Height-unmixed of Tensor Product of Lattices

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Abstract We investigate WB-height-unmixed of tensor product of distributive lattices: Cohen-Macaulay rings related to tensor products of distributive lattices are constructed using the method of Stanley and Reisner.

Keywords Distributive Lattices, Tensor Product, Cohen-Macaulay Rings, Stanley-Reisner Ring, Unmixedness

2010 AMS(MOS) Subject Classification: 13H10, 08B10, 08B25

1 Introduction

All rings and algebras considered in this paper are commutative with identity elements and $k$ stands for a perfect field. Suitable background on depth of algebraic geometry and Cohen-Macaulay rings is [3], [8], and [11].

We recall that tensor products were introduced in J. Anderson and N. Kimura [1] and G. A. Fraser [5]. Our aim in this paper is to prove that the Cohen-Macaulay property is inherited by tensor products of $k$-algebras and tensor products of distributive lattices. Tensor products of semilattices and related structures have already been the source of many publications ([5] and [7]). We denote by $A \otimes B$ the tensor product of $A$ and $B$, which $A$ and $B$ are $\{\vee, 0\}$-semilattices.

In this paper, we suppose $k$ is a perfect field and $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. Then we prove that if $L_1$ and $L_2$ are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2][X_1, X_2, \ldots]$ is WB-height-unmixed. Finally, WB-depth-unmixed is related to universally catenary and approximately Cohen-Macaulay ring.

2 Tensor Product and Cohen-Macaulay Rings

Let $A$ be a lattice. If $a, b \in A$, the join and meet of $a$ and $b$ are written as $a \vee b$ and $a \wedge b$, respectively. If $a_i \in A$ for $i \in I$, where $I$ is any non-empty set, then the join and meet of $\{a_i : i \in I\}$, if they exist, are denoted by $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$, respectively. The smallest and largest elements of $A$, if they exist, are denoted by $0$ and $1$, respectively. Tensor products of vector spaces are well-known, but it exists in many other categories equipped with a forgetful functor to Set [14]. Examples of such categories that matters to us are boolean algebra, distributive lattices, semilattices with zero, etc. Given $A, B$ and $X$, three object of the same category, a bimorphism from $A \times B$ to $X$ is a set theoretic map $f : A \times B \rightarrow X$ such that for all $a \in A$ and all $b \in B$, the mappings $f(a, -) : B \rightarrow X$ and $f(-, b) : A \rightarrow X$ are morphisms. Being given an object $X$ of the category and a bimorphism $i : A \times B \rightarrow X$, we say that $X$ is a tensor product of $A \times B$ if for every object $C$ and every bimorphism $f : A \times B \rightarrow C$, there exist a unique morphism $h : X \rightarrow C$, such that $f = h \circ i$. Tensor products are unique up to isomorphisms and they are denoted by $A \otimes B$.

The bimorphism $i$ is not surjective but its image generates $A \otimes B$, thus we call generating elements (of $A \otimes B$) those coming from $A \otimes B$ and we will write $i(a, b) = a \otimes b$. Now, we have the following definitions from [5].

Definition 2.1. Let $A$, $B$ and $C$ be distributive lattices. A function $f : A \times B \rightarrow C$

is a bihomomorphism if the functions $g_a : B \rightarrow C$ defined by $g_a(b) = f(a, b)$ and $h_b : A \rightarrow C$ defined by $h_b(a) = f(a, b)$ are homomorphisms for all $a \in A$ and $b \in B$.

Definition 2.2. Let $A$ and $B$ be distributive lattices. A distributive lattice $C$ is a tensor product of $A$ and $B$ (in the category $\mathcal{D}$), if there exists a bihomomorphism $f : A \times B \rightarrow C$, such that $C$ is generated by $f(A \times B)$ and for any distributive lattice $D$ and any bihomomorphism $g : A \times B \rightarrow D$, there is a homomorphism $h : C \rightarrow D$ satisfying $g = hf$. 
Note that since \( f(A \times B) \) generates \( C \), the homomorphism \( h \) is necessarily unique. Let \( A \) and \( B \) be distributive lattices. Then a tensor product of \( A \) and \( B \) in the category \( \mathcal{D} \) exists and is unique up to isomorphism (see [5]).

The tensor product of \( A \) and \( B \) is denoted by \( A \otimes B \) and the image of \( (a, b) \) under the canonical bihomomorphism \( f : A \times B \rightarrow A \otimes B \) is written as \( a \otimes b \). Now, \( A \otimes B \) is the distributive lattice generated by the elements \( a \otimes b \) \((a \in A, \ b \in B)\), subject to the bihomomorphic conditions:

\[
(a_1 \lor a_2) \otimes b = (a_1 \otimes b) \lor (a_2 \otimes b),
\]
\[
(a_1 \land a_2) \otimes b = (a_1 \otimes b) \land (a_2 \otimes b),
\]
\[
a \otimes (b_1 \lor b_2) = (a \otimes b_1) \lor (a \otimes b_2),
\]
and
\[
a \otimes (b_1 \land b_2) = (a \otimes b_1) \land (a \otimes b_2),
\]
for all \( a, a_1, a_2 \in A \) and \( b, b_1, b_2 \in B \). Every element of \( A \otimes B \) can be written in the form \( \bigvee_{i=1}^{n} (a_i \otimes b_i) \) for some \( a_i \in I \) and \( b_i \in B, i = 1, 2, \ldots, n \).

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [4] and Gratzer [10]. In this paper, relationship among tensor product of the universal algebra, consult Birkhoff [4] and Gratzer [10].

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We first recall the definition of an algebra with straightening lows in [6], then recall WB-height-unmixed in ideal of a commutative ring in [11].

**Definition 2.3.** Given a commutative ring \( R \) with identity element. If \( P \) is a finite partially ordered set, then we say that \( \mathfrak{A} \) is an ASL (algebra with straightening lows) on \( P \) over \( R \) if the followings hold:

ASL-0. An injective map \( P \hookrightarrow \mathfrak{A} \) is given, \( \mathfrak{A} \) is a graded \( R \)-algebra generated by \( P \), and each element of \( P \) is a homogeneous of positive degree. We call a product of elements of \( P \) a monomial in \( P \). In general, a monomial \( M \) is a map \( P \rightarrow \mathbb{N}_0 \) and we denote \( M = \prod_{x \in P} x^{M(x)} \) such that it also stands for an element of \( \mathfrak{A} \). A monomial in \( P \) of the form

\[
x_{i_1} \cdots x_{i_l}
\]

with \( x_{i_1} \leq \cdots \leq x_{i_l} \) is called standard.

ASL-1. The set of standard monomials in \( P \) is an \( R \)-free basis of \( \mathfrak{A} \).

ASL-2. For \( x, y \in P \) that \( x \nleq y \) and \( y \nleq x \), there is an expression of the form

\[
xy = \sum_{M} c_{M}^{xy} M \ (c_{M}^{xy} \in R)
\]

where the sum is taken over all standard monomials

\[
x_{1} \leq \cdots \leq x_{r_{M}}
\]

with \( x_1 < x, y \) and \( \deg M = \deg(xy) \).

The most simple example of an ASL on \( P \) over \( R \) is the Stanley-Reisner ring \( R[P] = R[x \mid x \in P]/(xy \mid x \nleq y, y \nleq x) \). The Stanley-Reisner rings play central role in theory of ASL. For the proof of the following theorem, see [6].

**Theorem 2.4.** If \( R \) is Cohen-Macaulay ring, and if \( P \) is a distributive lattice, then \( R[P] \) is Cohen-Macaulay.

Let \( I \) is ideal of the ring \( R \). Then we say that an ideal \( I \) is unmixed if \( I \) has no embedded prime divisors or, in modern language, if the associated prime ideals of \( R/I \) are the minimal prime ideals of \( I \). We know a prime ideal \( p \) is a weak Bourbaki associated prime of the ideal \( I \) of the ring \( R \) if for some \( a \in R \) it is a minimal ideal of the form \( I : a \).

We recall that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes, which is height-unmixed if all the associated primes of \( I \) have equal height.

Before proving main theorem let us note that a ring is regular if for all prime ideal \( p \) of \( R \), \( R/p \) is a regular local ring. Recently, in [2], Bouchiba and Kabbaj showed that if \( R \) and \( S \) are \( k \)-algebras such that \( R \otimes_k S \) is Noetherian then \( R \otimes_k S \) is a Cohen-Macaulay ring if and only if \( R \) and \( S \) are Cohen-Macaulay rings. Now we are ready to present our main theorem.

**Theorem 2.5.** Let \( k \) be a perfect field, \( F \) and \( K \) be two extension fields of \( k \) such that \( F \otimes K \) is Noetherian. If \( L_1 \) and \( L_2 \) are distributive lattices, then the ring \( (F \otimes K)[L_1 \otimes L_2][X_1, X_2, \ldots] \) is WB-height-unmixed.

**Proof.** By ([4], Theorem 2.6), \( L_1 \otimes L_2 \) is distributive lattice if \( L_1 \) and \( L_2 \) are distributive lattices. On the other hand, in [15] and [2], it is proved that the tensor product of two extension fields of \( k \) is not necessarily Noetherian. Here, for all two \( k \)-algebras \( F \) and \( K \) we have \( F \otimes K \) is Noetherian and \( k \) is perfect field. From the note on page 49 of [13] we will have that \( F \otimes K \) is regular ring. By using the following well-known chain we conclude \( F \otimes K \) is Cohen-Macaulay ring:

Regular \( \implies \) Complete intersection \( \implies \) Gorenstein \( \implies \) Cohen Macaulay

On the other hand, by applying that Theorem 2.4, if \( R \) is Cohen-Macaulay and \( P \) is distributive lattice, then \( R[P] \) is Cohen-Macaulay ring. Here, we will have \( (F \otimes K)[L_1 \otimes L_2][X_1, X_2, \ldots] \) is WB-height-unmixed. \( \square \)
We will say that a Noetherian ring $R$ is catenary if every saturated chain joining prime ideals $p$ and $q$ has (maximal) length height $q/p$ such that $p \subseteq q$. Also, we say that $R$ is universally catenary if all the polynomial rings $R[X_1, X_2, \ldots]$ are catenary.

**Theorem 2.6.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Proof.** In Theorem 2.5, we showed that $(F \otimes K)[L_1 \otimes L_2]$ Cohen-Macaulay ring. By applying that ([3], Theorem 2.1.12), we see that every Cohen-Macaulay ring is universally catenary. Therefore, $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Corollary 2.7.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is Cohen-Macaulay ring.

**Corollary 2.8.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Corollary 2.9.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any quotient of $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

Now, assume that $(R, m)$ is local ring with $\dim(R) = d$. We say that $R$ is an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element $r$ of $m$ such that $R/m^nR$ is a Cohen-Macaulay ring of dimension $d - 1$ for every integer $n > 0$ ([9]). A ring $R$ is called an approximately Cohen-Macaulay ring if the ring $R_0$ is an approximately Cohen-Macaulay ring, for all prime ideals $p$ of $R$.

**Corollary 2.10.** Let $k$ be a perfect field and $F$ and $K$ be nonzero $k$-algebras such that $F \otimes K$ is Noetherian. Assume that $F$ is not a Cohen-Macaulay ring. If $F \otimes K$ is an approximately Cohen-Macaulay ring, then $K[X_1, X_2, \ldots]$ is universally catenary.

Acknowledgements

The author is deeply grateful of Professor I. Zamani for his careful reading of the paper and valuable suggestions.

REFERENCES