Quasi-Chebyshevity in $L^p(\mu, X)$, $1 \leq p < \infty$

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Abstract This paper deals with Quasi-Chebyshevity in the Bochner function spaces $L^p(\mu, X)$, $1 \leq p < \infty$, where X is a Banach space. For W a nonempty closed subset of X and $x \in X$, an element w_0 in W is called "best approximation" to x from W , if $\|x - w_0\| \leq \|x - w\|$, for all w in W . All best approximation points of x from W form a set usually denoted by $P_W(x)$. The set W is called "proximal" in X if $P_W(x)$ is non empty, for each x in X . Now, W is said to be "Quasi-Chebyshev" in X whenever, for each x in X , the set $P_W(x)$ is nonempty and compact in X . This subject was studied in general Banach spaces by several authors and some results had been obtained. In this work, we study Quasi-Chebyshevity in the Bochner L^p - spaces. The main result in this paper is that: given W a Quasi-Chebyshev subspace in X then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, for $1 < p < \infty$, if and only if $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$. As a consequence, one gets that if W is reflexive in X such that X satisfies the sequential KK -property then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, for $1 \leq p < \infty$.

Keywords Quasi-Chebyshev Set, Proximal Set, Compact Set, Reflexive Subspace, Sequential KK -property

approximations to $x \in X$ has the notation: $P_W(x)$. If for each x in X , is nonempty, then W is said to be *proximal* in X and if $P_W(x)$ is singleton $\forall x \in X$, then W is called *Chebyshev* in X . The set $P_W(x)$ is always closed and it is convex whenever W is. $P_W(x)$ is also bounded but usually it is not compact. The concept of Quasi-Chebyshevity can be considered as compact proximality. By definition, a proximal subset W of X is Quasi-Chebyshev in X if $P_W(x)$ (which is nonempty) is a compact set, See [1]. On the other hand, if $P_W(x)$ is nonempty and weakly compact, for each x in X , then W is said to be Weakly-Chebyshev in X .

Several authors studied Quasi-Chebyshevity in Banach spaces, for example: Mohebi [1, 2] and Narayana and Rao [3]. In [1], Mohebi proved several characterizations of Quasi-Chebyshev subspaces in general Banach spaces in terms of elements in the dual spaces.

In this paper, we will study Quasi-Chebyshevity in $L^p(\mu, X)$, for $1 \leq p < \infty$. Our results and approaches have a different flavour. In this study, we aim to give some answers to the following question: What conditions can be imposed on W to get that $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 \leq p < \infty$.

1. Introduction

Let X be a Banach space and W nonempty subset in X . Let x an element in X , then a best approximation point to x from W say $w_0 \in W$ is that satisfies the following:

$$\|x - w_0\| \leq \|x - w\|, \forall w \in W.$$

The set of all these elements $w_0 \in W$ which are best

2. Preliminary Results

Throughout this section, let $(\mathcal{T}, \mathcal{M}, \mu)$ be a measure space: $\mathcal{T}$ denotes the underlying space, μ is the measure and $\mathcal{M}$ the algebra of all measurable subsets of $\mathcal{T}$. Let X be a Banach space, consider the spaces $L^p(\mu, X)$, $1 \leq p < \infty$, of Bochner p -integrable functions on $\mathcal{T}$, with the p -norm defined as follows:

$$\|f\|_p = \left(\int_{\mathcal{T}} \|f(t)\|_X^p dt \right)^{1/p}$$

Actually, members of $L^p(\mu, X)$ are “equivalence classes” of strongly measurable functions from $\mathcal{T}$ to X . By definition a strongly measurable function in $L^p(\mu, X)$ is the limit of a sequence of simple functions in $L^p(\mu, X)$.

Many works have been published towards the study of best approximation theory in $L^p(\mu, X)$. The first among these were those of Khalil [4], Khalil and Deeb [5] and that of Light and Cheney [6]. We start with some definitions and properties of some basic concepts in best approximation theory in general Banach or normed spaces. (See Singer [7])

Definition 2.1

Let $(X, \|\cdot\|)$ be a Banach space and W a subspace in X . Let $x \in X$, an element w_0 in W is called best approximation point to x from W , if

$$\|x - w_0\| \leq \|x - w\|, \text{ for all } w \text{ in } W.$$

Usually, the set of “best approximation” points (from W) to x , given by:

$$P_W(x) = \{w_0 \in W: \|x - w_0\| = d(x, W)\}$$

and

$$d(x, W) = \inf \{\|x - w\| : w \in W\}.$$

Finally, W is called *proximal* in X if $\forall x$ in X , the set $P_W(x)$ is nonempty.

Definition 2.2

Let X be a Banach space and W a subspace in X , then W is called Chebyshev set if for all elements $x \in X$, the set $P_W(x)$ is singleton.

Remarks 2.3

1. For any proximal set W , then $P_W(x)$ is closed and bounded subset, for each x in X . It is convex whenever W is.
2. In general, a closed convex set is not proximal.
3. Every compact subset of a Banach space X is proximal.
4. Every finite dimensional (or reflexive) subspace of a Banach space X is proximal.

Two other simple results concerning proximality in $L^p(\mu, X)$, $1 < p < \infty$, follow directly from the last remark and the properties of $L^p(\mu, X)$ spaces:

1. For H a reflexive subspace of X then $L^p(\mu, H)$ is reflexive, hence proximal in $L^p(\mu, X)$, $1 < p < \infty$.
2. For H a finite-dimensional subspace in X then it is reflexive, hence $L^p(\mu, H)$ is proximal in $L^p(\mu, X)$, $1 < p < \infty$.

For the case $p = 1$, the above two results are also valid though $L^1(\mu, H)$ is not reflexive in $L^1(\mu, X)$, see Light & Cheney [6] and Khalil [4]. On the other hand, the proximality of $L^p(\mu, H)$ in $L^p(\mu, X)$, $1 \leq p < \infty$, implies that H is proximal in X has also been proved in [6] as consequence of the following theorem.

Theorem 2.4 (Lemma 2.10 in [6])

Let $f \in L^p(\mu, X)$ and $h \in L^p(\mu, H)$, $1 \leq p < \infty$. Then h is a best approximation to f from $L^p(\mu, H)$ if and only if, for a.e t in $\mathcal{T}$, $h(t) \in H$ is best approximation to $f(t)$ in X .

Khalil and Deeb, in [5], proved the following important theorem, which will be used later in the proof of our main result.

Theorem 2.5 (Theorem 1.1 in [5])

Let X is a Banach space. Let the subspace H be closed in X . Then the followings equivalent:

1. $L^p(\mu, H)$ is proximal in $L^p(\mu, X)$, $1 < p < \infty$.
2. $L^1(\mu, H)$ is proximal in $L^1(\mu, X)$.

We mention here that using **Theorem 2.5**, one can get:

Corollary 2.6

If H is a finite dimensional or reflexive subspace in the Banach space X then $L^1(\mu, H)$ is proximal in $L^1(\mu, X)$.

In what follows, we present the definition of a Quasi-Chebyshev subspace in general Banach spaces and some related properties.

Definition 2.7 (Quasi-Chebyshev)

Let W be proximal in the Banach space X . W is said to be **Quasi-Chebyshev** in X if for each element x in X , the set of best approximations $P_W(x)$ is compact in X .

Remarks 2.8

1. Every Quasi-Chebyshev set is proximal, from definition.
2. In finite dimensional spaces every proximal subspace is Quasi-Chebyshev.
3. Every Chebyshev is Quasi-Chebyshev since every singleton set is compact.

Definition 2.9

A Banach space X possesses the **KK-property (Kadec-Klee)** if the weak and the norm topology of X coincide on S_X : the unit sphere of X . This means that weak convergence implies norm convergence on S_X . On the other hand, X has the sequential **KK-** property if every sequence in S_X that converges weakly in S_X also converges.

Narayana and Rao in [3], gave some conditions on the space in order that certain subspaces of X become Quasi-Chebyshev, as in the following proposition, (see **Proposition 2.9** in [3]).

Proposition 2.10

If W reflexive subspace in a Banach space X . Assume that X possesses the sequential **KK**- property, then W is Quasi-Chebyshev in X .

3. Quasi-Chebyshev Subspaces in $L^p(\mu, X)$, $1 \leq p < \infty$

In this section, we study Quasi-Chebyshev subspaces in L^p - Bochner function spaces. Our main result is that: If W Quasi-Chebyshev in X , then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 < p < \infty$, if and only if $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$. As an application of this result we investigate some other conditions on which $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 \leq p < \infty$. We first discuss the opposite direction: given $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, does this imply that W is a Quasi-Chebyshev in X ? Our first result gives a positive answer. For simplicity, we prove it for $L^1(\mu, W)$ in $L^1(\mu, X)$, then by **Theorem 3.3**, the general result follows.

Theorem 3.1

If $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$ then W is Quasi-Chebyshev in X .

Proof.

Let $L^1(\mu, W)$ be Quasi-Chebyshev in $L^1(\mu, X)$ then $L^1(\mu, W)$ is a proximal subspace in $L^1(\mu, X)$ (from **Definition 2.7** of Quasi-Chebyshevity), which again implies that W is proximal in X .

Hence, $P_{W(x)}$ is nonempty, for each x in X . Left to prove that $P_{W(x)}$ is compact for all x in X . Let $\{y_n\}$ be a sequence in $P_{W(x)}$. Define the function $f \in L^1(\mu, X)$ as $f(t) = x$, for all t in $\mathcal{T}$. Also, for each n , let $f_n(t) = y_n, \forall t$. So, $\{f_n\}$ is a sequence of constant functions in $L^1(\mu, W)$. But since $\{y_n\}$ is a sequence in $P_{W(x)}$ then

$$\|x - y_n\| \leq \|x - w\|, \text{ for all } w \text{ in } W, \forall n.$$

This gives by the choice of f and f_n above, that

$$\|f(t) - f_n(t)\| \leq \|f(t) - h(t)\|, \forall h \in L^1(\mu, W), \text{ a.e } t \text{ in } \mathcal{T}.$$

Hence,

$$\|f - f_n\|_1 \leq \|f - h\|_1, \forall h \in L^1(\mu, W).$$

And this implies that

$$f_n \in P_{L^1(\mu, W)}(f), \forall n.$$

Now, since $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$

then $P_{L^1(\mu, W)}(f)$ is a compact set and so there exists

$\{f_{n_k}\}$ a convergent subsequence of $\{f_n\}$ in $P_{L^1(\mu, W)}(f)$.

Also by the choice of $\{f_n\}$ above, we can take $\{y_{n_k}\}$ as a convergent subsequence of $\{y_n\}$ in $P_{W(x)}$. So, the set $P_{W(x)}$ is compact and W is a Quasi-Chebyshev in X .

In what follows, we give some results that discuss what conditions to be imposed on W in order that $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 \leq p < \infty$. One simple result is the following:

Theorem 3.2

Let W be reflexive subspace in X , then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 < p < \infty$, provided that $L^p(\mu, W)$ has the sequential **KK**- property.

Proof.

Let W be reflexive in X . This implies W to be proximal and $L^p(\mu, W)$ is also reflexive hence proximal in $L^p(\mu, X)$, for $1 < p < \infty$. On the other hand, since (from the given) $L^p(\mu, X)$ has the sequential **KK**-property then by **Proposition 2.10**, we are done.

The following theorem is our main result in this study.

Theorem 3.3

Let W be Quasi-Chebyshev subspace in X . The followings equivalent:

1. $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$.
2. $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 < p < \infty$.

Proof.

2.) $\Rightarrow$ 1.) Let $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, for $1 < p < \infty$. It follows, from **Definition 2.7**, that $L^p(\mu, W)$ is proximal in $L^p(\mu, X)$. This yields that $L^1(\mu, W)$ also proximal in $L^1(\mu, X)$ by **Theorem 2.5**. Next, we prove that the set of best approximations from $L^1(\mu, W)$, for each $f \in L^1(\mu, X)$ is compact, as follows.

Define the mapping $T: L^1(\mu, X) \rightarrow L^p(\mu, X)$ such that whenever $f(t) \neq 0$, a.e t in $\mathcal{T}$, let

$$T(f)(t) = \|f(t)\|^{\frac{1}{p}-1} f(t).$$

Otherwise, $T(f)=0$. Let $\{g_n\}$ be a sequence in

$P_{L^1(\mu, W)}(f)$. So, for each n ,

$$\|f - g_n\|_1 \leq \|f - k\|_1, \forall k \in L^1(\mu, W)$$

But this implies that, for a.e t in $\mathcal{T}$,

$$\|f(t) - g_n(t)\| \leq \|f(t) - w\|, \forall w \in W, \quad (1)$$

Considering the nontrivial case, take $h_n = T(g_n)$. Then $\{h_n\} \in L^p(\mu, W)$ and

$$h_n(t) = T(g_n)(t) = \|g_n(t)\|^{\frac{1}{p}-1} g_n(t).$$

Now, writing $g_n(t) = \|g_n(t)\|^{1-1/p} h_n(t)$ then from (1),

$$\begin{aligned} \|f(t) - g_n(t)\| &= \| |f(t)|^{1-1/p} T f(t) - |g_n(t)|^{1-1/p} h_n(t) \| \\ &\leq \| |f(t)|^{1-1/p} T f(t) - w \|, \forall w \in W. \end{aligned}$$

This implies, since W is a subspace that

$$\begin{aligned} \| |f(t)|^{1-1/p} T f(t) - |g_n(t)|^{1-1/p} h_n(t) \| \\ \leq \| |f(t)|^{1-1/p} T f(t) - |f(t)|^{1-1/p} w \|, \forall w \in W. \end{aligned}$$

Hence,

$$\| T f(t) - \left(\frac{\|g_n(t)\|}{\|f(t)\|} \right)^{1-1/p} h_n(t) \| \leq \| T f(t) - w \|, \forall w \in W,$$

for a.e t in $\mathcal{T}$.

Taking $G_n(t) = \left(\frac{\|g_n(t)\|}{\|f(t)\|} \right)^{1-1/p} h_n(t)$, we then have

$\{G_n\} \in P_{L^p(\mu, W)}(T(f))$. So, by the given that $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, there exists a convergent subsequence $\{G_{n_k}\}$ of $\{G_n\}$ in $P_{L^p(\mu, W)}(T(f))$. But, since we can write

$$G_{n_k}(t) = \left(\frac{\|g_{n_k}(t)\|}{\|f(t)\|} \right)^{1-1/p} h_{n_k}(t),$$

Then one gets that there is also a convergent subsequence $\{g_{n_k}\}$ of $\{g_n\}$ in $P_{L^1(\mu, W)}(f)$. By this the conclusion in this direction follows.

1.) $\Rightarrow$ 2.) Assume that $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$. Let $f \in L^p(\mu, X)$, then since $L^p(\mu, W)$ is proximal in $L^p(\mu, X)$, we have $P_{L^p(\mu, W)}(f) \neq \emptyset$ (from Definition 2.7 of Quasi-Chebyshevity and Theorem 2.5).

Now, let $\{h_n\}$ be a sequence in $P_{L^p(\mu, W)}(f)$, so we have

for each n ,

$$\|f - h_n\|_p \leq \|f - g\|_p, \forall g \in L^p(\mu, W). \quad (2)$$

Since the measure space is finite, then $L^p(\mu, X) \subset L^1(\mu, X)$, for $1 < p < \infty$. But also $f \in L^p(\mu, X)$ and $\{h_n\} \in L^p(\mu, W)$ implies that $f \in L^1(\mu, X)$ and $\{h_n\} \in L^1(\mu, W)$.

Actually, $\{h_n\} \in P_{L^1(\mu, W)}(f)$. Indeed, from (2) and

Theorem 2.4, we get;

$\|f(t) - h_n(t)\| \leq \|f(t) - w\|, \forall w \in W$, for a.e t in $\mathcal{T}$.

This gives $\|h_n(t)\| \leq 2\|f(t)\|$, for a.e t in $\mathcal{T}$ and for all n , hence: $\|h_n\|_1 \leq 2\|f\|_1, \forall n$.

Now by the above argument, we also have

$$\|f - h_n\|_1 \leq \|f - k\|_1, \forall k \in L^1(\mu, W).$$

But $P_{L^1(\mu, W)}(f)$ is compact since $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$, by assumption. So there exists $\{h_{n_k}\}$ a convergent subsequence of $\{h_n\}$ in $P_{L^1(\mu, W)}(f)$ that converges to an element say h in $P_{L^1(\mu, W)}(f)$. Left to prove that $h \in P_{L^p(\mu, W)}(f)$. We argue as above, since we have,

$$\|h_{n_k}(t)\| \leq 2\|f(t)\|, \text{ for a.e } t \text{ in } \mathcal{T}.$$

Then by taking limits, we get $\|h(t)\| \leq 2\|f(t)\|$, for a.e t in $\mathcal{T}$, which implies that

$$\|h\|_p \leq 2\|f\|_p,$$

Hence, $h \in P_{L^p(\mu, W)}(f)$.

So, we have proved that for any $f \in L^p(\mu, X)$ the set of best approximations $f \in L^p(\mu, X)$ in $L^p(\mu, W)$ is compact. This gives that $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$. Hence, the proof is complete.

Some corollaries follow,

Corollary 3.4

Let W a reflexive subspace in X , then $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$ provided that $L^p(\mu, X)$ has the sequential KK -property for some $p: 1 < p < \infty$.

Proof.

If W is reflexive in X and $L^p(\mu, X)$ satisfies the sequential KK -property then by Theorem 3.2, we get $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$. Hence, $L^1(\mu, W)$

is also Quasi-Chebyshev in $L^1(\mu, X)$ follows directly from **Theorem 3.3**.

Corollary 3.5

Let W be reflexive in X . Assume X has the sequential KK -property, then $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$.

Proof.

It follows from **Theorem 3.2** that $L^2(\mu, W)$ is Quasi-Chebyshev in $L^2(\mu, X)$, hence by the main theorem (**Theorem 3.3**) and **Corollary 3.4**, we have also $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$.

Corollary 3.6

If W is a finite dimensional subspace in X , then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, $1 \leq p < \infty$, provided that X has the sequential KK -property.

4. Conclusions

In this paper, Quasi-Chebyshevity in the Bochner spaces $L^p(\mu, X)$ was studied, for $1 \leq p < \infty$. We investigated when $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$ under some conditions on the subspace W of X . We proved the main result in this paper: given W a Quasi-Chebyshev subspace of X then $L^p(\mu, W)$ is Quasi-Chebyshev in $L^p(\mu, X)$, where $1 < p < \infty$, iff $L^1(\mu, W)$ is Quasi-Chebyshev in $L^1(\mu, X)$. Other results were obtained as consequences of the main theorem.

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