A Presentation of the Free Lie Algebra $M_{2,m}$

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Abstract Let $M_{2,m}$ be a free metabelian nilpotent Lie algebra of rank 2 and nilpotency class $m - 1$. It is shown that $M_{2,m}$ admits a minimal presentation whose set of defining relators consists of certain types of basic commutators of length at most $m$.

Keywords Free Lie Algebra, Metabelian, Nilpotent, Presentation

1 Introduction

Describing Lie algebras by generators and defining relations is one of the natural ways defining presentation of Lie algebras. Though being a natural way of presenting Lie algebras, the presentation by generators and relations does not reveal much about the Lie algebra when the sets of generators and relations are infinite. Finding a finite presentation defining a given Lie algebra and the minimalization of this presentation are fundamental problems of Lie algebra presentations. Research studies related of this aspect include the works of R. Bryant (Bryant, 1999)[1] and V. P. Gerdt and V. Kornyak (Gerdt and Kornyak, 1996)[2]. Group case of these problems has been dealt in a number of papers (Gupta and Levin, 1986; Searby and Wamsley, 1972; Moghaddam, 1999; Wamsley, 1973)[3],[5],[6],[7].

In this work it is considered that free metabelian nilpotent Lie algebras of rank two and of nilpotency class $m$, $m \geq 2$ and it is found that a finite presentation of such algebras.

2 Preliminaries

Let $F$ be a free Lie Algebra generated by a set $X = \{x, y\}$ over a field $K$ of characteristic zero. Denote by $F'$ and $\gamma_m(F)$ the derived subalgebra and the $m-th$ term of the lower central series of $F$ respectively. It is considered the elements of $X$ as elements of length 1. For $m \geq 2$ an element $v$ of $F$ is of length $m$ if $v = [v_1, v_2]$ such that $v_1$ and $v_2$ are of length $m_1$ and $m_2$ respectively and $m = m_1 + m_2$. The basic words of length 1 are $x, y$ with fixed order $x > y$.

Having defined and ordered basic words of length less than $m$ it is defined those of length $m$ as the following: let $v_1, v_2$ be basic words of length $m_1$ and $m_2$, where $m = m_1 + m_2$; then $v = [v_1, v_2]$ is said to be basic if $v_1 > v_2$ and if $v_1 = [v_1', v_1'']$ then $v_2 \leq v_2$. Basic words of the same length are put in a fixed but arbitrary order. A basic word of length $m$ is greater than any of weight less than $m$. The set of all basic words of length $m$ is denoted by $H_m$.

Consider the Lie algebra $M_{2,m}$, $m \geq 2$, defined by the presentation

$$M_{2,m} = \langle X \mid \gamma_m(F) + F'' \rangle.$$ Then $M_{2,m} \cong F/\gamma_m(F) + F''$ is the free metabelian nilpotent Lie algebra of rank two and of nilpotency class $m - 1$. In this work, it is given that a minimal presentation of $M_{2,m}$. By $(D)$ is denoted the ideal of $F$ generated by any subset $D$ of $F$.

3 A Presentation of $M_{2,m}$

In this section, a minimal presentation of $M_{2,m}$ is going to be found for arbitrary but fixed integer $m$.

It is defined the following subsets of $F$:

$$B_{2,m} = \{((xy)y^i)x^j \mid i + j + 2 = m\},$$

$$T_{2,c} = \{((xy)y^r)x^s(x)y \mid r + s + 4 = c\},$$

$$D_{2,m} = B_{2,m} \bigcup_{c=5}^{m-1} T_{2,c}.$$ It is need the following technical lemma.

3.1 Lemma Let $a, b$ be monomials in $F$, then

$$(a(bx^k)) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} \langle((ax^{s-k}b)x^k)\rangle.$$ Then $m \geq 6, 2 \leq r \leq m - 4$ and $a = ((xy)y^r)$. Then
i) \((ax) y \in (D_{2,m})\), if \(r = m - 4\).

**Proof** i) Let \(r = m - 4\). Applying the Jacobi identity, it is obtained that

\[
(ax)y = -(ax)a - (ay)x = (a(xy)) + (ay)x.
\]

Since \((a(xy)) = ((xy)y^r)(xy) \in T_{2,m}\), \((a(xy)) \in (D_{2,m})\). Clearly \((ax)y = ((xy)y^r)x \in B_{2,m}\) and so \((ay)x \in (D_{2,m})\). Hence, \((ax)y \in (D_{2,m})\).

ii) Let \(2 \leq r \leq m - 5\). By the Jacobi identity, \((a(xy))\) can be written as

\[
(ax)y = (a(xy)) + (ay)x.
\]

Since \(r + 4 \leq m - 1\), \((a(xy)) = ((xy)y^r)(xy) \in T_{2,c} \subset (D_{2,m})\) (\(c \leq m - 1\)). So,

\[
(ax)y = (a(xy)) + (ay)x \equiv (a(xy))(\text{mod}(D_{2,m})).
\]

Hence it is obtained that

\[
(ax)y \equiv (a(xy))(\text{mod}(D_{2,m})).
\]

### 3.3 Lemma

Let \(m \geq 6\), \(a = ((xy)y^r)x^s\) and \(r + s \leq m - 5\), where \(r \geq 0\), \(s \geq 1\),

\[
(ax)y \equiv (a(xy))(\text{mod}(D_{2,m})).
\]

**Proof** Clearly \(r + s + 4 \leq m - 1\). This leads \((a(xy)) \in T_{2,c}\) where \(6 \leq c \leq m - 1\). Hence \((a(xy)) \in D_{2,m}\). Therefore,

\[
(ax)y = (a(xy)) + (ay)x \equiv (a(xy))(\text{mod}(D_{2,m})).
\]

### 3.4 Lemma

Let \(u\) be an element of \(F\) of the form

\[
u = (((xy)y^r)x^s)z_1z_2\ldots z_k,
\]

where \(k + i + j = m\), \(m \geq 6\) and \(z_1, z_2, \ldots, z_k \in \{x, y\}\), \(k \geq 0\). Then \(u \in (D_{2,m})\).

**Proof** The lemma will be proved by induction on \(j\) and \(k\). Let \(j = 1\) and \(k = 0\).

In this case \(u = (((xy)y^r)x)z\) and \(i + 4 = m\). Hence \(u \in (D_{2,m})\) by part (i) of Lemma 3.2. Assume that the Lemma is true for \(r \leq j - 1\) and \(s \leq k - 1\). Then \(w = (((xy)y^r)x^s)z_1z_2\ldots z_k \in (D_{2,m})\).

It will be proved that it is true for \(r\) and \(s\), that is;

\[
u = (((xy)y^r)x^s)z_1z_2\ldots z_k \in (D_{2,m}).
\]

Let \(v = (((xy)y^r)x^s)\). So,

\[
u = (((xy)y^r)x^s)z_1z_2\ldots z_k = (((xy)y^r)x^s)z_1z_2\ldots z_k z_{s+1}\ldots z_k.
\]

In this expression, \(((xy)y^r)x^s)\) can be written as follows:

\[
((xy)y^r)x^s = (((xy)y^r)x^s-1)xy = (((xy)y^r)x^{s-1})xy.
\]

From the equality \(i + j + k + 3 = m\), \(i + j - 1 = m - k - 4\). Since \(k \geq 0\) then \(i + j - 1 \leq m - 4\). So, \(i + j - 1 + 4 = m\) then \(((xy)y^r)x \in (D_{2,m})\) by applying part(i) of Lemma 3.2. Hence \(u \in (D_{2,m})\). If \(i + j - 1 \leq m - 5\) then by Lemma 3.3, it is had that

\[
((xy)y^r)x \equiv (((xy)y^r)x^{s-1})xy(\text{mod}(D_{2,m})).
\]

Since

\[
(((xy)y^r)x^{s-1})xy = (((xy)y^r)x^{s-1})xy\text{ and } i + j - 1 \leq m - 5
\]

then \(i + j - 2 \leq m - 6 < m - 5\). So

\[
((xy)y^r)x \equiv (((xy)y^r)x^{s-2})xy^2(\text{mod}(D_{2,m})).
\]

Since \(i + j - 1 \leq m - 5\) and \(r \leq m - 1\), \(i + j - 1 - t \leq m - 5\). Then, it is obtained that \(((xy)y^r)x \equiv ((xy)y^r)x^2(\text{mod}(D_{2,m})).

Thus, \(u\) can be written as,

\[
u \equiv (((((xy)y^r)x^2)z_1z_2\ldots z_k).
\]

If for every \(r = 1, 2, \ldots, s\), \(z_r = x\) then \(u \in (D_{2,m})\). Since

\[
u \equiv (((xy)y^r)x^{s+1})z_k(\text{mod}(D_{2,m})
\]

and \(w \in (D_{2,m})\). If each \(z_i \in z_1, z_2, \ldots, z_k\) is either \(x\) or \(y\) then the proof is going to be obtained as follows:

Let \(f_i = (((xy)y^r)x^{s+1})x)\). Since \(i + j + k + 3 = m\) and \(i + r + j + r + k + 3 = m\) we have \(i + r = m - k - 3 - (j - r)\). This leads \(i + r + 1 = m - k - 4 - (j - r)\). Here \(k \geq 1\), \(r \leq j - 1\) and \(j - r \geq 1\). This leads \(i + r < m - 5\). Hence it is found that:

\[
f_i \equiv (((xy)y^r)x^{s+1})x(\text{mod}(D_{2,m})).
\]

By successive application of the Jacobi identity it is obtained that

\[
f_i \equiv (((xy)y^r)x^{s+1})x(\text{mod}(D_{2,m})).
\]

Then \(u \equiv (((((xy)y^r)x^{s+1})x)z_{s+k}(\text{mod}(D_{2,m})).\) If \(z_1 = y\), \(u\) can be written as follows:

\[
u \equiv ((((((xy)y^r)x^{s+1})x)z_{s+k}(\text{mod}(D_{2,m})).
\]

Since \(i + j = m - k - 3\) then \(i + j \leq m - 4\) for \(k \geq 1\).

Let \(i + j = m - 4\) and let \(((xy)y^r)x^{s+1}) = g. Then

\[
u \equiv (((((xy)y^r)x^{s+1})x)z_{s+k}(\text{mod}(D_{2,m})).
\]


Here \( i + j - 1 + 4 = i + j + 4 = m \) and \( ((xy)^{i+1}x^{j-1})(xy) \in T_{2,m} \). Since \( T_{2,m} \subset \langle D_{2,m} \rangle \)
\[ u \equiv (((((xy)^{i+1}x^{j-1})x)(xy))(D_{2,m})). \]
Since \( i + j = m - 4 \) then \( i + j - 1 = m - 5 \). By Lemma 3.3, it is found that \( u \equiv ((((((xy)^{i+1}x^{j-2})y)(xy))(D_{2,m})). \) Again using Lemma 3.3,
\[ u \equiv (((((xy)^{i+1}x^{j-2})y)(xy))(mod(D_{2,m})). \]
is obtained. It is written that \( u \equiv ((((((xy)^{i+1}x^{j-2})y)(xy))(D_{2,m})). \) Here \( i + 2 + j - 3 = i + j - 1 = m - 5 \). So by
Lemma 3.3,
\[ u \equiv ((((((xy)^{i+1}x^{j-2})y)(xy))(mod(D_{2,m})). \]
It is continued similarly,
\[ u \equiv ((((((xy)^{i+1}x^{j-2})y)(xy))(mod(D_{2,m})). \]
is obtained. Since \( i + j + 1 = m - 4 \) and \( h \equiv (((((xy)^{i+2})x^{j-1}) \) then
\[ u \equiv ((((((xy)^{i+2})(xy)^{j-1})(xy))(mod(D_{2,m})). \]
Since \( i + 2 + j - 1 + 4 = i + j + 1 + 4 = m - 4 + 4 = m \) and 
\( (((xy)^{i+2})x^{j-1})(xy)(D_{2,m}) \) then the first term is an
element of \( T_{2,m} \). Thus the first term is also an element of \( D_{2,m} \) and so,
\[ u \equiv ((((((xy)^{i+2})x^{j-1})(xy))(z_{2})(mod(D_{2,m})). \]
Step 2: (The case of \( i + j + 1 = m - 4 \)) If \( i + j + 1 = m - 4 \) then \( i + j = m - 5 \). By Lemma 3.3,
\[ u \equiv ((((((xy)^{i+2})(xy)^{j-2})(xy))(mod(D_{2,m})). \]
Similarly
\[ u \equiv ((((((xy)^{i+2})(xy)^{j-2})(xy))(mod(D_{2,m})). \]
is obtained. If the computations in the first step is made
\[ u \equiv ((((((xy)^{i+2})(xy)^{j-2})(xy))(mod(D_{2,m})). \]
is got and
\[ u \equiv ((((((xy)^{i+2})(xy)^{j-2})(xy))(mod(D_{2,m})). \]
Since \( w \in (D_{2,m}) \) then \( u \in (D_{2,m}) \).
Step 3: (The case of \( i + j + 1 = m - 5 \)) In this case it is clear
that \( u \in (D_{2,m}) \) by Lemma 3.3. When all the cases are considered, it is obtained that \( u \equiv 0(mod(D_{2,m})). \)

3.5 Corollary Let \( u = (((xy)^{i}x^{j})(xy)) \) for \( i \geq 0, j \geq 0 \). Then
\[ u = (((xy)^{i}x^{j})(xy)) \in (D_{2,m}). \]
where \( i + j + 3 = m \) and \( m \geq 6 \).

Proof If \( j = 0 \) then \( u = (((xy)^{i}y)) \). In this word, \( i + 1 + 2 = m \) yields \( i + 0 + 3 = m \) and so \( u \in B_{2,m} \).
Since \( B_{2,m} \subset (D_{2,m}) \) then \( u \in (D_{2,m}). \)
There $1 \leq j \leq m - 3$, $k = 0$ is written applying Lemma 3.4. Hence $i + j + k + 3 = m$ and $(((xy^jy^k)x^l)yz_i)z_k \in \langle D_{2,m} \rangle$.

3.6 Lemma For $c \geq m$, $T_{2,c} \subset \langle D_{2,m} \rangle$.

Proof Let $u \in T_{2,c}$ and $u = (((xy^jy^k)x^l)(xy))$. Hence $r + s + 4 = c$. Using the Jacobi identity for $u$, it is obtained

$$u = -((x(y(((xy^jy^k)x^l)yz_i)z_k))) - (y(((xy^jy^k)x^l)yz_i)z_k)) = -(((xy^jy^k)x^l)yz_i)z_k + (((xy^jy^k)x^l)yz_i)z_k)$$

Let $f = (((((xy^jy^k)x^l)yz_i)z_k)y)x$ and $g = (((((xy^jy^k)x^l)yz_i)z_k)x)y$. In this case $r + s + 4 = c \geq m$. If $c = m$ then $r + s + 4 = m$ and $r + s = m - 4$. So, $r + s - 1 = m - 5$. It is written that $f = (((((xy^jy^k)x^l)yz_i)z_k)y)x = (((((xy^jy^k)x^l)yz_i)z_k)x)y$.

Here, $r + s - 1 = m - 5$ then $r + s = m - 6$ and $r + s - t = m - 5 - (t - 1)$ where $t \geq 1$. For $1 \leq t \leq s$, $r + s - t = m - 5 - (t - 1)$. By the Lemma 3.3,

$$f = (((xy^jy^k)x^l)yz_i)z_k)(\mod D_{2,m})$$

is obtained. Since $f \in B_{2,m}$ then $f \in \langle D_{2,m} \rangle$ and it is found that $f = 0(\mod D_{2,m})$. In the equality $g = (((((xy^jy^k)x^l)yz_i)z_k)y)x = (((((xy^jy^k)x^l)yz_i)z_k)x)y$. Here, it is obtained that $g \in \langle D_{2,m} \rangle$ applying the Corollary 3.5. Since $u = -f + g$ then $u \in \langle D_{2,m} \rangle$. Hence for $c = m$, $T_{2,c} \subset \langle D_{2,m} \rangle$.

If $c = m + 1$ then $r + s + 4 = m + 1$ and $r + s + 3 = m$. Let $g_1 = (((xy^jy^k)x^l)yz_i)$. So $g_1 \in B_{2,m} \subset \langle D_{2,m} \rangle$. Hence $g = (g_1y)x = (((xy^jy^k)x^l)yz_i)x$ and $a = (((xy^jy^k)x^l)yz_i)$. Using the Jacobi identity for $f_1$, then

$$f_1 = ((ax)y) = -(x(a)y) - ((ya)x) = (a(xy)) + ((xy)x) = (((xy^jy^k)x^l)yz_i)(xy) + (((xy^jy^k)x^l)yz_i)(xy))$$

In this case $r + s + 4 = m + 1$ and $r + s + 3 = m$. So, $(((xy^jy^k)x^l)yz_i) \in T_{2,m} \subset \langle D_{2,m} \rangle$. Let $b = (((xy^jy^k)x^l)yz_i)x$. Then $r + s + 1 + 3 = r + s + 3 = m$ and by the Lemma 3.2, $u \in \langle D_{2,m} \rangle$. Hence $u = -(f_1x) + g = -f + g$. So $u \in \langle D_{2,m} \rangle$.

If $c = m + 2$ then $r + s + 4 \geq m + 2$ and $r + s + 2 \geq m$. When it is considered that $r + s + 2 = m + k$ for $0 \leq k \leq s$, $u$ is obtained as follows:

$$u = -(((xy^jy^k)x^l)yz_i)(xy) + (((xy^jy^k)x^l)yz_i)(xy))$$

Let $h = (((xy^jy^k)x^l)yz_i)$. Here $r + s + 2 = m + k$ then $r + s + k + 2 = m$ and $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$. So, $h = (((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$ and $h \in \langle D_{2,m} \rangle$. Since $u = -((hxy) + ((hx)y), u \in \langle D_{2,m} \rangle$. Hence, for $c \geq m$ $T_{2,c} \subset \langle D_{2,m} \rangle$.

3.7 Lemma The elements of the form of $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$.

Proof By applying Lemma 3.1 consecutively in the expression $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$, the following expression is obtained

$$\sum_{k = 0}^{t}(-1)^{t-k} \binom{t}{k} \(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$$

is obtained. Since $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$, $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$. So,

$$\sum_{i = 0}^{l}(-1)^{l-i} \binom{l}{i} \(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$$

is obtained. Hence $(((xy^jy^k)x^l)yz_i) \in \langle D_{2,m} \rangle$.

3.8 Corollary Every element of $F''$ belong to the ideal $\langle D_{2,m} \rangle$ of $F$.

Proof For every $w \in F''$, the form of $w$ is $\sum_{i} \alpha_i u_i$, where $\alpha_i \in k$ and $u_i$ is a product of commutators of the form $(((xy^jy^k)x^l)yz_i)$.

3.9 Theorem $M_{2,m}$ admits the following presentation with exactly $\frac{(m^2 - 3m + 2)}{2}$ relators:

$$M_{2,m} = \langle X | \langle D_{2,m} \rangle \rangle, m \geq 2.$$
If \( 2 \geq m \geq 5 \) then straightforward calculations show that \( M_{2,m} \) has a presentation \( \langle X \mid H_m \rangle \), where \( H_m \) is the set of basic monomials of length \( m \). Clearly for \( m = 2, 3, 4, 5 \), \( H_m = D_{2,m} \). Therefore \( M_{2,m} = \langle X \mid D_{2,m} \rangle, m \geq 2 \). Since the elements of the set of \( D_{2,m} \) are basic words then they are linearly independent. Hence the presentation of \( M_{2,m} \) is minimal.

Now, the number of the relations of the presentation of \( M_{2,m} \) will be determined. If \( 2 \leq m \leq 5 \) then \( M_{2,m} = \langle X \mid H_m \rangle \). Hence the number of elements of \( H_m \) is \( \frac{1}{m}(2^m - 2) \) by Witt’s formula. If \( m \geq 6 \) then the number of the elements of \( B_{2,m} \) and \( T_{2,c} \) are computed.

If \( (((xy)y^{i})x^{j}) \in B_{2,m} \) then \( i + j = m - 2 \). Different values of \( i \) and \( j \) is considered. Since \( i + j = m - 2 \) then the number of the two-tuples \( (i,j) \) is \( m - 1 \).

If \( (((xy)y^{r})x^{s})(xy) \in T_{2,c} \) then \( i + j = m - 4 \). For \( c = 5 \), \( i + j = 1 \). Therefore, the number of the two-tuples \( (i,j) \) is \( 2 \). For \( c = m - 1 \), \( i + j = m - 5 \). Therefore, the number of the two-tuples \( (i,j) \) is \( m - 4 \). So the number of element of \( \bigcup_{c=5}^{m-1} T_{2,c} \) is \( 2 + 3 + \ldots + (m-4)(m-5) - 1 = \frac{m^2 - 7m + 10}{2} \).

Hence the number of all elements of \( D_{2,m} \) are

\[
\frac{m^2 - 7m + 10}{2} + m - 1 = \frac{m^2 - 5m + 8}{2}.
\]

REFERENCES


