Remotest Points and Best Proximity Points in Metric Spaces

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Abstract In this paper, we give sufficient conditions for the existence of remotest points with recurrence relations. Then, we apply these relations to generalize recent best proximity point theorems. The findings of the present research contribute and enrich previous results reviewed in the literature.

Keywords Recurrence Relations, Remotest points, Property UC_f , Property UC , Best Proximity Points
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1 Introduction

Let A and B be nonempty bounded subsets of a metric space (X, d) . The set A is said to be remotest from a point $x \in X$, if there exists a point $x_0 \in A$ such that $\delta(x, A) = \sup\{d(x, y) : y \in A\} = d(x, x_0)$. The point x_0 is called a remotest point of A from x . If there is a pair $(x_0, y_0) \in A \times B$ for which $d(x_0, y_0) = \delta(A, B)$, that $\delta(A, B)$ is remotest distance of A and B , define by

$$\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}.$$

Then the pair (x_0, y_0) is called a remotest pair for A and B and put

$$F(A, B) = \{(x, y) \in A \times B : d(x, y) = \delta(A, B)\}$$

as the set of all remotest pairs (A, B) .

The problem of characterizing remotest sets has applications in the study of approximation theory and geometry of Banach spaces, see [5, 6, 13]. One could find more results about remotest points in [9, 11, 12].

In the following this research, in 2015, we introduced the concept of remotest points for a cyclic map $T : A \cup B \rightarrow A \cup B$ i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$. As follows.

Definition 1.1 [2] Let A and B be nonempty bounded subsets of a metric space X and $T : A \cup B \rightarrow A \cup B$ be a cyclic map. The point $x \in A \cup B$ is a remotest point of the map T , if $d(x, Tx) = \delta(A, B)$.

On the other hand, the study of existence of best proximity points for cyclic contraction mappings is very interesting. More precisely, for two given nonempty subsets A and B of a metric space (X, d) , a point $x \in A \cup B$ is called a best proximity point of cyclic map $T : A \cup B \rightarrow A \cup B$, if $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. The first result in this area was reported by Kirk and et al [10]. Later, many authors continued investigation and more results have been obtained, such as, [1, 3, 4, 7, 8].

In this paper, we first present recurrence relation as follows, $T : A \cup B \rightarrow A \cup B$ is a continuous cyclic contraction map, assuming $x_0 \in A$, define $x_{n+1} = Tx_n$ for every $n \in \mathbb{N} \cup \{0\}$ and let $d_n = d(x_n, x_{n+1})$. Then we find remotest points of the continuous cyclic map T , with recurrence relations. Finally, we give best proximity point results by using this relation. To prove our results in section 3, we recall following definition.

Definition 1.2 [14] Let A and B be nonempty bounded subsets of a metric space X . Then (A, B) is said to satisfy the property UC if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n \rightarrow \infty} d(z_n, y_n) = d(A, B)$, then $d(x_n, z_n) \rightarrow 0$ holds.

2 Remotest Points

We prove the following result which will be needed in what follows.

Proposition 2.1 Let A and B be nonempty bounded subsets of a metric space X and

$$d_{n+k} \geq \alpha^k d_n + (1 - \alpha^k) \delta(A, B), \quad (2.1)$$

where α, k are constant such that $0 < \alpha < 1$ and $k \in \mathbb{N}$. Then $d_{n+k} \rightarrow \delta(A, B)$ as $n \rightarrow \infty$.

Proof. Let $m \in \mathbb{N}$. By inequality (2.1),

$$d_{km+k-1} \geq \alpha^{km} d_{k-1} + (1 - \alpha^k) \left(\sum_{i=0}^{m-1} \alpha^{ki} \right) \delta(A, B),$$

$$d_{km+k-2} \geq \alpha^{km} d_{k-2} + (1 - \alpha^k) \left(\sum_{i=0}^{m-1} \alpha^{ki} \right) \delta(A, B),$$

$$d_{km+k-3} \geq \alpha^{km} d_{k-3} + (1 - \alpha^k) \left(\sum_{i=0}^{m-1} \alpha^{ki} \right) \delta(A, B),$$

continuing this process,

$$d_{km+1} \geq \alpha^{km} d_1 + (1 - \alpha^k) \left(\sum_{i=0}^{m-1} \alpha^{ki} \right) \delta(A, B) \text{ and}$$

$$d_{km} \geq \alpha^{km} d_0 + (1 - \alpha^k) \left(\sum_{i=0}^{m-1} \alpha^{ki} \right) \delta(A, B).$$

Letting $m \rightarrow \infty$, $\lim_{m \rightarrow \infty} d_{km+i} \geq \delta(A, B)$, $i = 0, 1, \dots, k-1$. On the other hand, $\delta(A, B) \geq \lim_{m \rightarrow \infty} d_{km+i}$, $i = 0, 1, \dots, k-1$. So

$$\lim_{m \rightarrow \infty} d_{km+i} = \delta(A, B), \quad i = 0, 1, \dots, k-1.$$

Therefore $d_{n+k} \rightarrow \delta(A, B)$ as $n \rightarrow \infty$. □

In following theorem give remotest point when one of the sets is boundedly compact.

Theorem 2.1 Let A and B be nonempty bounded subsets of a metric space X and relation (2.1) holds. If either A or B is boundedly compact, then there exists $x \in A \cup B$ with $d(x, Tx) = \delta(A, B)$.

Proof. Let k be an even positive integer. So $x_{2n+k} \in A$. Suppose A is boundedly compact. Since the sequence $\{x_{2n+k}\}$ is bounded and A is boundedly compact. Hence $\{x_{2n+k}\}$ has a convergent subsequence in A . Let $\{x_{2n_l+k}\}$ be a subsequence of $\{x_{2n+k}\}$ with $x_{2n_l+k} \rightarrow x \in A$ as $l \rightarrow \infty$. Since T is a continuous map and $d(x_{2n_l+k}, Tx_{2n_l+k}) \rightarrow d(A, B)$ as $l \rightarrow \infty$. Therefore $d(x, Tx) = \delta(A, B)$. The proof when k is an odd positive integer and B is boundedly compact is similar. □

Example 2.1 Given $0 < \alpha < 1, k \in \mathbb{N}$. let A and B be nonempty bounded subsets of l^p , $1 \leq p < \infty$ where p is odd number, defined by $A = \{(1 - \alpha^{2n})e_{2n} : n \in \mathbb{N}\}$ and $B = \{(1 - \alpha^{2m-1})e_{2m-1} : m \in \mathbb{N}\}$. Suppose

$$T((1 - \alpha^{2n})e_{2n}) = (1 - \alpha^{2n+1})e_{2n+1}$$

and

$$T((1 - \alpha^{2m-1})e_{2m-1}) = (1 - \alpha^{2m})e_{2m}.$$

Then

$$d_{n+k} \geq \alpha^k d_n + (1 - \alpha^k) \delta(A, B),$$

where α, k are constant such that $0 < \alpha < 1$ and $k \in \mathbb{N}$.

Proof. Here $\delta(A, B) = 2^{1/p}$.

$$\begin{aligned} \alpha^k ((1 - \alpha^{2m})^p + (1 - \alpha^{2n+1})^p)^{1/p} + (1 - \alpha^k) 2^{1/p} &= ((1 + \alpha^k - 1 - \alpha^{2m+k})^p \\ &+ (1 + \alpha^k - 1 - \alpha^{2n+1+k})^p)^{1/p} \\ &+ (1 - \alpha^k) 2^{1/p} \\ &\leq ((1 - \alpha^{2m+k})^p + (1 - \alpha^{2n+1+k})^p)^{1/p} \\ &- 2^{1/p} (1 - \alpha^k) \\ &+ (1 - \alpha^k) 2^{1/p} \\ &= ((1 - \alpha^{2m+k})^p \\ &+ (1 - \alpha^{2n+1+k})^p)^{1/p}. \end{aligned}$$

Which implies that

$$d_{n+k} \geq \alpha^k d_n + (1 - \alpha^k) 2^{1/p}.$$

□

Definition 2.1 Let A and B be nonempty bounded subsets of a metric space X . Then (A, B) is said to satisfy the property UC_f if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = \delta(A, B)$ and $\lim_{n \rightarrow \infty} d(z_n, y_n) = \delta(A, B)$, then $d(x_n, z_n) \rightarrow 0$ holds.

Theorem 2.2 Let A and B be nonempty closed and bounded subsets of a complete metric space X such that (A, B) has a property UC_f and relation (2.1) holds. Then T has a remotest point.

Proof. We show that for each $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$,

$$d(x_{2m+k}, x_{2n+k+1}) > \delta(A, B) - \epsilon. \quad (2.2)$$

On the contrary, assume that there exists $\epsilon_0 > 0$ such that for each $l \geq 1$, there is $m_l > n_l \geq l$ satisfying

$$d(x_{2m_l+k}, x_{2n_l+k+1}) \leq \delta(A, B) - \epsilon_0 \quad (2.3)$$

and

$$d(x_{2m_l-2+k}, x_{2n_l+k+1}) > \delta(A, B) - \epsilon_0. \quad (2.4)$$

It follows from (2.3) and (2.4) that

$$\begin{aligned} d(x_{2m_l-2+k}, x_{2n_l+k+1}) - d(x_{2m_l+k}, x_{2m_l-2+k}) &\leq d(x_{2m_l+k}, x_{2n_l+k+1}) \\ &\leq \delta(A, B) - \epsilon_0. \end{aligned}$$

Letting $l \rightarrow \infty$, property UC_f implies

$$\lim_{k \rightarrow \infty} d(x_{2m_l+k}, x_{2n_l+k+1}) = \delta(A, B) - \epsilon_0. \quad (2.5)$$

By relation (2.1),

$$\begin{aligned} \delta(A, B) - \epsilon_0 &\geq d(x_{2m_l+2k}, x_{2n_l+2k+1}) \\ &\geq \alpha^k d(x_{2m_l+k}, x_{2n_l+k+1}) + (1 - \alpha^k) \delta(A, B) \end{aligned}$$

Letting $l \rightarrow \infty$ and using (2.5), we obtain

$$\begin{aligned} \delta(A, B) - \epsilon_0 &\geq \alpha^k (\delta(A, B) - \epsilon_0) + (1 - \alpha^k) \delta(A, B) \\ &= \delta(A, B) - \alpha^k \epsilon_0, \end{aligned}$$

which is a contradiction. Therefore by (2.1) and property UC_f , $\{x_{2n+k}\}$ is a Cauchy sequence by property UC_f . Now the completeness of X and the closedness of A imply that $\lim_{n \rightarrow \infty} x_{2n+k} = z \in A$, whenever k is a even number. (if k is a odd number, $x_{2n+k} \rightarrow z \in B$ as $n \rightarrow \infty$.) Since T is a continuous map and $d(x_{2n+k}, x_{2n+k+1}) \rightarrow \delta(A, B)$ as $n \rightarrow \infty$, hence $d(z, Tz) = \delta(A, B)$. Therefore, T has a remotest point. □

3 Best Proximity Points

First, we give a useful approximation result.

Proposition 3.1 Let A and B be nonempty subsets of a metric space X and

$$d_{n+k} \leq \alpha^k d_n + (1 - \alpha^k) d(A, B), \quad (3.1)$$

where α, k are constant such that $0 < \alpha < 1, k \in \mathbb{N}$. Then $d_{n+k} \rightarrow d(A, B)$ as $n \rightarrow \infty$.

Proof. Let $m \in \mathbb{N}$ By inequality (3.1),

$$\begin{aligned} d_{km+k-1} &\leq \alpha^{km} d_{k-1} + (1 - \alpha^{km}) d(A, B), \\ d_{km+k-2} &\leq \alpha^{km} d_{k-2} + (1 - \alpha^{km}) d(A, B), \\ d_{km+k-3} &\leq \alpha^{km} d_{k-3} + (1 - \alpha^{km}) d(A, B), \end{aligned}$$

continuing this process,

$$\begin{aligned} d_{km+1} &\leq \alpha^{km} d_1 + (1 - \alpha^{km}) d(A, B) \text{ and} \\ d_{km} &\leq \alpha^{km} d_0 + (1 - \alpha^{km}) d(A, B), \end{aligned}$$

Letting $m \rightarrow \infty$,

$$d_{km+i} \rightarrow d(A, B), \quad i = 0, 1, \dots, k-1.$$

So $d_{n+k} \rightarrow d(A, B)$ as $n \rightarrow \infty$. □

Next, we are ready to state our main results in this section.

Proposition 3.2 *Let A and B be nonempty subsets of a metric space X and relation (3.1) holds. Then the sequences $\{x_{2n+k}\}$ and $\{x_{2n+k+1}\}$ are bounded.*

Proof. Since the sequence $\{d(x_{2n+k}, x_{2n+k+1})\}$ converges to $d(A, B)$ as $n \rightarrow \infty$, it is enough to prove that $\{x_{2n+k+1}\}$ is bounded. Suppose $\{x_{2n+k+1}\}$ is not bounded, then there exists a N_0 such that

$$d(x_{1+k}, x_{2N_0+k+1}) > M$$

and

$$d(x_{1+k}, x_{2N_0+1}) \leq M,$$

where $M > \max\{\frac{\alpha^k}{1-\alpha^k} d(x_1, x_{1+k}) + d(A, B), d(x_{1+k}, x_{1+2k})\}$. By relation (3.1),

$$\begin{aligned} M &< d(x_{1+k}, x_{2N_0+k+1}) \\ &\leq \alpha^k d(x_1, x_{2N_0+1}) + (1 - \alpha^k) d(A, B) \\ &\leq \alpha^k d(x_1, x_{1+k}) + \alpha^k d(x_{1+k}, x_{2N_0+1}) + (1 - \alpha^k) d(A, B) \\ &\leq \alpha^k d(x_1, x_{1+k}) + \alpha^k M + (1 - \alpha^k) d(A, B) \end{aligned}$$

Thus $M < \frac{\alpha^k}{1-\alpha^k} d(x_1, x_{1+k}) + d(A, B)$, which is a contradiction. □

The following Theorem explain a result when one of the sets is boundedly compact.

Theorem 3.1 *Let A and B be nonempty subsets of a metric space X and relation (3.1) holds. If either A or B is boundedly compact, then there exists $x \in A \cup B$ with $d(x, Tx) = d(A, B)$.*

Proof. Let k be even number. So $x_{2n+k} \in A$. Suppose A is boundedly compact. Since the sequence $\{x_{2n+k}\}$ is bounded and A is boundedly compact. Hence $\{x_{2n+k}\}$ has a convergent subsequence in A . Let $\{x_{2n_l+k}\}$ be a subsequence of $\{x_{2n+k}\}$ with $x_{2n_l+k} \rightarrow x \in A$ as $l \rightarrow \infty$. Since T is a continuous map and $d(x_{2n_l+k}, Tx_{2n_l+k}) \rightarrow d(A, B)$ as $l \rightarrow \infty$. Therefore $d(x, Tx) = d(A, B)$. The proof when k is odd number and B is boundedly compact is similar. □

Example 3.1 *Given $0 < \alpha < 1, k \in \mathbb{N}$. let A and B be nonempty subsets of l^p , $1 \leq p \leq \infty$ defined by $A = \{(1 + \alpha^{2n})e_{2n} : n \in \mathbb{N}\}$ and $B = \{(1 + \alpha^{2m-1})e_{2m-1} : m \in \mathbb{N}\}$. Suppose*

$$T((1 + \alpha^{2n})e_{2n}) = (1 + \alpha^{2n+1})e_{2n+1}$$

and

$$T((1 + \alpha^{2m-1})e_{2m-1}) = (1 + \alpha^{2m})e_{2m}.$$

Then relation (2.1) holds.

Proof. The case when $p = \infty$ is easy to check, so we consider $1 \leq p < \infty$.

$$\begin{aligned} ((1 + \alpha^{2m+k})^p + (1 + \alpha^{2n+1+k})^p)^{1/p} &= ((1 + \alpha^k - \alpha^k + \alpha^{2m+k})^p \\ &\quad + (1 + \alpha^k - \alpha^k + \alpha^{2n+1+k})^p)^{1/p} \\ &\leq ((\alpha^k + \alpha^{2m+k})^p + (\alpha^k + \alpha^{2n+1+k})^p)^{1/p} \\ &\quad + 2^{1/p}(1 - \alpha^k) \\ &\leq \alpha^k((1 + \alpha^{2m})^p + (1 + \alpha^{2n+1})^p)^{1/p} \\ &\quad + 2^{1/p}(1 - \alpha^k). \end{aligned}$$

Which implies that

$$d_{n+k} \leq \alpha^k d_n + (1 - \alpha^k) 2^{1/p},$$

where $d(A, B) = 2^{1/p}$. □

Theorem 3.2 Let A and B be nonempty closed subsets of a complete metric space X such that (A, B) has a property UC and relation (3.1) holds. Then T has a best proximity point.

Proof. We show that for each $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$,

$$d(x_{2m+k}, x_{2n+k+1}) < d(A, B) + \epsilon. \quad (3.2)$$

On the contrary, assume that there exists a $\epsilon_0 > 0$ such that for each $l \geq 1$, there is $m_l > n_l \geq l$ satisfying

$$d(x_{2m_l+k}, x_{2n_l+k+1}) \geq d(A, B) + \epsilon_0 \quad (3.3)$$

and

$$d(x_{2m_l-2+k}, x_{2n_l+k+1}) < d(A, B) + \epsilon_0. \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} d(A, B) + \epsilon_0 &\leq d(x_{2m_l+k}, x_{2n_l+k+1}) \\ &\leq d(x_{2m_l+k}, x_{2m_l-2+k}) + d(x_{2m_l-2+k}, x_{2n_l+k+1}) \\ &< d(x_{2m_l+k}, x_{2m_l-2+k}) + d(A, B) + \epsilon_0. \end{aligned}$$

Letting $l \rightarrow \infty$, and with property UC implies

$$\lim_{k \rightarrow \infty} d(x_{2m_l+k}, x_{2n_l+k+1}) = d(A, B) + \epsilon_0. \quad (3.5)$$

By relation (3.1),

$$\begin{aligned} d(A, B) + \epsilon_0 &\leq d(x_{2m_l+2k}, x_{2n_l+2k+1}) \\ &\leq \alpha^k d(x_{2m_l+k}, x_{2n_l+k+1}) + (1 - \alpha^k) d(A, B) \end{aligned}$$

Letting $l \rightarrow \infty$ and using (3.5), we obtain

$$\begin{aligned} d(A, B) + \epsilon_0 &\leq \alpha^k (d(A, B) + \epsilon_0) + (1 - \alpha^k) d(A, B) \\ &= d(A, B) + \alpha^k \epsilon_0 \\ &< d(A, B) + \epsilon_0, \end{aligned}$$

which is a contradiction. Therefore by (3.1) and property UC_f , $\{x_{2n+k}\}$ is a Cauchy sequence by property UC . Now the completeness of X and the closedness of A imply that $\lim_{n \rightarrow \infty} x_{2n+k} = z \in A$, whenever k is a even number. (if k is a odd number, $x_{2n+k} \rightarrow z \in B$ as $n \rightarrow \infty$.) Since T is a continuous map and $d(x_{2n+k}, x_{2n+k+1}) \rightarrow d(A, B)$ as $n \rightarrow \infty$, hence $d(z, Tz) = d(A, B)$. Therefore, T has a best proximity point. $\square$

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