{v_i : n+1 \leq i \leq 2n\} \text{ and}$$

$$E(S_n) = \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\} \cup \{e'_i : 1 \leq i \leq n\}$$

where the edge e_i is between the nodes $v_i v_{i+1}$ ($1 \leq i \leq n-1$), the edge e_n is between $v_n v_1$ and the edges e'_i are between the nodes $v_i v_{n+i}$ ($1 \leq i \leq n-1$).

Taking TP for the above two graphs, the resultant graph has $2mn$ nodes and $4mn$ edges, therefore

$$V(C_m \otimes S_n) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq 2n\}$$

$$\begin{aligned}
E(C_m \otimes S_n) = & \{(u_i v_j, u_{i+1} v_{j+1}) \\
& \cup (u_i v_{j+1}, u_{i+1} v_j) : 1 \leq i \leq m, \\
& \quad 1 \leq j \leq n\} \\
& \cup \{(u_i v_1, u_{i+1} v_n) : 1 \leq i \leq m-1\} \\
& \cup \{(u_{i+1} v_1, u_i v_n) : 1 \leq i \leq m\} \\
& \cup \{(u_i v_j, u_{i+1} v_{j+n}) \\
& \cup (u_i v_{j+n}, u_{i+1} v_j) : 1 \leq i \leq m-1, \\
& \quad 1 \leq j \leq n\} \\
& \cup \{(u_1 v_j, u_m v_{j+1}) \\
& \cup (u_1 v_{j+1}, u_m v_j) \\
& \cup (u_1 v_j, u_m v_{j+n}) \\
& \cup (u_m v_j, u_1 v_{j+n}) : 1 \leq j \leq n\} \\
& \cup \{(u_1 v_1, u_m v_n)\} \cup \{(u_m v_1, u_1 v_n)\}
\end{aligned}$$

The node coloring is processed by defining $f : P \rightarrow \mathcal{C}$,

where $P = V(C_m \otimes S_n)$ and $\mathcal{C} = \{1, 2, 3\}$.

Case(i): When $m = \text{odd}$

$$\begin{aligned}
f(u_i, v_j) = 1, \text{ where } i = 2k-1, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\
j = 1, 2, \dots, 2n
\end{aligned}$$

$$\begin{aligned}
f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\
y = 1, 2, \dots, 2n
\end{aligned}$$

$$f(u_m, v_l) = 3, \text{ where } l = 1, 2, \dots, 2n$$

The coloring on the TP of odd cycle C_m and sunlet S_n , $m, n \geq 3$ is diagrammed in Figure 3.

Figure 3. Tensor Product of odd cycle C_m and S_n .

Case(ii): When $m = \text{even}$

$$\begin{aligned}
f(u_i, v_j) = 1, \text{ where } i = 2k-1, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\
j = 1, 2, \dots, 2n
\end{aligned}$$

$$\begin{aligned}
f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\
y = 1, 2, \dots, 2n
\end{aligned}$$

Further the coloring on the generalized structure of tensor product with even cycle $C_m, m > 3$ and sunlet $S_n, n \geq 3$ is portrayed in Figure 4.

Figure 4. Tensor Product of even cycle C_m and S_n .

Hence it is observed that, while taking tensor product with cycles it requires 3 colors for odd cycles and 2 colors for even cycles.

$$\text{Therefore } \chi(C_m \otimes S_n) = \begin{cases} 3, & \text{if } m \text{ is odd} \\ 2, & \text{if } m \text{ is even} \end{cases}$$

Remark 3 The tensor product between the cycle C_m and Sunlet graph $S_n, (m, n \geq 3)$ has the chromatic number 3 if $m = \text{odd}$ and 2 if $n = \text{even}$. The maximum degree is $\Delta = 6$ and the clique number $\omega = 2$. It is evident that Reed's conjecture holds for this product graph in both odd and even cases, which implies $\chi(C_m \otimes S_n) \leq 5$.

Theorem 4 For any positive integers of $m, n \geq 3$,

$$\chi(C_m \otimes CH_n) = \begin{cases} 3, & \text{when } m = \text{odd} \\ 2, & \text{when } m = \text{even} \end{cases}$$

Proof

The cycle is formulated as

$$V(C_m) = \{u_i : 1 \leq i \leq m\} \text{ and}$$

$$E(C_m) = \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{u_m u_1\} \text{ respectively.}$$

The closed helm CH_n has $2n+1$ nodes and $4n$ edges and defined as

$$V(CH_n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v_i : n+1 \leq i \leq 2n\}$$

$$\begin{aligned}
E(CH_n) = & \{e_i \cup e'_i : 1 \leq i \leq n\} \cup \{e'_k\} \\
& \cup \{e''_i \cup e'''_i : n+1 \leq i \leq 2n\} \cup \{e'''_k\}
\end{aligned}$$

where the edges e_i are between the nodes v_0v_i ($1 \leq i \leq n$), similarly the edges e'_i are between the nodes v_iv_{i+1} ($1 \leq i \leq n-1$), the edge e'_n is between the nodes v_nv_1 , the edges e''_i are between the nodes v_iv_{n+i} ($1 \leq i \leq n$), the edges e'''_i are between the nodes v_iv_{i+1} ($n+1 \leq i \leq 2n-1$) and the edge e'''_n is between the nodes $v_{2n}v_{n+1}$.

By taking the tensor product for these two graphs, the extended new graph with $m(2n+1)$ nodes and $8mn$ edges which has

$$V(C_m \otimes CH_n) = \{u_iv_j : 1 \leq i \leq m, 0 \leq j \leq 2n\}$$

$$\begin{aligned} E(C_m \otimes CH_n) = & \{(u_iv_0, u_{i+1}v_j) \cup (u_{i+1}v_0, u_iv_j) \\ & \cup (u_iv_j, u_{i+1}v_{j+1}) : 1 \leq i \leq m-1, \\ & 1 \leq j \leq n\} \\ & \cup \{(u_iv_{j+1}, u_{i+1}v_j) : 1 \leq i \leq m-1, \\ & 0 \leq j \leq n\} \\ & \cup \{(u_iv_1, u_{i+1}v_n) : 1 \leq i \leq m-1\} \\ & \cup \{(u_{i+1}v_1, u_iv_n) : 1 \leq i \leq m\} \\ & \cup \{(u_iv_j, u_{i+1}v_{j+n}) \\ & \cup (u_{i+1}v_j, u_iv_{j+n}) : 1 \leq i \leq m-1, \\ & 1 \leq j \leq n\} \\ & \cup \{(u_1v_0, u_mv_j) \cup (u_mv_0, u_1v_j) \\ & \cup (u_1v_j, u_mv_{j+1}) \cup (u_1v_{j+1}, u_mv_j) \\ & \cup (u_1v_j, u_mv_{j+n}) \\ & \cup (u_mv_j, u_1v_{j+n}) : 1 \leq j \leq n\} \\ & \cup \{(u_1v_1, u_mv_n)\} \cup \{(u_mv_1, u_1v_n)\} \end{aligned}$$

The node coloring is obtained by considering a function $f : P \rightarrow \mathcal{C}$,

$$\text{where } P = V(C_m \otimes CH_n) \text{ and } \mathcal{C} = \{1, 2, 3\}.$$

Case(i): when m is odd

$$\begin{aligned} f(u_i, v_j) = 1, \text{ where } i = 2k-1, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ j = 0, 1, 2, \dots, 2n \end{aligned}$$

$$\begin{aligned} f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ y = 0, 1, 2, \dots, 2n \end{aligned}$$

$$f(u_m, v_l) = 3, \text{ where } l = 0, 1, 2, \dots, 2n$$

The coloring on the TP of odd cycle C_m and closed helm CH_n $m, n \geq 3$ is presented in Figure 5.

Figure 5. Tensor Product of odd cycle C_m and CH_n .

Case(ii): When m is even

$$\begin{aligned} f(u_i, v_j) = 1, \text{ where } i = 2k-1, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ j = 0, 1, 2, \dots, 2n \end{aligned}$$

$$\begin{aligned} f(u_x, v_y) = 2, \text{ where } x = 2k, k = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor, \\ y = 0, 1, 2, \dots, 2n \end{aligned}$$

The summarizing of the coloring on the generalized construction of tensor product with even cycle $C_m, m > 3$ and sunlet $CH_n, n \geq 3$ is exhibited in Figure 6.

Figure 6. Tensor Product of even cycle C_m and CH_n .

Hence it is clear that while taking TP of cycles with closed helm graph, it requires 3 colors for odd cycles and 2 colors for even cycles.

$$\text{Therefore } \chi(C_m \otimes CH_n) = \begin{cases} 3, & \text{when } m = \text{odd} \\ 2, & \text{when } m = \text{even} \end{cases}$$

Remark 4 The TP between the cycle C_m and closed helm $CH_n, (m, n \geq 3)$ has the chromatic number 3 if m is odd and

2 if m is even. The maximum degree is $\Delta = 8$ and the clique number $\omega = 3$. It is clear that Reed's conjecture is true for this graph $\chi(C_m \otimes S_n) \leq 6$.

3 Conclusions

In this paper we have applied vertex coloring to obtain the chromatic number of TP of graphs pondered. We have shown the minimum colors required for optimal allocation to the tensor product between different graphs like path, cycle, helm, closed helm and sunlet graph. Furthermore bounds of chromatic number could be explored and achieved for tensor products between different graphs.

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