Parametric Solutions to Equation $(pa^n + qb^n = pc^n + qd^n)$ Where 'n' Stands for Degree 2, 3, 4, 5, 6, 7, 8 & 9

Seiji Tomita^{1,*}, Oliver Couto²

¹Tokyo Software Company (Inc.), Tokyo, Japan
²University of Waterloo, Canada

Copyright©2017 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract Historically equation $(pa^n + qb^n + rc^n = pu^n + qv^n + rw^n)$ has been studied for degree 2, 3, 4 etc., and equation $(pa^n + qb^n = pc^n + qd^n)$ herein called equation (1) has been published for $n=4$, $p=1, q=4$ (Ref.no. 1) by Ajai Choudhry. Also Tito Piezas & others has discussed about equation (1) (Ref. no. 3 & 2). While Ref. no. (1, 2 & 3) deals with equation no. (1) for degree $n=4$ this paper has provided parametric solutions for degree $n=2, 3, 4, 5, 6, 7, 8$ & 9. Also there are instances in this paper where parametric solutions have been arrived at using different methods.

Keywords Diophantine Equations, Equal Sums of Powers, Pure Mathematics

Parametric Solutions to Equation

$$(pa^n + qb^n) = (pc^n + qd^n)$$

First we consider:

Degree two

For $n=2$ In equation (1) above;
We get $(pa^2 + qb^2 = pc^2 + qd^2)$
Hence we have after simplification:

$$\frac{p}{q} = \frac{d^2 - b^2}{a^2 - c^2}$$

We take $p = d^2 - b^2$ & $q = a^2 - c^2$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$(p, q) = (w, u)$$

Hence we get the Identity given below:

$$w(u+v)^2 + u(v-w)^2 = w(u-v)^2 + u(v+w)^2$$

Let: $(u, v, w) = (5, 3, 2)$

$$(p, q, a, b, c, d) = (2, 5, 8, 1, 2, 5)$$

We get:

$$2(8)^2 + 5(1)^2 = 2(2)^2 + 5(5)^2$$

Solution to: $pa^2 + qb^2 = pc^2 + qd^2$ where $(p, q) = (1, 3)$,

We have:

$$(1)a^2 + (3)b^2 = (1)c^2 + (3)d^2$$

The above equation has numerical solution $(a, b, c, d) = (1, 5, 7, 3)$

Using the substitution: $(a, b, c, d) = (1+t, 5+t, 7+kt, 3+t)$ we get after simplifying

$$t = \left(-\frac{14}{k+1}\right)$$

Substituting value of 't' in (a, b, c, d) we get the below mentioned parametric equation:

$$(1)(k-13)^2 + 3(5k-9)^2 = (1)(7k-7)^2 + (3)(3k-11)^2$$

For $k=3$ we have numerical solution as:

$$(1)5^2 + 3(3)^2 = (1)(7)^2 + (3)(1)^2$$

(Next): Degree (n=3)

There is a parametric solution for:

$$pa^3 + qb^3 = pc^3 + qd^3$$

$$a = u^2(qv^6 + pu^6)$$

$$b = (-2pu^6 + qv^6)$$

$$c = v^2(qv^6 + pu^6)$$

$$d = -(2qv^6 - pu^6)$$

a, b, p, q : arbitrary

$$pa^3 + qb^3 = pc^3 + qd^3$$

$$\text{let } a = u^2, b = t + v^2, c = mt + u^2, d = v^2$$

$$(q - pm^3)t^3 + (-3pu^2m^2 + 3qv^2)t^2 + (-3pu^4m + 3qv^4)t = 0$$

Then we obtain, $m = \frac{qv^4}{pu^4}$ &
 $t = \frac{-3v^2pu^6}{qv^6 + pu^6}$.

Substitute m and t to (2), and we obtain a solution.

Case. $(p, q, v) = (2, 3, 1)$ (After removing common factors) we get parametric form:

$$a = u^2(3 + 2u^6)$$

$$b = (-4u^6 + 3)$$

$$c = 2u^2(u^6 - 3)$$

$$d = (3 + 2u^6)$$

$$2(u^2(3 + 2u^6))^3 + 3(-4u^6 + 3)^3 \\ = 2(2u^2(u^6 - 3))^3 + 3(3 + 2u^6)^3$$

For $u=2, p=2, q=3$ we get numerical solution:

$$(2)(260)^3 + (3)(-253)^3 = (2)(488)^3 + (3)(131)^3$$

Alternate solution:

We have $(pa^3 + qb^3 = pc^3 + qd^3)$

We get:

$$\frac{p}{q} = \frac{d^3 - b^3}{a^3 - c^3}$$

Let $a=u+v, b=v-w, c=u-v, d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_3(u+v)^3 + q_3(v-w)^3 = p_3(u-v)^3 + q_3(v+w)^3$$

Where:

$$p_3 = w(3v^2 + w^2)$$

$$q_3 = v(3u^2 + v^2)$$

Let: $(u, v, w) = (3, 2, 1)$

$$(p, q, a, b, c, d) = (31, 126, 5, 1, 1, 3)$$

$$13(5)^3 + 62(1)^3 = 13(1)^3 + 62(3)^3$$

Additional method:

We have below mentioned equation:

$$(pa^3 + qb^3 = pc^3 + qd^3)$$

For $(p, q) = (1, 86)$ above equation becomes

$$(1)a^3 + (86)b^3 = (1)c^3 + (86)d^3$$

The above has numerical solution $(a, b, c, d) = (9, 2, 11, 1)$

We substitute $(a, b, c, d) = (9+t, 2+kt, 11+t, 1+kt)$

We also have $(b-d) = [(2+kt) - (1+kt)] = 1$ and

$$(c-a) = [(11+t)-(9+t)] = 2$$

After simplification we get

$$t = \left(\frac{20 - 129k}{43k^2 - 1} \right)$$

After substituting for t in (a, b, c, d) we get the below parameterization:

$$a = 387k^2 - 129k + 1$$

$$b = -43k^2 + 20k - 2$$

$$c = 473k^2 - 129k + 9$$

$$d = -86k^2 + 20k - 1$$

For $k=1$, we get numerical solution:

$$(1)(269)^3 + (86)(67)^3 \\ = (1)(353)^3 + 86(25)^3 \quad (1)$$

Degree (n=4)

We show solutions of quartic equation with coefficient 'h' for degree $n=4$

$$a^4 + hb^4 = c^4 + hd^4$$

Gerardin gave a solution of $a^4 + hb^4 = c^4 + hd^4$ (Ref. no. 4)

$$(2k^2)^4 + h(k-1)^4 = (2k)^4 + h(k+1)^4$$

For $h = 2k^3(k^2 - 1)$

Let $a = px+w, b = qx-1, c = px-w, d = qx+1$, then (3) becomes (4) given below.

$$(8wp^3 - 8hq^3)x^3 + (-8hq + 8w^3p)x$$

To obtain a solution of (4), we have to find the rational solution of (5).

$$v^2 = -64w^4p^4 + 64whqp^3 + 64hq^3w^3p - 64h^2q^4$$

Let us consider eqn. (5) as a quadratic equation in h.

We find two solutions of (5) as follows.

$$h = \left(\frac{w}{q} \right) (p)^3$$

Substitute $h = p^3(w/q)$ to (5), then

$$v^2 = -64p^4w^2(p-w)(p+w)(q-1)(q+1).$$

Let: $w = m^2 + n^2, p = m^2 - n^2, q = \frac{m^2+n^2}{2mn}$, then $h = 2(m^2 - n^2)^3(mn)$

From eqn. (4) above we get: $x = \frac{4m^2n^2}{-2m^2n^2+n^4+m^4}$

We obtain a solution below: (Form A)

$$a = (n-m)(n+m)(n^4 - 4m^2n^2 - m^4)$$

$$b = 2m^3n + 2n^3m + 2m^2n^2 - n^4 - m^4$$

$$c = (n-m)(n+m)(n^4 + 4m^2n^2 - m^4)$$

$$d = 2m^3n + 2n^3m - 2m^2n^2 + n^4 + m^4$$

$$h = 2mn(m^2 - n^2)^3$$

By using this new solution as a known solution, we obtain more new solutions.

Next we take $h = \frac{pw}{q}$, Similarly, (5) becomes to

$$v^2 = 64w^2p^2(w-1)(w+1)(p-q)(p+q)$$

$$v^2 = (8wp)^2(w^2 - 1)(p^2 - q^2)$$

Let $w = \frac{m^2+n^2}{2mn}, q = m^2 + n^2, p = m^2 - n^2$

Then

$$h = 8(m^2 - n^2)m^3n^3 \text{ and } x = -\frac{1}{4}\left(\frac{-m^2+n^2}{m^2n^2}\right)$$

We obtain another solution below: (Form B)

$$a = 2mn(m^4 - 2m^2n^2 + n^4 + 2m^3n + 2mn^3)$$

$$b = m^4 - n^4 - 4m^2n^2$$

$$c = 2mn(m^4 - 2m^2n^2 + n^4 - 2m^3n - 2mn^3)$$

$$d = m^4 - n^4 + 4m^2n^2$$

$$h = 8m^3n^3(m^2 - n^2)$$

Alternate solution:

To get equation of the type below:

$$pa^4 + qb^4 = pc^4 + qd^4$$

We put $n=1$ in the equation [Form (B) above] and then substitute $m=x/y$, giving us the parametric form below.

$$a = 2x(x^4 - 2x^2y^2 + y^4 + 2x^3y + 2xy^3)$$

$$b = y(x^4 - y^4 - 4x^2y^2)$$

$$c = 2x(x^4 - 2x^2y^2 + y^4 - 2x^3y - 2xy^3)$$

$$d = y(x^4 - y^4 + 4x^2y^2)$$

$$p = y^5$$

$$q = 8x^3(x^2 - y^2)$$

For $x=1$ & $y=2$ we get the numerical solution:

$$(4)(29)^4 + 3(1)^4 = (4)(11)^4 + 3(31)^4 \quad (2)$$

Alternate solution:

We have $(pa^4 + qb^4 = pc^4 + qd^4)$

We get:

$$\frac{p}{q} = \frac{d^4 - b^4}{a^4 - c^4}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_4(u+v)^4 + q_4(v-w)^4 = p_4(u-v)^4 + q_4(v+w)^4$$

Where:

$$p_4 = w(v^2 + w^2)$$

$$q_4 = u(u^2 + v^2)$$

Let: $(u, v, w) = (5, 3, 2)$ & we get: $(p, q, a, b, c, d) = (26, 170, 8, 1, 2, 5)$

$$26(8)^4 + 170(1)^4 = 26(2)^4 + 170(5)^4 \quad (3)$$

(See Table below for parametric solutions)

Table (A). (degree fourth power)

$$(pa^4 + qb^4 = pc^4 + qd^4)$$

For $p=f(m)$ & $q=f(m, n)$, where $f(n)$ & $f(m, n)$ are a function in variable 'n' & '(m, n)' respectively

The following parametric solutions were found:

$(p, q) = [f(m), f(m, n)]$	$a \& b$	$c \& d$
$m^4, (n^2 + 4m^2)(n^2 + 2m^2)$	$\frac{n^2 - nm + 2m^2}{(n + m)m}$	$n^2 + nm + 2m^2, (n - m)m$
$m^4, 2(n^4 + 6n^2m^2 + m^4)$	$\frac{n^2 - 2nm + m^2}{(n + m)m}$	$n^2 + 2nm + m^2, (n - m)m$
$m^4, 2(n^2 + 9m^2) * (n^2 + 3m^2)$	$\frac{n^2 - 2nm + 3m^2}{(n + m)m}$	$n^2 + 2nm + 3m^2, (n - m)m$
$m^4, 3(n^2 + 4m^2) * (n^2 - 2m^2)$	$\frac{n^2 - 3nm - 2m^2}{(n + m)m}$	$n^2 + 3nm - 2m^2, (n - m)m$
$m^4, 3(n^4 + 11n^2m^2 + m^4)$	$\frac{n^2 - 3nm + m^2}{(n + m)m}$	$n^2 + 3nm + m^2, (n - m)m$
$m^6, (n^6 + 2n^4m^2 + n^2m^4 + m^4)$	$\frac{n^3 + nm^2 - m^3}{(n + m)m^2}$	$n^3 + nm^2 + 1, (n - m)m^2$

Next for fifth power:

Degree (n=5)

We have eqn. $(pa^5 + qb^5 = pc^5 + qd^5)$

We get:

$$\frac{p}{q} = \frac{d^5 - b^5}{a^5 - c^5}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_5(u+v)^5 + q_5(v-w)^5 = p_5(u-v)^5 + q_5(v+w)^5$$

Where:

$$\begin{aligned} p_5 &= w(5v^4 + 10v^2w^2 + w^4) \\ q_5 &= v(5u^4 + 10u^2v^2 + v^4) \end{aligned}$$

Let: $(u, v, w) = (5, 3, 2)$ & we get:

$$(p, q, a, b, c, d) = (1562, 16368, 8, 1, 2, 5)$$

$$1562(8)^5 + 16368(1)^5 = 1562(2)^5 + 16368(5)^5 \quad (4)$$

Degree (n=6)

We have $(pa^6 + qb^6 = pc^6 + qd^6)$

We get:

$$\frac{p}{q} = \frac{d^6 - b^6}{a^6 - c^6}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_6(u+v)^6 + q_6(v-w)^6 = p_6(u-v)^6 + q_6(v+w)^6$$

Where:

$$\begin{aligned} p_6 &= w(v^2 + 3w^2)(3v^2 + w^2) \\ q_6 &= u(u^2 + 3v^2)(3u^2 + v^2) \end{aligned}$$

Let: $(u, v, w) = (5, 3, 2)$

$$(p, q, a, b, c, d) = (1302, 21840, 8, 1, 2, 5)$$

$$1302(8)^6 + 21840(1)^6 = 1302(2)^6 + 21840(5)^6$$

Degree (n=7)

We have $(pa^7 + qb^7 = pc^7 + qd^7)$

We get:

$$\frac{p}{q} = \frac{d^7 - b^7}{a^7 - c^7}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_7(u+v)^7 + q_7(v-w)^7 = p_7(u-v)^7 + q_7(v+w)^7$$

Where:

$$\begin{aligned} p_7 &= [w(7v^6 + 35v^4w^2 + 21v^2w^4 + w^6)] \\ q_7 &= [v(7u^6 + 35u^4v^2 + 21u^2v^4 + v^6)] \end{aligned}$$

Let: $(u, v, w) = (5, 3, 2)$ & we get for degree $n=7$ the below mentioned numerical solution:

$$(p, q, a, b, c, d) = (19531, 524256, 8, 1, 2, 5)$$

$$\begin{aligned} 19531(8)^7 + 524256(1)^7 \\ = 19531(2)^7 + 524256(5)^7 \quad (5) \end{aligned}$$

Degree n=8

We have $(pa^8 + qb^8 = pc^8 + qd^8)$

We get:

$$\frac{p}{q} = \frac{d^8 - b^8}{a^8 - c^8}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_8(u+v)^8 + q_8(v-w)^8 = p_8(u-v)^8 + q_8(v+w)^8$$

Where: $p_8 = w(v^2 + w^2)(w^4 + 6v^2w^2 + v^4)$

$$q_8 = u(v^2 + u^2)(v^4 + 6u^2v^2 + u^4)$$

Let: $(u, v, w) = (5, 3, 2)$ & we get numerical solution:

$$(p, q, a, b, c, d) = (4069, 174760, 8, 1, 2, 5)$$

$$4069(8)^8 + 174760(1)^8 = 4069(2)^8 + 174760(5)^8 \quad (6)$$

Degree n=9

We have $(pa^9 + qb^9 = pc^9 + qd^9)$

We get:

$$\frac{p}{q} = \frac{d^9 - b^9}{a^9 - c^9}$$

Let $a=u+v$, $b=v-w$, $c=u-v$, $d=v+w$

By resolving above polynomial into factors, we get $\{p, q\}$.

$$p_9(u+v)^9 + q_9(v-w)^9 = p_9(u-v)^9 + q_9(v+w)^9$$

Where: $p_9 = w(3v^2 + w^2)(w^6 + 33v^2w^4 + 27v^4w^2 + 3v^6)$

$$q_9 = v(3u^2 + v^2)(v^6 + 33u^2v^4 + 27u^4v^2 + 3u^6)$$

Let: $(u, v, w) = (5, 3, 2)$

$$(p, q, a, b, c, d) = (976562, 266304, 8, 1, 2, 5)$$

$$\begin{aligned} 976562(8)^9 + 266304(1)^9 \\ = 976562(2)^9 + 266304(5)^9 \quad (7) \end{aligned}$$

On a side note, the interested reader may look at the below identities (A, B, C & D) that were derived for the equation (2) given below:

$$ma^n + nb^n = uc^n + vd^n \quad (2)$$

Here the coefficients for eqn.(2) are different than that used for equation (1) at the top of the page with the coefficients on the left hand side of equation (2) being different from the coefficients on the right hand side.

$$(m, n, u, v) = (1, 3, 1, 4)$$

$$[A] \quad (1)(12x^8 + 6x^4 - 1)^4 + 3(2x)^4 = \\ = (1)(12x^8 - 6x^4 - 1)^4 + 4(12x^7)^4$$

$$(m, n, u, v) = (1, 6, 1, 2)$$

$$[B] \quad (1)(48x^8 + 12x^4 - 1)^4 + 6(2x)^4 = \\ = (1)(48x^8 - 12x^4 - 1)^4 + 2(48x^7)^4$$

$$(m, n, u, v) = (1, 24, 1, 8)$$

$$[C] \quad (1)(3x^8 + 3x^4 - 1)^4 + 24(x)^4 = \\ = (1)(3x^8 - 3x^4 - 1)^4 + 8(3x^7)^4$$

$$(m, n, u, v) = (1, 27, 1, 36)$$

$$[D] \quad (1)(972x^8 + 54x^4 - 1)^4 + 27(2x)^4 = \\ (1)(972x^8 - 54x^4 - 1)^4 + 36(324x^7)^4$$

[See below for numerical solutions for equation (1) for different degree 'n']

Equation (1) $(pa^n + qb^n = pc^n + qd^n)$ – for $n=2, 3, 4, 5, 6, 7, 8 \& 9$

Table (B). (Solutions degree 'n')

Degree (n)	p	q	a	b	c	d
2	2	3	11	1	1	9
3	13	62	5	1	1	3
4	5	3	5	4	1	6
5	25	4	34	9	12	49
6	19	5	4	2	1	5
7	547	32	6	1	2	9
8	353	60	4	3	2	5
9	729	19	4	3	1	6

REFERENCES

- [1] Ajai Choudhry, A Note on the Quartic Diophantine Equation, $a^4 + hb^4 = c^4 + hd^4$. <https://arxiv.org/pdf/1608.06220v1>
- [2] Farzali & Mehdi, (Is the quartic equation $(a^4 + hb^4 = c^4 + hd^4)$, solvable for any integer h?) Journal Arxiv, 1701 Jan. 2017
- [3] Tito Piezas-Online collection algebraic identities, equation $a^4 + hb^4 = c^4 + hd^4$. equation for fourth power, <http://sites.google.com/site/tpiezas>
- [4] Gerardin on the eqn. $a^4 + hb^4 = c^4 + hd^4$, pg.648, vol. (20, Dickson, history of theory of number theory, Chelsea publication, year 2000
- [5] Parametric solutions to (six) n^{th} powers equal to another (six) n^{th} powers for degree 'n' = 2,3,4,5,6,7,8,& 9, Universal Journal of applied mathematics Vol. (4), (3), pages 55-65, 2016, <http://www.hrpub.org>
- [6] Published math paper, Oliver Couto & Seiji Tomita, Sum of three biquadratics a multiple of a n^{th} power, Universal Journal of applied mathematics, Vol. (4)(1), pages 22-31, 2016, <http://www.hrpub.org>
- [7] Published math paper, Oliver Couto & Seiji Tomita, Generalized parametric solution to multiple sums of powers, Universal Journal of applied mathematics, Jan 2016, Volume 3(5), <http://www.hrpub.org>
- [8] Published math paper, Oliver Couto & Seiji Tomita, Solution of Polynomial Equation of any degree 'n' with Special Emphasis For $n = (2, 3, 4, 5 \& 6)$, Journal "SOP transactions on applied mathematics" ISSN (Print): 2373-8472, July 2015. <http://www.scipublish.com/journals/am>
- [9] Published math paper, Oliver Couto, Taxicab Equations for power (2, 3, 4 & 5), Journal, International Math Forum, Hikari Ltd., Vol.9, Jan 2014.n0.12, pages 561-577. <http://www.m-hikari.com/imf.html>
- [10] Math paper, Seiji Tomita & Oliver Couto, Sum $(k+3)$ & $(k+5)$ bi-quadratics, Scheduled to be posted on-line, Universal Journal of applied mathematics, Vol. (4) (2), 2016, <http://www.hrpub.org>
- [11] Seiji Tomita, webpage <http://www.maroon.dti.ne.jp/fermat> Web site "Computation number theory"
- [12] Oliver Couto, webpage, <http://www.celebrating-mathematics.com> Web site "Number theory Math"
- [13] Ramanujan lost notebook, Narosa publishing house, New Delhi, India
- [14] Euler Leonhard, Opera Omnia books, 1984 publication.