Height-unmixed of Tensor Product of Lattices

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Abstract We investigate WB-height-unmixed of tensor product of distributive lattices: Cohen-Macaulay rings related to tensor products of distributive lattices are constructed using the method of Stanley and Reisner.

Keywords Distributive Lattices, Tensor Product, Cohen-Macaulay Rings, Stanley-Reisner Ring, Unmixedness

2010 AMS(MOS) Subject Classification: 13H10, 08B10, 08B25

1 Introduction

All rings and algebras considered in this paper are commutative with identity elements and k stands for a perfect field. Suitable background on depth of algebraic geometry and Cohen-Macaulay rings is [3], [8], and [11].

We recall that tensor products were introduced in J. Anderson and N. Kimura [1] and G. A. Fraser [5]. Our aim in this paper is to prove that the Cohen-Macaulay property is inherited by tensor products of k -algebras and tensor products of distributive lattices. Tensor products of semilattices and related structures have already been the source of many publications ([5] and [7]). We denote by $A \otimes B$ the tensor product of A and B , which A and B are $\{\vee, 0\}$ -semilattices.

In this paper, we suppose k is a perfect field and F and K be two extension fields of k such that $F \otimes K$ is Noetherian. Then we prove that if L_1 and L_2 are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2][X_1, X_2, \dots]$ is WB-height-unmixed. Finally, WB-height-unmixed is related to universally catenary and approximately Cohen-Macaulay ring.

2 Tensor Product and Cohen-Macaulay Rings

Let A be a lattice. If $a, b \in A$, the join and meet of a and b are written as $a \vee b$ and $a \wedge b$, respectively. If $a_i \in A$ for $i \in I$, where I is any non-empty set, then the join and meet of $\{a_i : i \in I\}$, if they exist, are denoted by $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$, respectively. The smallest and largest elements of A , if they exist, are denoted by 0 and 1 , respectively. Tensor products of vector spaces are well-known, but it exists in many other categories equipped with a forgetful functor to **Set** [14]. Examples of such categories that matters to us are boolean algebra, distributive lattices, semilattices with zero, etc. Given A, B and X , three object of the same category, a bimorphism from $A \times B$ to X is a set theoretic map $f : A \times B \rightarrow X$ such that for all $a \in A$ and for all $b \in B$, the mappings $f(a, -) : B \rightarrow X$ and $f(-, b) : A \rightarrow X$ are morphisms. Being given an object X of the category and a bimorphism $i : A \times B \rightarrow X$, we say that X is a tensor product of $A \times B$ if for every object C and every bimorphism $f : A \times B \rightarrow C$, there exist a unique morphism $h : X \rightarrow C$, such that $f = h \circ i$. Tensor products are unique up to isomorphisms and they are denoted by $A \otimes B$. The bimorphism i is not surjective but its image generates $A \otimes B$, thus we call generating elements (of $A \otimes B$) those coming from $A \otimes B$ and we will write $i(a, b) = a \otimes b$. Now, we have the following definitions from [5].

Definition 2.1. Let A, B and C be distributive lattices. A function

$$f : A \times B \rightarrow C$$

is a bihomomorphism if the functions $g_a : B \rightarrow C$ defined by $g_a(b) = f(a, b)$ and $h_b : A \rightarrow C$ defined by $h_b(a) = f(a, b)$ are homomorphisms for all $a \in A$ and $b \in B$.

Definition 2.2. Let A and B be distributive lattices. A distributive lattice C is a tensor product of A and B (in the category $\mathfrak{D}$), if there exists a bihomomorphism $f : A \times B \rightarrow C$, such that C is generated by $f(A \times B)$ and for any distributive lattice D and any bihomomorphism $g : A \times B \rightarrow D$, there is a homomorphism $h : C \rightarrow D$ satisfying $g = hf$.

Note that since $f(A \times B)$ generates C , the homomorphism h is necessarily unique. Let A and B be distributive lattices. Then a tensor product of A and B in the category $\mathfrak{D}$ exists and is unique up to isomorphism (see [5]).

The tensor product of A and B is denoted by $A \otimes B$ and the image of (a, b) under the canonical bihomomorphism $f : A \times B \rightarrow A \otimes B$ is written as $a \otimes b$. Now, $A \otimes B$ is the distributive lattice generated by the elements $a \otimes b$ ($a \in A, b \in B$), subject to the bihomomorphic conditions:

$$(a_1 \vee a_2) \otimes b = (a_1 \otimes b) \vee (a_2 \otimes b),$$

$$(a_1 \wedge a_2) \otimes b = (a_1 \otimes b) \wedge (a_2 \otimes b),$$

$$a \otimes (b_1 \vee b_2) = (a \otimes b_1) \vee (a \otimes b_2),$$

and

$$a \otimes (b_1 \wedge b_2) = (a \otimes b_1) \wedge (a \otimes b_2),$$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Every element of $A \otimes B$ can be written in the form $\bigvee_{i=1}^n (a_i \otimes b_i)$ for some $a_i \in A$ and $b_i \in B, i = 1, 2, \dots, n$.

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [4] and Gratzner [10]. In this paper, relationship among tensor product of the Cohen-Macaulay rings, tensor product of the distributive lattices, WB-height-unmixed and Stanley-Reisner ring are considered.

We first recall the definition of an algebra with straightening lows in [6], then recall WB-height-unmixed in ideal of a commutative ring in [11].

Definition 2.3. Given a commutative ring R with identity element. If P is a finite partially ordered set, then we say that $\mathfrak{A}$ is a ASL (algebra with straightening lows) on P over R if the followings hold:

ASL-0. An injective map $P \hookrightarrow \mathfrak{A}$ is given, $\mathfrak{A}$ is a graded R -algebra generated by P , and each element of P is a homogeneous of positive degree. We call a product of elements of P a monomial in P . In general, a monomial M is a map $P \rightarrow \mathbb{N}_0$ and we denote $M = \prod_{x \in P} x^{M(x)}$ such that it also stands for an element of $\mathfrak{A}$. A monomial in P of the form

$$x_{i_1} \cdots x_{i_l}$$

with $x_{i_1} \leq \cdots \leq x_{i_l}$ is called standard.

ASL-1. The set of standard monomials in P is an R -free basis of $\mathfrak{A}$.

ASL-2. For $x, y \in P$ that $x \not\leq y$ and $y \not\leq x$, there is an expression of the form

$$xy = \sum_M c_M^{xy} M \quad (c_M^{xy} \in R)$$

where the sum is taken over all standard monomials

$$M = x_1 \cdots x_{r_M} \quad (x_1 \leq \cdots \leq x_{r_M})$$

with $x_1 < x, y$ and $\deg M = \deg(xy)$.

The most simple example of an ASL on P over R is the Stanley-Reisner ring $R[P] = R[x \mid x \in P]/(xy \mid x \not\leq y, y \not\leq x)$. The Stanley-Reisner rings play central role in theory of ASL. For the proof of the following theorem, see [6].

Theorem 2.4. If R is Cohen-Macaulay ring, and if P is a distributive lattice, then $R[P]$ is Cohen-Macaulay.

Let I is ideal of the ring R . Then we say that an ideal I is unmixed if I has no embedded prime divisors or, in modern language, if the associated prime ideals of R/I are the minimal prime ideals of I . We know a prime ideal $\mathfrak{p}$ is a weak Bourbaki associated prime of the ideal I of the ring R if for some $a \in R$ it is a minimal ideal of the form $I : a$.

We recall that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes, which an ideal is height-unmixed if all the associated primes of I have equal height.

Before proving main theorem let us note that a ring is regular if for all prime ideal $\mathfrak{p}$ of R , $R_{\mathfrak{p}}$ is a regular local ring. Recently, in [2], Bouchiba and Kabbaj showed that if R and S are k -algebras such that $R \otimes_k S$ is Noetherian then $R \otimes_k S$ is a Cohen-Macaulay ring if and only if R and S are Cohen-Macaulay rings. Now we are ready to present our main theorem.

Theorem 2.5. Let k be a perfect field, F and K be two extension fields of k such that $F \otimes K$ is Noetherian. If L_1 and L_2 are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2][X_1, X_2, \dots]$ is WB-height-unmixed.

Proof. By ([4], Theorem 2.6), $L_1 \otimes L_2$ is distributive lattice if L_1 and L_2 are distributive lattices. On the other hand, in [15] and [2], it is proved that the tensor product of two extension fields of k is not necessarily Noetherian. Here, for all two k -algebras F and K we have $F \otimes K$ is Noetherian and k is perfect field. From the note on page 49 of [13] we will have that $F \otimes K$ is regular ring. By using the following well-known chain we conclude $F \otimes K$ is Cohen-Macaulay ring:

$$\text{Regular} \implies \text{Complete intersection} \implies \text{Gorenstein} \\ \implies \text{Cohen Macaulay}$$

On the other hand, by applying that Theorem 2.4, if R is Cohen-Macaulay and P is distributive lattice, then $R[P]$ is Cohen-Macaulay ring. Here, we will have $(F \otimes K)[L_1 \otimes L_2]$ is Cohen-Macaulay ring. By ([12], Theorem 2.3), $(F \otimes K)[L_1 \otimes L_2][X_1, X_2, \dots]$ is WB-height-unmixed. $\square$

We will say that a Noetherian ring R is catenary if every saturated chain joining prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ has (maximal) length $\text{height } \mathfrak{q}/\mathfrak{p}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Also, we say that R is universally catenary if all the polynomial rings $R[X_1, X_2, \dots]$ are catenary.

Theorem 2.6. *Let k be a perfect field, F and K be two extension fields of k such that $F \otimes K$ is Noetherian. If L_1 and L_2 are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.*

Proof. In Theorem 2.5, we showed that $(F \otimes K)[L_1 \otimes L_2]$ is a Cohen-Macaulay ring. By applying that ([3], Theorem 2.1.12), we see that every Cohen-Macaulay ring is universally catenary. Therefore, $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary. $\square$

Corollary 2.7. *Let k be a perfect field, F and K be two extension fields of k such that $F \otimes K$ is Noetherian. If L_1 and L_2 are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is Cohen-Macaulay ring.*

Corollary 2.8. *Let k be a perfect field, F and K be two extension fields of k such that $F \otimes K$ is Noetherian. If L_1 and L_2 are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.*

Corollary 2.9. *Let k be a perfect field, F and K be two extension fields of k such that $F \otimes K$ is Noetherian. If L_1 and L_2 are distributive lattices, then any quotient of $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.*

Now, assume that $(R, \mathfrak{m})$ is a local ring with $\dim(R) = d$. We say that R is an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element r of $\mathfrak{m}$ such that $R/a^n R$ is a Cohen-Macaulay ring of dimension $d - 1$ for every integer $n > 0$ ([9]). A ring R is called an approximately Cohen-Macaulay ring if the ring $R_{\mathfrak{p}}$ is an approximately Cohen-Macaulay ring, for all prime ideals $\mathfrak{p}$ of R .

Corollary 2.10. *Let k be a perfect field and F and K be nonzero k -algebras such that $F \otimes K$ is Noetherian. Assume that F is not a Cohen-Macaulay ring. If $F \otimes K$ is an approximately Cohen-Macaulay ring, then $K[X_1, X_2, \dots]$ is universally catenary.*

Acknowledgements

The author is deeply grateful of Professor I. Zamani for his careful reading of the paper and valuable suggestions.

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