Renormalization Group Limit of Anderson Models

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Abstract We present the adaptive feedback scaling method for the Anderson localization analysis of several large classes of random Hamiltonians in discrete and continuous disordered media. We also give a constructive scale-free criterion of localization with asymptotically exponential decay of eigenfunction correlators, which can be verified in applications with the help of numerical methods.

Keywords Multi-particle Anderson Localization, Eigen-value Concentration Estimates

1 Introduction

This paper focuses on the rigorous mathematical treatment of the Anderson localization phenomena, very important in physics of solid state, mostly for the respective Green functions. The physical foundations of this theory have been laid down by P. W. Anderson in his Nobel prize winning paper [1]. Derivation of stronger forms of localization is only briefly discussed in Section 2.3, since here we rely on techniques and results developed earlier and well-understood by now. The main novelty of this paper concerns the asymptotical rate of decay of the Green functions (GFs), which is essentially reproduced in the eigenfunctions (EFs) and averaged eigenfunction correlators (EFCs); the latter characterize the dynamical properties of localized eigenstates. Specifically, we show that a particularly organized scaling procedure, building on the ideas and techniques from [28, 97, 47, 49, 16, 20, 18] and close in spirit to the renormalization group approach, results in a fairly elementary proof of *exponential scaling limit* (ESL), i.e., asymptotically exponential decay of all functionals involved (GFs, EFs and EFCs). This provides stronger localization bounds than those obtained with the help of the Germinet–Klein Bootstrap Multi-Scale Analysis [47, 49] and extends the results of [18] from discrete models to several large classes of disordered systems.

We study various types of random Hamiltonians $H(\omega)$ on configuration spaces $\mathcal{X}$ which can be of different geometrical

nature. We assume that $\mathcal{X}$ is endowed with the structure of a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and of a measure space $(\mathcal{X}, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is a completion of the Borel σ -algebra of $(\mathcal{X}, d_{\mathcal{X}})$ and μ is a σ -finite σ -additive positive real measure. This class covers periodic lattices in $\mathbb{R}^d$, e.g., $\mathbb{Z}^d$, $d \geq 1$, as well as more general combinatorial graphs (here $d_{\mathcal{X}}$ is the graph-distance and μ is the counting measure on the complete sigma-algebra of subsets of $\mathcal{X}$), metric graphs over combinatorial graphs (often called quantum graphs in the context of spectral theory of quantum systems), and Euclidean spaces. We assume that $H(\omega)$, relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, is a.s. (almost surely) defined as a self-adjoint operator in the Hilbert space $\mathcal{H} = L^2(\mathcal{X}, \mu)$, with some common domain $D_H \subset \mathcal{H}$.

Long ago, Pólya and Szegő [88] noticed: “... *one must not forget that there are two kinds of generalization, one facile and one valuable. One is generalization by dilution, the other is generalization by concentration. Dilution means boiling the meat in a large quantity of water into a thin soup; concentration means condensing a large amount of nutritive material into an essence. The unification of concepts which in the usual view appear to lie far removed from each other is concentration.*”

Hopefully, the reader would find the generalization used in the present paper concentrating and not diluting: it allows one to avoid proving several times similar results by similar arguments. Instead, we prove just once main general results which apply to various particular models of disordered quantum systems.

The Multi-Scale Analysis (MSA), Bn general, relies on deep analytic and probabilistic results, first of all the resolvent inequalities of Simon-Lieb type, Wegner estimate, and a Lifshitz-type asymptotics of the Integrated Density of States (IDS; cf., e.g., [13, Chapter VI], [56, Section 5]) near the spectral edges.

The main goal of this paper is to provide a fairly general Renormalization Group type analysis, focusing on deriving the flow of the key scaling exponents. Its main distinctive feature is a systematic use of *self-improving* inductively obtained scaling exponents, and not merely reproducing their values obtained at earlier stages of scaling induction. Surprisingly, that is all: to achieve stronger decay estimates than

with earlier developed variants of the Multi-Scale Analysis, one needs merely *not to discard* the surplus of decay, obtained at a stage k – surplus compared to what is required for reproducing at the next stage $k + 1$ the decay of the stage k .

Specifically, we show that one can *easily* obtain the exponential renormalization group limit for the Green functions, and consequently for the EFs and EFCs; the limiting decay rate of all these objects is exponential, provided the local probability distribution encoded in $H(\omega)$ is "almost" Hölder continuous. The precise condition is formulated in (1).

Recall that the Fractional-Moment Method (FMM) [6, 3, 7, 4], when applicable, allows one to establish a genuine exponential decay of EFCs under the assumption of Hölder continuity of the local disorder distribution of any positive order $\beta > 0$. This also implies exponential decay of EFs. The MSA, too, leads to the proof of exponential decay of the EFs, but can only establish a fractional-exponential decay of the EFCs, of the form $L \mapsto e^{-L^\zeta}$, and the Germinet-Klein BMSA [47] shows that any fixed $\zeta \in (0, 1)$ can be achieved, at sufficiently large scales L .

◆ Note, however, that there are important classes of models where the FMM applies but provides only a *slower* (sub-exponential) decay of the EFCs and, consequently, of the EFs, while the MSA proves the exponential decay of the latter. Cf. [37, 19]. Moreover, there are important models of disordered quantum systems to which the FMM cannot be applied; these include systems with a singular local disorder (modeling a substitution disorder in imperfect crystals; cf. e.g. [10, 49]), and almost-periodic operators (cf., e.g., [9]).

Technically, the main localization analysis is performed in a model system, where the structure of Hilbert space and many details of the specific random Hamiltonian are lost (or better to say, ignored), and only the graph structure of expanding subsets of the underlying configuration space $\mathcal{X}$ is exploited. This degree of abstraction proves very useful for unifying the proofs of strong dynamical localization in various models studied in the literature over the last thirty years. Simply put, it is easier to perform scaling analysis just once, in a rather general discrete model, than to do it every time again in a large number of more complex models describing physically different disordered systems.

Introduce some notation. Let be given a finite or locally finite countable graph with the vertex set $\mathcal{Z}$ and edge set $\mathcal{E}$, endowed with the canonical graph distance $d = d_{\mathcal{Z}}$; a ball of radius $L \geq 0$ centered at $x \in \mathcal{Z}$ is denoted by $B_L(x)$. We fix arbitrary $d \in [1, +\infty)$, $C_d \in (0, +\infty)$ and assume that $\mathcal{Z}$ belongs to the class $\mathfrak{Z}(d, C_d)$ of graphs with the following property:

$$\forall L \geq 1 \quad \sup_{x \in \mathcal{Z}} \text{card}(B_L(x)) \leq C_d L^d.$$

The assumption $\mathcal{Z} \in \mathfrak{Z}(d, C_d)$ will always be in force.

In the presented approach, the graph $\mathcal{Z}$ may serve only as a geometrical substrate to an auxiliary analytic problem which is *not a spectral* problem; one aims here to obtain quantitative decay bounds on the solutions to a large or infinite system of inequalities for non-negative functions. One of the most important examples is the periodic lattice $\mathcal{Z} = \mathbb{Z}^d \hookrightarrow \mathbb{R}^d = \mathcal{X}$,

canonically injected in the Euclidean space; the latter is covered by cubes (or balls in any suitable metric) $C_R(x)$, centered at the lattice vertices x and with sufficiently large radius R . The L^2 -norm of a square-summable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ can be upper-bounded by those of its restrictions to the cubes $C_R(x)$, effectively replacing f for the estimation purposes with a non-negative function $F_f : \mathbb{Z}^d \rightarrow \mathbb{R}_+$, $F_f(x) = \|\mathbf{1}_{C_R(x)} f\|_2$. Once the kernel of the resolvent of a given operator in $L^2(\mathbb{R}^d)$ is replaced in a similar manner by an infinite matrix, with arguments in $\mathbb{Z}^d$, standard techniques (resolvent inequalities) allow one to proceed with these matrices, which do *not* represent a Hamiltonian and are *not* studied from the point of view of spectral analysis.

However, in the simplest case of a discrete Anderson model, the graph $\mathcal{Z}$ can itself be the configuration space $\mathcal{X}$, hosting the Hilbert space $\mathcal{H} = \ell^2(\mathcal{X})$ and a random ensemble of Hamiltonians $H(\omega)$ acting in $\mathcal{H}$. In this particular (but quite instructive) case, we work with a random Hamiltonian $H(\omega)$ which is a discrete Schrödinger operator of the form

$$(H\psi)(x) = \sum_{d(x,y)=1} (\psi(x) - \psi(y)) + V(x; \omega)\psi(x),$$

where $V : \mathcal{X} \times \Omega \rightarrow \mathbb{R}$ is an IID random field relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

Remark 1.1. *For the sake of clarity and brevity, we usually assume that marginal probability distribution function (PDF) of the random field V , $F_V(t) := \mathbb{P}\{V(0; \omega) \leq t\}$, $t \in \mathbb{R}$, is Hölder continuous of some order $\beta \in (0, 1]$, i.e., the common marginal probability distribution function (PDF) of the random field $V(x, \omega)$ generating the potential (or otherwise incorporated into the Hamiltonian $H(\omega)$),*

$$F_V(t) := \mathbb{P}\{V(x, \omega) \leq t\}, \quad x \in \mathcal{X}, \quad t \in \mathbb{R},$$

has the continuity modulus $\mathfrak{s}_V(s) := \sup_{a \in \mathbb{R}} (F_V(a + s) - F_V(a))$ obeying $\mathfrak{s}_V(s) \leq C s^\beta$, $s \in [0, 1]$, $\beta \in (0, 1]$. However, as shown in [18], the assumption of Hölder continuity can be slightly relaxed and replaced by (cf. [18, Eqn. (6.1)])

$$\mathfrak{s}(\epsilon) \leq C' \epsilon^{\frac{C}{\ln |\ln \epsilon|}}, \quad C, C' \in (0, +\infty). \quad (1)$$

If the graph $\mathcal{Z}$ is finite or V is a.s. bounded, then $H(\omega)$ is a.s. well-defined as a bounded self-adjoint operator in the Hilbert space $\ell^2(\mathcal{Z})$. In any case, $H(\omega)$ is a.s. self-adjoint with the core formed by compactly supported functions. A more general and much more challenging case, where the "physical" configuration space locally looks like a Riemannian manifold (for we usually need some kind of Laplacian), is reduced – on the level of inequalities for the Green functions – to the discrete setting with the help of the well-known and versatile Simon–Lieb Inequality.

Speaking formally, the mathematical novelty of the paper is conveyed by Sections 4 and 5, which are rather short, and fits into three or four, half-page elementary calculations focusing on the key scaling exponents. The new technique results in stronger localization bounds proved in a shorter way than with the help of earlier methods. However, non-experts in Anderson localization theory may need a considerable time to go through numerous publications in the area

of spectral analysis of random operators, appeared over the last three decades, to get a panorama of accomplished or ongoing research projects in this exciting field of mathematical physics.

The second part of the paper (Section 7) aims precisely at presenting such a panorama, albeit without in-depth discussions of the fundamental techniques – in order to keep the size of the paper within reasonable limits.

Speaking of the FMM approach, the availability of the exponential scaling limit, following and further developing the progress made by Germinet and Klein [47], significantly narrows the gap between the two leading techniques, FMM and MSA, of which the latter is applicable to a larger class of models of disordered systems.

2 Origins of the finite-volume criteria of localization

The first finite-volume criteria for the onset of Anderson localization appeared shortly after the pioneering works by Fröhlich and Spencer [45] and Fröhlich, Martinelli, Scoppola and Spencer [44]. Von Dreifus, in his Ph.D. project [28] under the advisorship by Spencer, derived exponential decay of Green functions in the lattice Anderson model with diagonal disorder from a power-law decay of the GFs at some initial scale L_0 ; cf. (2). The required upper bound on the probability in Eqn. (2) depends upon the scale L_0 . Spencer [97] developed a scale-independent version of the localization criterion (cf. Proposition 2.2): the RHS of Eqn (4) just needs to be small enough, while L_0 may be (and in some applications, is required to be) large enough. The analytical and probabilistic schemes employed in [28, 97] share a number of common features. However, the idea of bootstrapping the MSA estimates did not appear until the paper by Germinet and Klein [47].

The authors of [28, 97] acknowledge the role of the ideas and suggestions by Jennifer and Lincoln Chayes [14].

2.1 A scale-dependent criterion

Von Dreifus studied in his Ph.D. thesis [28] two models of disordered systems, including a random lattice Schrödinger operator. The techniques used for the proof of exponential decay of the Green functions are quite close to those required for the proof of a scale-free criterion, established by Spencer [97], but the reformulation of the geometrical scaling scheme carried out in [97] had far-going ramifications, as was shown later by Germinet and Klein [47].

Proposition 2.1 (Cf. [28, Theorem 4.1]). *Assume that for some $\rho_0 \in (0, 1)$ and $p > d$*

$$\mathbb{P} \left\{ \sum_{y \in \partial^- \Lambda_{L_0}(0)} |G_{\Lambda_{L_0}(0)}(0, y)| < \rho_0 \right\} \geq 1 - L_0^{-p}, \quad (2)$$

with $\partial^-(\cdot)$ defined as in (6), and let $L_k = L_{k-1}^\alpha$, for all

$k \geq 1$, and $m = \frac{2 \ln \rho_0^{-1}}{L_0}$. Then for all $k \geq 1$ and $x \in \mathbb{Z}^d$

$$\mathbb{P} \left\{ \sum_{\partial^- \Lambda_{L_k}(x)} |G_{\Lambda_{L_k}(x)}(0, y; E)| < \rho_0 \right\} \geq 1 - L_k^{-p}. \quad (3)$$

2.2 A scale-free criterion

Spencer [97] proposed a different variant of the scale induction used in [28], resulting in a significant reduction of the combinatorial "entropy" entering the probabilistic analysis of the GF decay in cubes of growing size L_k .

Proposition 2.2 (Cf. [97, Theorem 1]). *There exists $\delta_0 > 0$ such that if for some $l \geq 4$ and $p > d$*

$$\mathbb{P} \left\{ \max_{a \in \Lambda_{l/2}(0)} \sum_{b \in \partial^- \Lambda_l(0)} |G_{\Lambda_l(0)}(a, y; E_0)| l^2 \geq \frac{1}{2} \right\} \leq \delta_0, \quad (4)$$

then there exist an interval $I = [E_0 - \epsilon, E_0 + \epsilon]$, $\epsilon > 0$, and $m > 0$ such that for all $E \in I$ the Green functions exponentially with rate m .

2.3 On derivation of stronger forms of localization

The finite-volume variant of the variable-energy MSA from [44] was developed by von Dreifus and Klein. At that time, the only general method for derivation of spectral localization from the fixed-energy MSA, available to the authors of [28, 97], was the Simon–Wolff spectral averaging [96]. As to the proof of dynamical localization building on the results of the MSA, one had to wait ten years after [28, 97], for the works by Germinet and De Bièvre [46], Damanik and Stollmann [26] and Germinet and Klein [47]. It was the latter work where it has been fully realized that the finite-volume fixed-energy criteria lay ground for a "bootstrapped", multi-stage strategy of localization analysis, gradually deriving the strongest features of Anderson localization from their weakest manifestations. The dependencies between the key parameters of random operators were made explicit in [48].

Once the fixed-energy MSA bounds are obtained, by any method, one can transform them into more informative, variable-interval MSA bounds on the Green functions, employing the techniques originally developed by Elgart *et al.* [31, 30], as discussed in [18, Section 5, Proposition 1]. The general idea was originally used by Martinelli and Scoppola [80] who thus inferred absence of a.c. spectrum from a fast decay of the GFs at fixed energy.

Finally, dynamical localization can be inferred from the variable-energy bounds, with the help of a "soft" argument proposed by Germinet and Klein [47], as explained, e.g., in [18, Section 5, Theorem 3].

Summarising, in a large class of Anderson (and some other disordered) models, the fundamental analytic and probabilistic work can be done on the level of fixed-energy scaling analysis, and existing techniques ultimately put the fixed energy

GF localization, variable-energy GF localization, and strong dynamical localization on essentially equal footage.

3 Main results

As was said, we mainly work with an auxiliary, large or infinite system of recursive inequalities for non-negative functions on a finite or locally finite, unordered combinatorial graph $\mathcal{Z}$ with graph-distance $d = d_{\mathcal{Z}}$.

The main scaling procedure is performed with the help of a growing sequence of length scales L_k , starting with a sufficiently large positive integer L_0 and setting recursively

$$L_{k+1} = \lfloor L_k^\alpha \rfloor. \quad (5)$$

It is readily seen that for any $\alpha > 1$ and $L_0 \geq 2^{\frac{1}{\alpha-1}}$, the sequence L_k is unbounded, and one has $L_k \geq 2^k L_0$. Speaking informally, L_k grow approximately as $\sim 2^{\alpha^k}$.

The two most popular classes of Anderson models are discrete (finite-difference) and differential Schrödinger operators $H(\omega) = H_0 + V(\omega)$, where H_0 is the kinetic energy operator (usually a Laplacian or graph Laplacian), and $V(\omega)$ is the operator of multiplication by a sample of a random field $V : \mathcal{X} \times \Omega \rightarrow \mathbb{R}$, relative to a probability space $(\Omega, \mathfrak{B}, \mathbb{P})$, defined on the configuration space $\mathcal{X}$.

Analytically, the simplest case is that of a combinatorial graph $\mathcal{Z}$ with the respective graph distance $(x, y) \mapsto d_{\mathcal{Z}}(x, y)$ and the graph Laplacian $H_0 = -\Delta_{\mathcal{Z}}$, so we consider it first to provide the motivation for introducing the notion of a specification. Given a finite connected subgraph $\Lambda \subset \mathcal{Z}$, we introduce the finite-volume, or restricted, Hamiltonian $H_\Lambda = \mathbf{1}_\Lambda H \mathbf{1}_\Lambda$, where the indicator function $\mathbf{1}_\Lambda$ of Λ is identified with the operator of multiplication by $\mathbf{1}_\Lambda$. Next, introduce the resolvent, defined for E outside the spectrum $\Sigma(H_\Lambda)$ of H_Λ ,

$$G_\Lambda(E) = (H_\Lambda - E)^{-1},$$

and its matrix elements in the standard delta-basis, called the Green functions:

$$G_\Lambda(x, y; E) = \langle \mathbf{1}_x, (H_\Lambda - E)^{-1} \mathbf{1}_y \rangle.$$

One usually operates with upper bounds, so only the absolute values $|G_\Lambda(x, y; E)|$ are actually used. Further, the inductive analysis of decay of the GFs for $d_{\mathcal{Z}}(x, y) \gg 1$ can be reduced to that of the functions of one spatial argument,

$$\mathbf{G}_\Lambda(x) = \|\mathbf{1}_x (H_\Lambda - E)^{-1} \mathbf{1}_{\partial^- \Lambda}\| \in \overline{\mathbb{R}}_+$$

with

$$\partial^- \Lambda = \{x \in \Lambda : d_{\mathcal{Z}}(x, \mathcal{Z} \setminus \Lambda) = 1\}, \quad (6)$$

where it is convenient to set, formally, $\mathbf{G}_\Lambda(x) = +\infty$ whenever $H_\Lambda - E$ is not invertible. In inequalities involving only non-negative quantities, with the convention $t < +\infty$ for any $t \in \mathbb{R}$, this is unambiguous.

In the case of a continuous configuration space, say, in $\mathbb{R}^d$, it is well-known by now that the Green functions can be effectively replaced (again, in inequalities operating with upper bounds on absolute values) by their discretized counterparts,

$$\mathbb{Z}^d \times \mathbb{Z}^d \ni (a, b) \mapsto \|\mathbf{1}_{\mathbf{C}(a)} (H_\Lambda - E)^{-1} \mathbf{1}_{\mathbf{C}(b)}\|$$

where $\mathbf{C}(a)$ and $\mathbf{C}(b)$ are, e.g., unit cubes centered at $a, b \in \mathbb{Z}^d$. (A similar construction can be used for the metric graphs.) Once such a discretization is performed, the crucial part of scaling analysis is carried out in terms of the functions $\mathbf{G}$ of a discret argument ranging in a combinatorial graph.

Starting from the next section we shall call the functions $\mathbf{G}_\bullet$, labeled by finite subsets of a combinatorial graph, Dominated Random Decay (DRD, in short) specifications. The motivation for such specifications comes from an important analytical property of the Green functions, relating the resolvents in two embedded domains $\Lambda \subsetneq \Lambda'$ is given by a direct analog of the Simon–Lieb inequality (SLI):

$$\|\mathbf{1}_{\mathbf{C}(b)} G_{\Lambda'}(E) \mathbf{1}_{\mathbf{C}(a)}\| \leq C \|\mathbf{1}_{\mathbf{C}(b)} G_{\Lambda'}(E) \mathbf{1}_{\partial \Lambda}\| \|\mathbf{1}_{\partial \Lambda} G_\Lambda(E) \mathbf{1}_{\mathbf{C}(a)}\|,$$

with a suitable definition of the boundary layer $\partial \Lambda$.

3.1 Abstract DRD-specifications

Definition 3.1. A DRD-specification (or simply specification) on a graph $\mathcal{Z}$ is a pair of families of a random functions $\mathbf{G}_\Lambda : \Lambda \times \Omega \rightarrow \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$ and $\mathbf{C}_\Lambda^G : \Omega \rightarrow \overline{\mathbb{R}}_+$, relative to a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and defined for all finite connected subgraphs $\Lambda \subset \mathcal{Z}$, which fulfills the following conditions:

- (G1) [“All-or-none” principle] for any subgraph $\Lambda \subset \mathcal{Z}$ and any $\omega \in \Omega$, if $\mathbf{G}_\Lambda(x'; \omega) = +\infty$ for at least one vertex $x' \in \Lambda$, then $\mathbf{G}_\Lambda(x; \omega) = +\infty$ for any $x \in \Lambda$.
- (G2) [Dominated decay property] for any pair of embedded subgraphs $\Lambda \subsetneq \Lambda' \subseteq \mathcal{Z}$, and any pair of vertices $x \in \Lambda$, $y \in \Lambda' \setminus \Lambda$

$$\mathbf{G}_{\Lambda'}(x; \omega) \leq \mathbf{C}_\Lambda^G(\omega) \mathbf{G}_\Lambda(x; \omega) \max_{z \in \partial^- \Lambda} \mathbf{G}_{\Lambda'}(z; \omega).$$

- (G3) [Exhaustivity] $\mathbb{P} \{ \mathbf{C}_\Lambda^G(\omega) < +\infty \} = 1$.

A specification is called uniform iff

$$\sup_{\Lambda \subset \mathcal{Z}} \|\mathbf{C}_\Lambda^G(\cdot)\|_{L^\infty(\Lambda \times \Omega)} =: \widehat{\mathbf{C}}^G < +\infty.$$

Naturally, in the case of a uniform specification it suffices in practice to replace variable factors $\mathbf{C}_\Lambda^G(\omega)$ by the nonrandom upper bound $\widehat{\mathbf{C}}^G$. The principal motivation for allowing $\mathbf{C}_\Lambda^G$ to be random comes from the Anderson-type models in a continuous space with Gaussian (hence lower unbounded) potentials (cf. Section 6).

The argument $\omega \in \Omega$ will often be omitted. Since we focus here on the specifications, a preamble of the form “Let be given a DRD-specification ...” will usually be assumed but sometimes omitted from the formulations of the statements,

unless it is necessary or instructive. All the graphs appearing in our statements are assumed to be of the class $\mathfrak{Z}(d, C_d)$ for some $d, C_d \in (0, +\infty)$.

I would like to stress that the goal of the above definition is not to make things more bureaucratically formal, but rather to render notations and calculations more intuitive and less cumbersome, making abstraction of irrelevant specifics.

Definition 3.2. Let be given a graph $\mathcal{Z}$, an integer $L \geq 0$ and a real number $\gamma > 0$.

- A ball $B_L(x) \subset \mathcal{Z}$ is called γ -NR (non-resonant), iff $\|G_{B_L(x)}\|_\infty \leq \gamma^{-1}$.
- A ball $B_L(x)$ is called γ -CNR (completely non-resonant), iff for all $0 \leq r \leq L-1$ the ball $B_r(x)$ is γ -NR.

The role of the balls $B_r(x)$ will be clarified in the proof of Lemma 4.2.

Definition 3.3. We say that a specification $\mathbf{G}$ admits a continuity modulus $t \mapsto \mathfrak{s}(t)$ iff for any finite connected subgraph $\Lambda \subset \mathcal{Z}$ and some $C_W \in (0, +\infty)$

$$\mathbb{P}\{\|G_\Lambda\|_\infty \geq t^{-1}\} \leq C_W |\Lambda| \mathfrak{s}(t). \quad (7)$$

In particular, we say that $\mathbf{G}$ is β -Hölder continuous (resp., Lipschitz continuous) iff it admits a continuity modulus of the form $\mathfrak{s}(t) = C_W t^\beta$ (resp., with $\beta = 1$).

The above terminology is slightly abusive, but convenient. In the simplest case, where the Hamiltonian $H(\omega)$ is the multiplication by the sample of an IID random field $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$, $\Lambda = \{x\}$, and $G_{\{x\}}(x) = |V(x, \omega) - E|^{-1}$, it is readily seen that $\mathfrak{s}$ -continuity of the common marginal probability distribution function (PDF) $F_V(t) := \mathbb{P}\{V(x, \omega) \leq t\}$ implies the respective property of the specification $\mathbf{G}$. For $H(\omega) = H_0 + V(x, \omega)$ and $G_\Lambda(x, \omega) = \|1_{\partial-\Lambda}(H(\omega) - E)^{-1}1_x\|$, the celebrated Wegner estimate [106] translates the continuity modulus of the PDF F_V into that of the specification $\mathbf{G}$, in the particular case where F_V is Lipschitz continuous.

A number of deep results extended the original Wegner estimate to various types of random Hamiltonians and a large class of continuous PDFs; we discuss these results below, where they are used. In some cases, the dependence upon $|\Lambda|$ is not linear (cf., e.g., [99, 35]), and the resulting continuity modulus for a specification $\mathbf{G}$, such as defined by (7), effectively becomes scale-dependent.

To include such cases, one could introduce an additional parameter $\mathfrak{b} \geq 1$ in Definition 3.3, replacing the RHS of (7) with $|\Lambda|^\mathfrak{b} \mathfrak{s}(t)$. We do not do so, for two reasons. Firstly, incorporating $\mathfrak{b} > 1$ is a fairly simple exercise which an interested reader can carry out easily, if and when this is needed. Secondly, the appearance of the exponent $\mathfrak{b} > 1$ is often only a temporary situation (as was the case with the works [99, 35]). The most notable case is that of the acoustic-type Hamiltonians (cf. [39, 41]), but even here, there has been recently some progress made by Kitagaki [62]. Basically, the exact value of $\mathfrak{b}$ has a visible impact only on the quantitative scale-free criteria of localization.

On the other hand, note that there is an elementary way [17] to automatically extend a Wegner-type estimate proved

for a given type of random Hamiltonians with "diagonal disorder" and Lipschitz continuous marginal PDF to an arbitrary continuous PDF (within the same class of Hamiltonians).

In what follows, we focus on the models admitting exponential scaling limit for the eigenfunction correlators (EFC), and the most natural condition upon the marginal PDF of local disorder is Hölder continuity of some order $0 < \beta \leq 1$; the same condition is also used in the proof of genuine exponential decay of the EFCs via the Fractional-Moment Method (FMM), whenever the latter can be successfully applied. For this reason, we will usually assume the specifications $\mathbf{G}$ Hölder continuous.

Remark 3.4. As shown in [18], exponential scaling limit can be achieved for PDF F_V with a slightly weaker regularity than Hölder continuity, viz. for $\mathfrak{s}(\epsilon) \leq C\epsilon^{\frac{c}{\ln|\ln \epsilon|}}$.

3.2 How the Exponential Scaling Limit is achieved

Definition 3.5. Let be given an integer $L \geq 0$, a real number $\epsilon > 0$, and a sample of a specification $\mathbf{G}(\cdot; \omega)$. A ball $B_L(x)$ is called ϵ -NS (non-singular), iff $\hat{C}_\Lambda^\mathbf{G}(\omega)G_{B_L(x)}(x; \omega) \leq \epsilon$, and ϵ -S (singular), otherwise.

In the case of a uniform specification (which will always be assumed until Section 6), the random factor $C_\Lambda^\mathbf{G}(\omega)$ is to be replaced by the global upper bound $\hat{C}^\mathbf{G}$. Usually the SLI-related factor $C_\Lambda^\mathbf{G}$, be it sample-dependent (when V can be negative and large) or not, along with combinatorial factors, is not included in the definition of a "singular" ball, but treated as they appear in the RHS of inequalities. We prefer to have a norm-factor in the SLI related to a non-singular ball to be smaller than 1; this gives rise to a simpler geometrical induction.

◆ Our first general result on DRD-specifications is a variant of [18, Theorem 1] formulated in a more abstract form.

The underlying graph $\mathcal{Z}$ can be finite, in which case the sequence of radii L_k of the balls $B_{L_k}(\cdot)$ is to be defined for $L_k \leq \text{diam } \mathcal{Z}$. This suffices for the main intended application to the localisation analysis in configuration spaces of arbitrarily large but finite size.

The assumption (10) is scale-dependent; this is a compromise made to accommodate the graphs $\mathcal{Z}$ which, unlike periodic lattices embedded in $\mathbb{R}^d$, do not feature a property which we call uniform scalability (cf. Definition 3.7 and Fig. 1). Here we follow [18].

Theorem 3.6. Consider a uniform DRD-specification $\mathbf{G}$ on a graph $\mathcal{Z} \in \mathfrak{Z}(d, C_d)$ and assume that the marginal PDF F_V of an IID random potential $V(\cdot; \omega)$ is Hölder-continuous of some order $\beta \in (0, 1]$. Given a positive integer L_0 , define recursively

$$Y_{k+1} = \lfloor L_k^{\frac{1}{16d}} \rfloor, \quad L_{k+1} = Y_{k+1} L_k, \quad k \geq 0. \quad (8)$$

Next, given some $b_0 \geq 16(d+1)/\beta$, define recursively positive sequences $\{b_k\}, \{s_k\}$

$$b_{k+1} = \frac{1}{4} Y_{k+1} b_k, \quad s_k = \frac{1}{4} b_k, \quad k \geq 0. \quad (9)$$

Then there exists $L_0^* \in \mathbb{N}$ with the following property: if for some integer $L_0 \geq L_0^*$ and some $b_0 \geq 16(d+1)/\beta$ one has

$$\sup_{x \in \mathcal{Z}} \mathbb{P} \left\{ B_{L_0}(x) \text{ is } L_0^{-b_0} \text{-S} \right\} \leq L_0^{-\sigma_0}, \quad \sigma_0 \geq 4d, \quad (10)$$

then there exist some $a, c > 1$ such that

$$\forall k \geq 1 \quad \sup_{x \in \mathcal{Z}} \mathbb{P} \left\{ B_{L_k}(x) \text{ is } L_k^{-b_0 a^k} \text{-S} \right\} \leq L_k^{-\sigma_0 c^k}. \quad (11)$$

Consequently, there exists some positive sequences $m_k, m'_k, \delta_k, \kappa_k$ such that

$$\forall k \geq 1 \quad \sup_{x \in \mathcal{Z}} \mathbb{P} \left\{ B_{L_k}(x) \text{ is } e^{-m_k L_k^{1-\delta_k}} \text{-S} \right\} \leq e^{-m'_k L_k^{1-\kappa_k}}, \quad (12)$$

with $m_k, m'_k \rightarrow +\infty$ and $\delta_k, \kappa_k \leq Ch^k$, $h \in (0, 1)$.

The derivation of (12) from (11) is an elementary exercise: it suffices to re-write the expressions like $L_k^{-CA^k}$, with $C > 0$ and $A > 1$, in the exponential form.

The threshold $\sigma_0^* = 4d$ can be improved (made smaller), but its optimal value is of little importance in applications.

◆ The second result on the DRD-specifications concerns the graphs $\mathcal{Z}$ sharing with the periodic lattices $\mathbb{Z}^d$ and Euclidean spaces $\mathbb{R}^d$ a property which we call below *uniform scalability* (cf. Definition 3.7). In this case, one can follow more closely the strategy of [97, 47] and obtain a scale-free criterion of localization.

Definition 3.7. A locally finite graph $\mathcal{Z}$ is called *uniformly scalable*, with scaling factor $Y \mapsto M_Y(\mathcal{Z}) < +\infty$ (in short, M_Y -scalable) iff for any integers $Y \geq 2$, $\ell \geq 1$, any ball in $\mathcal{Z}$ of radius $Y \cdot 3\ell$ can be covered by at most $M_Y(\mathcal{Z})$ balls of radius ℓ .

Theorem 3.8. Let $\mathcal{Z} \in \mathfrak{Z}(d, C_d)$ be a uniformly scalable graph with scaling factor $Y \mapsto M_Y$, and $\mathbf{G}$ a uniform specification on $\mathcal{Z}$, which is Hölder continuous of order $\beta \in (0, 1]$. Let be given $b_0 > d\beta^{-1}$; denote $\sigma = \frac{\beta b_0 - d}{7d} > 0$, and suppose that for some integers $Y \geq 11$ and

$$3\mathbb{N} \ni L_0 \geq L_0^* = \max \left[Y^{\frac{1}{\sigma}}, Y^{\frac{1}{2\sigma}} (2Y M_Y^2)^{\frac{2}{3\sigma}} \right] \quad (13)$$

the following condition is fulfilled:

$$\sup_{u \in \mathcal{Z}} \mathbb{P} \left\{ \mathbf{G}_{B_{L_0}(u)}(u) > L_0^{-b_0} \right\} < \left(\frac{2}{M_Y} \right)^2. \quad (14)$$

Let $L_k = Y^k L_0$, $b_k = (1 + 2\sigma)^k b_0$, $\epsilon_k = L_k^{-b_k}$, $\gamma_k = L_{k+1}^{-(1-2\sigma)b_k}$, $k \geq 0$. Then for some $k_0 \geq 0$

$$\forall k \geq k_0 \quad \sup_{u \in \mathcal{Z}} \mathbb{P} \left\{ \mathbf{G}_{B_{L_k}(u)}(u) > \epsilon_k \right\} \leq (\gamma_k)^\beta.$$

See the proof in Section 5.

The above statement results already in a fast (fractional-exponential) decay of the GFs. However, it is to be stressed that it also gives rise to power-law bounds on GFs, of the

Figure 1. Examples of (a) an infinite non-uniformly scalable graph embedded in $\mathbb{Z}^2$, and (b) a non-uniformly scalable domain in $\mathbb{R}^2$, composed of strips, with a power-law growth of balls. Example (b) has some features of realistic circuits used in microelectronics.

form L^{-b} with arbitrarily large $b > 0$ (for $L = L_k$ and k large enough). Thus the scale-free criterion can actually be enhanced, by bootstrapping it with the help of Theorem 3.6, to the asymptotically exponential decay of the Green functions, eigenfunctions and their correlators.

Simply put, an efficient, optimized analysis of localization requires not rigid, fixed scaling rules but an adaptive algorithm revealing a non-linear flow of critical exponents, as usual in the renorm-group method.

4 A scale-dependent criterion of ESL for DRD specifications

4.1 Inductive dominated decay estimate

Definition 4.1. Let be given a sample of a DRD-specification $\mathbf{G}$ on a graph $\mathcal{Z}$. A ball $B_L(u) \subset \mathcal{Z}$ is called (ℓ, ϵ, S) -good iff it contains no collection of pairwise disjoint ϵ -S balls of radius ℓ , and (ℓ, ϵ, S) -bad, otherwise.

Clearly, the notion of singularity depends upon the specific sample $\mathbf{G}(\cdot, \omega)$. The next statement is a fairly standard result of the MSA, in essence going back to [104]. but for the readers' convenience, we provide below its detailed proof.

Lemma 4.2. Let be given a uniform DRD-specification on a graph $\mathcal{Z} \in \mathfrak{Z}(d, C_d)$, and positive numbers $\sigma \in (0, 1)$, $\epsilon \in (0, 1)$, $\gamma \geq \epsilon^{1-\sigma}$. Suppose that a ball $B_L(u)$ is γ -CNR and (ℓ, ϵ, S) -good. If

$$N := \lfloor L/\ell \rfloor - 6S + 3 \geq 1 \quad (15)$$

then $B_L(u)$ is $\epsilon^{N+\sigma}$ -NS.

◆ A word of caution: the uniformity of a DRD-specification is *unrelated* to uniform scalability of the underlying graph $\mathcal{Z}$, but connected to the functional-analytic nature of the Hamiltonian from which the specification arises in applications.

Proof. Take a maximal collection of ϵ -S balls $B_\ell(v_i)$, $1 \leq i \leq S' \leq S-1$, consider the respective collection of S' larger balls $B_{3\ell}(v_i)$, and the minimal covering $\mathcal{A}$ of $\cup_{i=1}^{S'} B_{3\ell}(v_i)$ by spherical layers $\mathcal{L}_r(u) = \{x : d_{\mathcal{Z}}(u, x) = r\}$. For future use, note that

$$\forall x \in B_{L-3\ell}(u) \setminus \mathcal{A} \quad \forall y \in B_{3\ell}(x) \text{ the ball } B_\ell(y) \text{ is } \epsilon\text{-NS}. \quad (16)$$

Decompose $\mathcal{A}$ into a union of disjoint annuli $A_i = B_{b_i} \setminus B_{a_i}$, $b_i < a_{i+1}$, $1 \leq i \leq S'' \leq S'$. Observe that $b_i - a_i \geq 6\ell$, since $[a_i, b_i]$ is the radial projection of one or (a union of) several balls of diameter 3ℓ . Next,

• if $b_{S''} \leq L - \ell - 1$, set $\tilde{L} = L - \ell - 1$, $n = S''$,

• otherwise, set $\tilde{L} = a_{S''} - 1$, $n = S'' - 1$,

and let $\mathbb{A} = \{A_i, 1 \leq i \leq n\}$, $w(\mathbb{A}) := \sum_{i=1}^n (b_i - a_i)$.

Call the layers $\mathcal{L}_r(u)$ with $r \leq L - \ell - 1$ and disjoint from $\mathbb{A}$ *regular*. Note that $\mathcal{L}_{r_n}(u)$ is always regular, since $b_{n''} - a_{n''} \geq 6\ell > \ell$ (that was the goal of the above two-case definition of n and S''). Thus each $\mathcal{L}_r(u) \subset B_{\tilde{L}}(u)$ is either regular or enclosed in a larger regular layer, viz. $\mathcal{L}_{b_{i+1}}(u)$, where

$$i = \min\{j \in [1, S''] : b_j \geq r\} \leq n.$$

We claim that $\lfloor \frac{\tilde{L} - w(\mathbb{A})}{\ell} \rfloor \geq N + 1$ (cf. (15)). The proof is as follows:

• If $n = S''$, then, recalling $S'' \leq S' \leq S - 1$,

$$\begin{aligned} \tilde{L} - w(\mathbb{A}) &= L - \ell - 1 - w(\mathbb{A}) \geq L - S'' \cdot 6\ell - \ell - 1 \\ &\geq L - 6(S - 1)\ell - 2\ell = L - (6S - 4)\ell. \end{aligned}$$

• If $n = S'' - 1$, then $\mathbb{A} = \{A_i, i \leq S'' - 1\}$, and one has $b_{S''} \geq L - \ell$, hence $a_{S''} \geq (L - \ell) - (b_{S''} - a_{S''})$, yielding again

$$\begin{aligned} \tilde{L} - w(\mathbb{A}) &= a_n - \sum_{i=1}^{S''-1} (b_i - a_i) \\ &\geq L - \ell - 1 - \sum_{i=1}^{S''} (b_i - a_i) \geq L - (6S - 4)\ell. \end{aligned}$$

Note 1: in either case, for every point $x \in B_{\tilde{L}}(u)$ there is exactly one regular spherical layer $\mathcal{L}_{R_x}(u)$ of *minimal* radius $R_x \geq d(u, x)$.

It follows that

$$\begin{aligned} \left\lfloor (\tilde{L} - w(\mathbb{A})/\ell) \right\rfloor &\geq \lfloor (L - (6S - 4)\ell)/\ell \rfloor \\ &= \lfloor L/\ell \rfloor - 6S + 4 = N + 1 \end{aligned}$$

as claimed. Therefore, we can define a sequence of $N + 1$ radii, $r_1 < r_2 < \dots < r_{N+1} = \tilde{L}$, by the recursion

$$r_{i-1} = \max\{r \leq r_i - \ell : \mathcal{L}_r(u) \text{ is regular}\}.$$

Note 2: given radii r_i, r_{i+1} , either $r_i = r_{i+1} - \ell$ and all $\mathcal{L}_r$ with $r \in [r_i, r_{i+1}] := [r_i, r_{i+1}] \cap \mathbb{Z}$ are regular, hence NS, or $r_{i+1} - r_i > 6\ell$, since r_i, r_{i+1} are separated by at least one radial projection of a ball $B_{3\ell}(v_j)$.

Define on the integer interval $\llbracket 0, \tilde{L} \rrbracket$ the functions

$$f : r \mapsto \max_{x \in \mathcal{L}_r(u)} \mathbf{G}(x), \quad F : r \mapsto \max_{r' \leq r} f(r'). \quad (17)$$

Every layer $\mathcal{L}_{r_i}(u)$ is regular by construction, hence each ball $B_\ell(x)$ with $x \in \mathcal{L}_{r_i}(u)$ is ϵ -NS, along with each ball $B_\ell(y)$ with $y \in B_\ell(x)$, so by the SLI,

$$\begin{aligned} \max_{x \in \mathcal{L}_{r_i}(u)} f(x) &\leq \max_{x \in \mathcal{L}_{r_i}(u)} \max_{z_1 \in B_\ell(x)} \epsilon \cdot f(z_1) \\ &\leq \epsilon \cdot F(r_i + \ell) \leq \epsilon \cdot F(r_{i+1}). \end{aligned}$$

Furthermore, on account of Note 2, either $r_i = r_{i+1} - \ell$ and for all $r \in \llbracket r_i, r_{i+1} \rrbracket$ (which are regular then) we have a similar recursive bound

$$\begin{aligned} \max_{x \in \mathcal{L}_r(u)} f(x) &\leq \max_{x \in \mathcal{L}_r(u)} \max_{z_1 \in B_\ell(x)} \epsilon \cdot f(z_1) \\ &\leq \epsilon \cdot F(r_i + \ell) \leq \epsilon \cdot F(r_{i+1}), \end{aligned}$$

or $r_{i+1} - r_i > 6\ell$, and then we argue as follows.

$B_L(u)$ is CNR, i.e., every ball $B_\ell(u)$ with $r \leq \tilde{L}$ is γ -NR, and on account of (16), by a 3-fold application of the SLI, we have for every $r' < r_i$

$$\begin{aligned} F(r') &= \max_{r \leq r_i - 1} \max_{x \in \mathcal{L}_r(u)} f(x) \leq \gamma^{-1} \max_{y \in \mathcal{L}_{r_i}(u)} f(y) \\ &\leq \gamma^{-1} \epsilon \max_{r \leq r_i + \ell} \max_{z_1 \in \mathcal{L}_r(u)} f(z_1) \leq \dots \\ &\leq \epsilon^{-(1-\sigma)} \cdot \epsilon^3 F(r_i + 3\ell) \leq \epsilon^2 F(r_{i+1}), \end{aligned}$$

2 more iterations of SLI

since here we assumed $r_{i+1} - r_i > 6\ell$. By induction in $j = N, \dots, 0$,

$$f(u) = F(0) \leq \epsilon^{N+1} \|\mathbf{G}_{B_L(u)}\| \leq \epsilon^{N+\sigma},$$

since $\|\mathbf{G}_{B_L(u)}\| \leq \gamma^{-1} \leq \epsilon^{-(1-\sigma)}$ by the assumed γ -CNR property of $B_L(u)$. $\square$

Lemma 4.3. Let be given numbers $b_k = 4s_k \geq 16(d + 1)$, $L_k \in \mathbb{N}^*$, $Y_{k+1} = \lfloor L_k^{1+\tau} \rfloor$, $L_{k+1} = Y_{k+1}L_k$, $S_{k+1} \leq \frac{1}{8}Y_{k+1}$. Assume $N_{k+1} = Y_{k+1} - 6(S_{k+1} - 1) \geq 1$ and consider a ball $B_{L_{k+1}}(u)$. Denote

$$\epsilon_k = L_j^{-b_k}, \quad \epsilon_{k+1} = L_j^{-b_{k+1}}, \quad \gamma_k = L_{k+1}^{-s_k}.$$

If $B_{L_{k+1}}(u)$ is γ_k -CNR and (ϵ_k, S_{k+1}) -good, then it is also ϵ_{k+1} -NS.

Proof. Since $S_{k+1} \leq \frac{1}{8}Y_{k+1}$, by Lemma 4.2, we know that

$$\mathbf{G}(u) \leq \epsilon_k^{N_{k+1}} \leq L_k^{-b_k(N_{k+1})} \leq \left(\widehat{\mathbf{C}}^{\mathbf{G}} L_{k+1}^{2d}\right)^{-1} L_{k+1}^{-N_{k+1}b_k} \quad (18)$$

(recall $b_k = 4s_k$, $s_k \geq s_0 \geq 4(d + 1)$), where

$$N_{k+1} \geq \frac{Y_{k+1}L_k}{L_k} - 6S_{k+1} \geq Y_{k+1} \left(1 - \frac{6S_{k+1}}{Y_{k+1}}\right) > \frac{1}{4}Y_{k+1},$$

thus

$$N_{k+1}b_k \geq \frac{1}{4}Y_{k+1}b_k = A_{k+1}b_k = b_{k+1}.$$

Thanks to the factor $(\widehat{\mathbf{C}}^{\mathbf{G}} L_{k+1}^{2d})^{-1}$ in the RHS of (18), we finally obtain

$$\widehat{\mathbf{C}}^{\mathbf{G}} L_{k+1}^{2d} \mathbf{G}(u) \leq L_{k+1}^{-b_{k+1}},$$

i.e., $B_{L_{k+1}}(u)$ is ϵ_{k+1} -NS. $\square$

4.2 Scaling of the probabilities

In this subsection, we assume the following relations:

$$Y_{j+1} = \lfloor L_j^\tau \rfloor, \quad L_{j+1} = L_0^{1+\tau}, \quad \tau = \frac{1}{16d},$$

$$A_j = \frac{1}{4}Y_j, \quad b_j = A_j b_{j-1}, \quad b_0 \geq 16(d+1), \quad s_j = \frac{1}{4}b_j. \quad (19)$$

Theorem 4.4 (Cf. [18, Lemma 4]). *Let be given a uniform, β -Hölder continuous DRD-specification $\mathbf{G}$ on a graph $\mathcal{Z} \in \mathfrak{Z}(d, C_d)$, assume the relations (19) Let*

$$\widehat{p}_j := \sup_{u \in \mathcal{Z}} \mathbb{P} \left\{ B_{L_j}(u) \text{ is } L_j^{-b_j}\text{-S} \right\},$$

$$\sigma_j := A_1 \cdots A_j \sigma_0, \quad \sigma_0 := \frac{|\ln \widehat{p}_0^{-1}|}{\ln L_0} \geq 4d.$$

If L_0 is large enough, then the assumption $\widehat{p}_0 \leq L_0^{-\sigma_0}$ implies $\widehat{p}_k \leq L_k^{-\sigma_k}$ for all $k \geq 1$.

The above statement is merely a notational adaptation of [18, Lemma 4], and so is its proof, which is omitted from the present paper. The main difference is that in [18] we worked with the actual Green functions, whereas now the key analytic and probabilistic properties of the latter are encoded in the uniform, β -Hölder continuous specification $\mathbf{G}$.

4.3 Exponential scaling limit. Proof of Theorem 3.6

Theorem 3.6. It suffices to merely re-write in a suitable way the decay bounds from Theorem 4.4:

$$b_k \geq b_0 4^{-k} \prod_{j=1}^k Y_j = \frac{b_0}{L_0} 4^{-k} L_k = L^{1 - \frac{\ln \frac{b_0}{L_0} + k \ln 4}{\ln L_k}} = L_k^{1 - \delta_k},$$

where $\delta_k \leq h^k$, $h \in (0, 1)$, since $\ln L_k \geq \alpha^k$, $\alpha > 1$, thus $k / \ln L_k \leq C \alpha^{-k/2}$. Consequently,

$$L_k^{-b_k} = e^{-\ln L_k \cdot L_k^{1 - \delta_k}} = e^{-c_k L_k^{1 - \delta_k}}$$

with $c_k \rightarrow \infty$ as $k \rightarrow \infty$.

Similarly, for $\widehat{p}_k \leq L_k^{-\sigma_k}$ we have

$$\ln \widehat{p}_k^{-1} \geq \sigma_0 \ln L_k \geq \sigma_0 A_1 \cdots A_k \geq C' 4^{-k} Y_1 \cdots Y_k$$

$$\geq L_k^{1 - \kappa_k},$$

whence $\widehat{p}_k \leq e^{-c'_k L_k^{1 - \kappa_k}}$, $\kappa_k \leq (h')^k$, with some $h' \in (0, 1)$ and $c'_k \rightarrow \infty$. $\square$

5 A scale-free criterion of ESL for specifications

5.1 Preliminary remarks and constructions

Recall that we introduced the notion of uniformly scalable graph in Definition 3.7.

Clearly, any finite graph can be rendered uniformly scalable by setting, e.g., $M_Y(\mathcal{Z}) = \text{card} \mathcal{Z}$, but for a given function $Y \mapsto M_Y$, not every finite graph is uniformly scalable with *this* particular scaling factor M_Y . All lattices $\mathbb{Z}^d$ are uniformly scalable with $M_Y = C(d)Y^d$.

Given a ball $B_{3\ell}(u)$, with $\ell \geq 1$, we call $B_\ell(u)$ the *core* and the annulus $B_{3\ell}(u) \setminus B_\ell(u)$ the *shell* of $B_{3\ell}(u)$. When convenient, we shall also say, by a slight abuse of terminology, that $B_{L_k}(u) \setminus B_{\ell_k}(u)$ the *shell* of the core $B_{L_k}(u)$. Further, given an M_Y -scalable graph $\mathcal{Z}$ and a ball $B_{Y L}(u) \subset \mathcal{Z}$, $L = 3\ell$, $Y \geq 2$, we can find a cover of the shell of $B_{Y L}(u)$ by at most M_Y cores (of balls) of radius ℓ . Without repeating every time again, we will always assume that such a covering is assigned canonically to each ball of radius $L = 3\ell$. The respective cores $B_\ell(x)$ will be referred to as *admissible*.

The efficiency of the scale-free criteria going back to [97, 47] relies on the fact that in the situation where $\mathcal{Z}$ is uniformly scalable, one can effectively reduce the analytic and probabilistic arguments relative to [non-]singularity of $B_{L_{k+1}}(u)$ to a smaller non-oriented graph which we call the *skeleton* graph of $B_{L_{k+1}}(u)$, and denote $\mathcal{B}_u^{k+1}$. Its vertices are all the admissible cores covering $B_{L_{k+1}}(u)$; two non-identical admissible cores $B_{\ell_k}(x)$, $B_{\ell_k}(y)$ form an edge iff one of them overlaps with the shell of the other. The net result is a dramatic reduction of the combinatorial, entropy-type "expenses" in the scaling procedure, which can now be started with some sufficiently small probability $\widehat{p}_0$ (like the one figuring in Section 4.2, or δ_0 in Spencer's criterion (cf. Proposition 2.2) *independent* of the scale L_0).

◆ We shall work with a sequence $L_k = Y^k L_0$, with $L_0 = 3\ell_0$, $\ell_0 \in \mathbb{N}^*$, so for any $k \geq 0$, $\ell_k := L_k/3 \in \mathbb{N}$. Changing the power-law growth $L_{k+1} = L_k^\alpha$ for the multiplicative one is crucial for starting with a weak, scale-free decay assumption, while the exponential growth $L_k \sim L_0^k$ is instrumental for proving exponential decay of the GFs, as in [104] or in the fourth multi-scale analysis stage of the BMSA [47]. ◀

Consider an admissible core, $\mathbf{c} = B_{\ell_k}(x')$, and all admissible cores $\mathbf{c}'_i = B_{\ell_k}(x'_i)$ with $(\mathbf{c}, \mathbf{c}')$ forming an edge. By the SLI for the specification on the original graph $\mathcal{Z}$, we have

$$\max_{x \in \mathbf{c}} \mathbf{G}(x) \equiv \max_{x \in B_{\ell_k}(x')} \mathbf{G}(x)$$

$$\leq \left\| \mathbf{G}_{B_{L_k}(x')} \right\|_{L_k, x'}^\wedge \max_{y \in \partial^- B_{L_k}(y)} \mathbf{G}(y)$$

$$\leq \left\| \mathbf{G}_{B_{L_k}(x')} \right\|_{L_k, x'}^\wedge \max_{\mathbf{c}': d(\mathbf{c}, \mathbf{c}')=1} \max_{y \in \mathbf{c}'} \mathbf{G}(y)$$

Introduce the function $\mathbf{G}' : \mathcal{B} \rightarrow \overline{\mathbb{R}}_+$ given by $\mathbf{G}'(\mathbf{c}) := \max_{x \in \mathbf{c}} \mathbf{G}_{B_{L_{k+1}}(u)}(x)$. It follows from the above that if $\mathbf{c}$ is the core of an ϵ -NS ball, then

$$\mathbf{G}'(\mathbf{c}) \leq \epsilon \max_{\mathbf{c}': d(\mathbf{c}, \mathbf{c}')=1} \mathbf{G}'(\mathbf{c}').$$

Further, if $B_{L_{k+1}}(u)$ is γ -CNR, it also follows by SLI for the specification $\mathbf{G}$ that

$$\mathbf{G}'(\mathbf{c}) = \max_{x \in \mathbf{c}} \mathbf{G}(x) \leq \left\| \mathbf{G}_{B_{r\ell_k}(x')} \right\|_{r\ell_k, x'}^\wedge \max_{y \in \partial^- B_{r\ell_k}(y)} \mathbf{G}(y)$$

$$\leq \gamma^{-1} \max_{\mathbf{c}': d(\mathbf{c}, \mathbf{c}')=r} \mathbf{G}'(\mathbf{c}').$$

5.2 A scale-free criterion. Proof of Theorem 3.8

Working with the skeleton graphs $\mathcal{B}$ of the balls $B \subset \mathcal{Z}$, in order to avoid any confusion, we denote the balls relative to $\mathcal{B}$ by $\Lambda_R(\mathbf{c}) := \{\mathbf{c}' : d(\mathbf{c}, \mathbf{c}') \leq R\}$. The spherical layer $\{\mathbf{c}' : d(\mathbf{c}, \mathbf{c}') = R\}$ is denoted by $\mathcal{L}_R(\mathbf{c})$. We will need the following statement which is a simpler variant of Lemma 4.2: here we have to treat just one singular "belt", while Lemma 4.2 has to deal with $S \geq 1$ such belts.

Lemma 5.1. *Consider a uniform specification $\mathbf{G}$ on a uniformly scalable graph $\mathcal{Z}$, a ball $B_{Y_L}(u)$, with $L = 3\ell$ and $Y \geq 11$, its skeleton graph $\mathcal{B}$ and the belts $\mathbb{B}_1 = \cup_{r=1}^5 \mathcal{L}_r(\mathbf{c})$, $\mathbb{B}_2 = \cup_{r=8}^{12} \mathcal{L}_r(\mathbf{c})$. If at least one of these two belts is ϵ -regular and $\mathcal{B}$ is γ -CNR, with $\gamma \geq \epsilon^{1-\sigma}$, $\sigma > 0$, then*

$$\mathbf{G}_{B_{Y_L}(u)}(u) \leq \epsilon^{1+\sigma}.$$

Proof. Consider first the case where the belt $\mathbb{B}_1$ is regular, and define the functions F and f as in (17). Then all vertices $x \in \mathcal{L}_r(u)$ with $r \in [[Y-3, Y+1]]$ are regular, and for any $r' \in [[0, Y-1]]$, arguing essentially as in the proof of Lemma 4.2, we obtain by a four-fold application of the SLI,

$$\begin{aligned} F(r') &= \max_{r \leq r'} \max_{x \in \mathcal{L}_r(u)} \mathbf{G}(x) \leq \gamma^{-1} \max_{y \in \mathcal{L}_{Y-1}(u)} \mathbf{G}(y) \\ &\leq \gamma^{-1} \epsilon \max_{Y-2 \leq r \leq Y} f(r) \leq \gamma^{-1} \epsilon^2 \max_{Y-3 \leq r \leq Y+1} f(r) \\ &\leq \gamma^{-1} \epsilon^3 \max_{Y-4 \leq r \leq Y+2} f(r) \leq \gamma^{-1} \epsilon^3 \max_{y \in B_L(u)} \mathbf{G}(y) \\ &\leq \epsilon^3 \cdot \gamma^{-2} \leq \epsilon^{1+2\sigma}. \end{aligned} \quad (20)$$

It remains to treat $f(Y)$ (**not** $F(Y)$!); the final estimate is the same, but the calculation is simpler, for $\mathcal{L}_Y(u)$ is regular, so we skip the first SLI in the above argument (thus avoiding an unwanted factor of γ^{-1} it produces):

$$\begin{aligned} f(Y) &= \max_{x \in \mathcal{L}_Y(u)} \mathbf{G}(x) \leq \epsilon \max_{Y-1 \leq r \leq Y+1} f(r) \\ &\leq \epsilon^2 \max_{Y-2 \leq r \leq Y+1} f(r) \\ &\leq \epsilon^2 \max_{y \in B_L(u)} \mathbf{G}(y) \leq \epsilon^2 \cdot \gamma^{-1} \\ &\leq \epsilon^{1+\sigma}. \end{aligned} \quad (21)$$

Collecting (20)–(21), the asserted bound follows in the case where $\mathbb{B}_1$ is regular.

The case of $\mathbb{B}_2$ is similar and even slightly simpler. In fact, the two cases would be completely similar if we shifted by 1 the positions of the two belts, away from the center. Our choice is motivated by a slight numerical improvement it provides in the explicitly calculated case where $\mathcal{Z} = \mathbb{Z}^d$. So, by a four-fold application of the SLI, we obtain, as in (20), $F(Y) \leq \epsilon^{1+2\sigma} \leq \epsilon^{1+\sigma}$. This completes the proof. $\square$

Before turning to the main result of this section, Lemma 5.2, we need to make some combinatorial observations. Specifically, let us assess the probability of the event (under which we have actually operated in the deterministic Lemma 5.1) that neither of the two belts $\mathbb{B}_1, \mathbb{B}_2$ is regular. This

happens only if each belt contains at least one admissible core corresponding to a singular ball. Given a uniform upper bound p on the probability for a ball to be singular, the event at hand has probability at most $|\mathbb{B}_1| \cdot |\mathbb{B}_2| \cdot p^2$, where $|\mathbb{B}_1| + |\mathbb{B}_2| \leq M_Y$, thus by the inequality for the arithmetical and geometrical mean,

$$\mathbb{P} \{ \text{both } \mathbb{B}_1 \text{ and } \mathbb{B}_2 \text{ are singular} \} \leq \left(\frac{M_Y}{2} p \right)^2. \quad (22)$$

Lemma 5.2. *Let be given a real number $b_0 > d/\beta$, and set $\sigma := \min \left[\frac{b_0\beta-d}{7d}, \frac{1}{49} \right]$. Suppose that for some integer L_0 obeying*

$$3N \ni L_0 \geq L_0^* = \max \left[Y^{\frac{1}{\sigma}}, Y^{\frac{1}{2\sigma}} (2C_W M_Y^2)^{\frac{2}{3\sigma}} \right] \quad (23)$$

it holds that

$$\sup_{u \in \mathcal{Z}} \mathbb{P} \left\{ \mathbf{G}_{B_{L_0}(u)}(u) > L_0^{-b_0} \right\} < \left(\frac{2}{M_Y} \right)^2. \quad (24)$$

Let $L_k = Y^k L_0$, $b_k = (1+\sigma)^k b_0$, $\epsilon_k = L_k^{-b_k}$, $s_k = (1-2\sigma)b_k$, $\gamma_k = L_{k+1}^{-s_k}$, $k \geq 0$. Then

$$\exists k_0 \in \mathbb{N} \quad \forall k \geq k_0 \quad \sup_{u \in \mathcal{Z}} \mathbb{P} \left\{ \mathbf{G}_{B_{L_k}(u)}(u) > \epsilon_k \right\} \leq \gamma_k^{-\beta}.$$

Proof. We are going to make use of Lemma 5.1, and to verify its hypotheses, show first that if $L_0 \geq Y^{1/\sigma}$, then for all $k \geq 1$, $\gamma_k \geq \epsilon^{1-\sigma}$. Indeed, since $L_k \geq L_0$, we have

$$\begin{aligned} (L_0 \geq Y^{\frac{1}{\sigma}}) &\Rightarrow (L_k^{1-\sigma-(1-2\sigma)} \geq Y \geq Y^{1-2\sigma}) \\ &\Rightarrow (L_k^{b_k(1-\sigma)} L_k^{-(1-2\sigma)b_k} \geq Y^{(1-2\sigma)b_k}) \\ &\Rightarrow (L_k^{-b_k(1-\sigma)} \leq L_{k+1}^{-(1-2\sigma)b_k}) \\ &\Rightarrow (\epsilon_k^{1-\sigma} \leq \gamma_k). \end{aligned}$$

Now it follows from Lemma 5.1 that

$$\begin{aligned} &\mathbb{P} \left\{ B_{L_{k+1}}(u) \text{ is } L_k^{-b_k(1+\sigma)}\text{-S} \right\} \\ &\leq \mathbb{P} \{ B_{L_{k+1}}(u) \text{ is not } \gamma\text{-CNR} \} \\ &\quad + \mathbb{P} \{ B_{L_{k+1}}(u) \text{ has 2 bad belts} \}. \end{aligned}$$

By the assumed β -Hölder continuity of the specification $\mathbf{G}$,

$$\begin{aligned} w_{k+1} &:= \mathbb{P} \{ B_{L_{k+1}}(u) \text{ is not } \gamma\text{-CNR} \} \\ &\leq C_W M_Y L_{k+1}^d L_{k+1}^{-\beta s_k} =: q_{k+1}. \end{aligned} \quad (25)$$

Suppose for some $k \geq 0$ and all $j \in \llbracket 0, k \rrbracket$ we have $p_{j+1} > 2q_{j+1}$, then with $a^2 = 2M_Y^2$, on account of (22), $\frac{1}{2}a^2 p_j^2 \geq p_{j+1} - q_{j+1} \geq q_{j+1}$, hence

$$p_{j+1} \leq \frac{1}{2}a^2 p_j^2 + \frac{1}{2}a^2 p_j^2 = a^2 p_j^2.$$

To simplify the dynamics of $\{p_j\}$, introduce $\rho_j = a^2 p_j$, then $a^{-2} p_{k+1} \leq a^{-2-4} \rho_k^2$, i.e.,

$$\rho_{k+1} \leq \rho_k^2 \leq (\rho_0)^{2^k}, \text{ by induction.}$$

It is readily seen that $(\rho_0)^{2^{k+1}} \ll (L_0 Y^k)^{-\text{const}(1+\sigma)^k}$ for k large enough and $1 + \sigma < 2$, so there must exist k_o such that $p_{k_o} \leq 2q_{k_o}$. Fix such k_o , and suppose we have $a^2 q_k^2 \leq q_{k+1}$ for some $k \geq k_o$, then $a^2 p_k^2 \leq a^2 q_k^2 \leq q_{k+1}$, hence, starting from $k = k_o$, we obtain

$$p_{k+1} \leq q_{k+1} + q_{k+1} = 2q_{k+1},$$

so by induction, $p_j \leq q_j$ for all $j \geq k_o$.

Observe that $s_j = (1 - 2\sigma_0)b_j$ (hence q_j) has been defined so far for $j \geq 1$ (cf. (25)); now set, formally, $b_{-1} = b_0/(1 + \sigma)$, $s_{-1} = (1 - 2\sigma)b_{-1}$, and $q_0 = C_W M_Y L_0^{-\beta s_{-1}}$.

With this definition of q_0 , a sufficient condition for $a^2 q_k^2 \leq q_{k+1}$, $k \geq 0$, is:

$$a^2 C_W^2 L_k^{-2(1-2\sigma)b_{k-1}\beta+2d} \leq C_W L_{k+1}^{-(1-2\sigma)b_k\beta+d},$$

with $b_k = b_{k-1}(1 + \sigma)$. In a stronger form, obtained by straightforward calculations and some simplifications (using $\sigma \leq 1/49$), we get, on account of $L_{k+1} = Y L_k$,

$$L_k^{(1-3\sigma)b_{k-1}\beta-d} \geq C_W a^2 Y^{(1-\sigma)b_{k-1}\beta-d}, \quad (26)$$

or, equivalently,

$$L_k \geq Y^{\frac{(1-\sigma)b_{k-1}\beta-d}{(1-3\sigma)b_{k-1}\beta-d}} \cdot (C_W a^2)^{\frac{1}{(1-3\sigma)b_{k-1}\beta-d}}. \quad (27)$$

The numerators and denominators of the fractions figuring in (27) are strictly positive owing to the definition of σ , viz. $b_0\beta \geq (1 + 7\sigma)d$. It is easy to check that the exponents of Y and of $C_W a^2 \equiv 2C_W M_Y^2$ in the RHS of (27) are monotone decreasing in k (since b_k is monotone increasing in k), so the strongest condition to be verified is achieved for $k = 0$, and the latter stems from a slightly stronger and less cumbersome inequality

$$L_k \geq (a^2 Y)^{\frac{1-\sigma}{2\sigma}} (2C_W M_Y^2)^{\frac{1+\sigma}{3\sigma}}, \quad (28)$$

stemming from (23), since $1 - \sigma < 1$, $1 + \sigma < 2$, and L_k is monotone increasing. $\square$

6 Adaptation to tempered non-uniform specifications

Here we aim to extend the above techniques and results to the continuous models where the random potential is Gaussian or Gaussian-like, from the perspective of the Fernique theory [38, 2], so that the maximum of a stationary random field $V : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ on a unit cube features, an almost-Gaussian, faster than exponential decay of the right-tail probabilities (cf. [2]).

◆ Consider first the scale induction starting with a sub-exponential ILS bound. Here we can allow for the factor $\widehat{C}_L^G = \widehat{C}_L^G(\omega)$ with values up to $O(L)$ (in fact, even larger, polynomial in L), with

$$\mathbb{P}\left\{\widehat{C}_L^G(\omega) > L\right\} \leq e^{-ML}, \quad (29)$$

where M can be chosen as large as we please, for $L \geq L_0$ and L_0 large enough. Clearly, the event complementary to the one figuring makes only a negligible contribution to the probabilistic estimates given in Sections 4 and 5, for the latter are sub-exponential.

For the lack of a better standard word, below we shall call non-uniform specifications obeying (29) *tempered*.

Under the condition $\widehat{C}_L^G(\omega) \leq L$, the decay factors $\epsilon_k \leq e^{-m_k L_k^{1-\delta_k}}$ are perturbed by a factor of $L_{k+1} = L_k^\alpha$, so that

$$L_k^\alpha e^{-m_k L_k^{1-\delta_k}} = e^{-m_k L_k^{1-\delta_k} - \alpha \ln L_k} = e^{-(m_k - o(1)) L_k^{1-\delta_k}}.$$

It is readily seen that such a modification of the decay factors does not change the entire scheme, for $m_k \rightarrow \infty$ as $k \rightarrow \infty$.

◆ Now turn to the situation, which is considered here only in passing, for it is treated in detail by Germinet and Klein [47] in the framework of the first multi-scale analysis, both for discrete and continuous models. Specifically, suppose we start with a power-law ILS bound, so $\epsilon_k \leq L_k^{-b}$, $b > d$ (as for the Anderson Hamiltonians) or $b > 2d$ (as for acoustic and Maxwell Hamiltonians). Now we can tolerate $\widehat{C}_L^G(\omega) \leq A \ln L$, with arbitrarily large but fixed $A > 0$, for then we have

$$A \ln L_k \cdot L_k^{-b} = L_k^{-b + \ln(A \ln L_k)} = L_k^{-(b - o(1))} = L_k^{-b'},$$

with $b' > d$ (resp., $b' > 2d$, for acoustic/Maxwell Hamiltonians). Again, this bound is compatible with the first stage of the BMSA, hence results in sub-exponential bounds at some large scale where the previously considered method does the rest and results in exponential scaling limit. Probabilistically, we have

$$\mathbb{P}\left\{\widehat{C}_L^G(\omega) > A \ln L\right\} \leq e^{-cA^2 \ln^2 L} \leq L^{-cA^2 \ln L},$$

i.e., better than polynomial bounds required for the first stage of the BMSA.

Summarising these observations, we come to the following conclusion.

Theorem 6.1. *The assertion of Theorem 3.6 remains valid for the tempered non-uniform specifications (cf. (29)).*

Theorem 6.2. *The assertion of Theorem 3.8 remains valid for the tempered specifications, except for the explicit lower bound on the initial scale length which is replaced by the following one: $L_0 \geq L_0^*$, where $L_0^* \in \mathbb{N}$ depends on M_Y and on the probability distribution of the random variable $\widehat{C}_L^G(\cdot)$.*

◆ This marks the end of the scaling analysis of DRD-specifications. Below we comment on applications of the above "abstract" results to the decay analysis of the Green functions in specific models. As said in the Introduction, we do not discuss the derivation of stronger forms of localization (viz. decay of the EFs and of the EFCs) which can be obtained by well-known technics, developed by Germinet and Klein [47] and Elgart *et al.* [31]. In the case of discrete Anderson models, the road map to strong dynamical localization was explained in [18]. ◆

7 Classes of random media admitting exponential scaling limit

Our goal in this section is to fill a rather abstract notion of a DRD-specification over a combinatorial graph $\mathcal{Z}$ with a more specific content, connecting it to spectral problems for random ensembles of operators.

Put simply, the models satisfying the assumptions listed in [47, Section 2] and in some other works, formulated in very general terms, are suitable for the proof of exponential scaling limit. Apparently there is no need reproduce here such assumptions, even in slightly more general terms so as to include quantum graphs, since providing such a list with no bibliographical comments would not be very helpful to a reader unfamiliar with the state-of-the art in the rigorous Anderson localization theory.

7.1 Discrete random Schrödinger operators

The usual form of the SLI/GRI employed in the scaling analysis of the discrete Anderson models can be easily obtained with the help of a one-line application of the second resolvent identity; cf., e.g., [56, Section 5.3]. (See also a detailed discussion of various boundary conditions in [56, Section 5.2].)

7.1.1 Lifshitz tails

The so-called Lifshitz tails asymptotic is named after a well-known physicist I. M. Lifshitz who obtained it in [78]. A rigorous mathematical justification of Lifshitz tails is not an easy task, for the proof of an accurate *asymptotics* for the integrated density of states (IDS) actually depends upon specific details of the model at hand. However, upper and/or lower bounds on the IDS can be much simpler to establish rigorously, with the help of the Large Deviations theory.

Apparently, the first rigorous result in this direction was proved long ago by Donsker and Varadhan [27]. Pastur [90] pointed out the role of the long-range correlations for the quantitative form of the (fast) decay of the IDS at the spectral edges.

See also the works by Kirsch and Martinelli [58], Kirsch and Simon [59], Simon [93], Stollmann [98], Klopp [68, 70].

There is also a rich literature on internal Lifshitz tails (i.e., Lifshitz-type asymptotics at internal spectral edges); see the works by Mezincescu [81], Simon [93, 94, 95], Klopp [69], Klopp and Wolff [77], and Najjar [82].

The initial length scale estimate stemming from the Large Deviations theory for IID (and some other correlated) random fields is very robust and holds even for Bernoulli potentials. However, carrying out the scale induction is more delicate; it requires either a log-Hölder regularity of disorder (insufficient for the exponential scaling limit) or a much more elaborate scheme, originally developed by Bourgain and Kenig [10] for Bernoulli potentials in $\mathbb{R}^d$. We do not discuss the latter, for it does not lead to a sub-exponential decay of EF correlators. Building on the techniques of [10, 5], Germinet and

Klein [49] extended the MSA to any non-constant IID scatterers' amplitudes in $\mathbb{R}^d$. A comprehensible exposition can be found in a review by Kirsch [56] and in the Stollmann's book [99].

7.2 Discrete magnetic Hamiltonians

7.2.1 Lifshitz tails

Nakamura [86] established Lifshitz asymptotics in the situation where disorder, in a two-dimensional lattice model, is introduced through a random magnetic field. Later, Klopp *et al.* [73] complemented the results of [86].

7.3 Off-diagonal disorder. Acoustic and electromagnetic operators in $\mathbb{Z}^d$

Faris [36] proved a linear Wegner-type estimate for the acoustic operators with off-diagonal disorder on a lattice. A particularity of such operators is that the decay exponent of the Green functions vanishes at $E = 0$, since a constant is a (generalized) eigenfunction.

7.3.1 SLI

The dominated decay property (G2) for the resolvents of Schrödinger operators in $\mathbb{R}^d$ has been established and used in a number of earlier works. In the context of the Anderson localization in Euclidean space it was used by Holden and Martinelli [55] who applied the Green's formulae to do an explicit integration over the geometrical boundary of the open cube Λ embedded with its boundary into a larger open cube Λ' . Later on, a different technique was adopted to prove the SLI in various analytic situations, namely, the commutator identities. See, e.g., [99, Section 2.5].

Fischer *et al.* [43] studied the case where the random potential does not have a discrete alloy-type structure, but is rather generated by a Gaussian random field with continuous argument on $\mathbb{R}^d$, under some technical conditions. Simply put, for any stationary Gaussian random field it suffices to assume a.s. continuity of its samples on a some cube of positive size. More explicit hypotheses can be found in the monograph by Adler [2]. In fact, it holds also for some non-Gaussian stationary random fields under the assumption of Gaussian-like decay of the tail probabilities for the single-point marginal distribution; see [2, Chapter 5].

Later, Ueki [101, 102] reconsidered the Gaussian Anderson Hamiltonians in $\mathbb{R}^d$, restricting his analysis to stationary Gaussian random potentials with compactly supported covariance function.

Germinet and Klein [49] established explicit finite-volume criteria for the onset of complete localization in Anderson-type models, focusing particularly on some important competition mechanisms in continuous-space models. A subtle point here is that the factor C^G figuring in the SLI depends itself upon the energy. More precisely, it depends upon the maximum difference $E - V(x)$ in a given finite domain Λ . This dependence is implicit in a number of papers proving localization only near spectral edge(s), in presence of

a bounded (possibly very weak) random potential; it is absorbed in some inconspicuously looking constants (which, as it is well-known, have a nasty habit to be parameter-dependent).

7.3.2 Wegner estimates

Fischer *et al.* [42] established the Wegner estimate for a large class of stationary Gaussian random fields on Euclidean spaces. We refer the interested reader to [42, Section 2] and [57].

7.3.3 Lifshitz tails for lower-bounded potentials

A comprehensive presentation can be found in the Stollmann's monograph [99].

7.3.4 Lifshitz tails for lower-unbounded potentials

The Lifshitz asymptotics near the lower edge of spectrum, used in combination with the Combes-Thomas estimate (cf. [25]) provide a tool for proving the ILS in the considered models, at least for sufficiently low energies. An alternative mechanism is the strong disorder; see the discussion in [49, pp.1206-1207]. Since the Wegner estimate for the Gaussian fields studied in [103] and in [42, 43] is linear (in the length of the energy interval), one has a sufficient input for the proof of exponential scaling limit of Gaussian Anderson models in $\mathbb{R}^d$ with finite-range correlations (as in [103]).

7.4 Localization of acoustic waves

Figotin and Klein [39, 40] studied the localization phenomena in propagation of acoustic waves in $\mathbb{Z}^d$ and in $\mathbb{R}^d$. Later on, Germinet and Klein [47] showed that the general assumptions, required for a successful application of the bootstrap MSA, allow for the acoustic operators from [40]. See also [65, 99, 66].

7.5 Surface models

The interest to the surface models has been essentially motivated by physical applications, where experimental and/or technological techniques create and make use of one- or two-dimensional media incorporated in three-dimensional samples.

The non-optimal volume dependence in the Wegner estimate in [39], employed by Germinet–Klein in the BMSA scheme and affecting their scale-free localization criterion, was later improved by Kitagaki [62].

Kirsch and Veselić [60] considered a number of different Anderson-type models, including those with a regular alloy potential, alloys with sparse scatterers, random displacements models, surface potentials, and several others.

See also a paper by Kirsch and Warzel [61] on the random surface models.

7.6 Localization of electromagnetic waves

Apart from the propagation of acoustic waves in disordered media, Figotin and Klein [39, 41], also considered electromagnetic waves, both in $\mathbb{Z}^d$ [39] and in $\mathbb{R}^d$ [41].

As in the case of acoustic operators, Germinet and Klein [47] checked their general assumptions, guaranteeing the applicability of the BMSA techniques and results, for the Maxwell operators in inhomogeneous media with random dielectric permittivity ε .

The Wegner estimate for such models was proved in [41, Theorem 34].

7.7 Schrödinger operators with magnetic field in $\mathbb{R}^d$

7.7.1 Wegner estimates: Non-random magnetic field

See the works by Combes *et al.* [24, 22, 21], Hislop and Klopp [54]

7.7.2 Wegner estimates: Random magnetic field

We refer the reader to the works by Broderix *et al.* [11], Erdős [32], Erdős and Hasler [33], Ueki [101, 103], Warzel [105], Erdős and Hasler [33]

7.7.3 Lifshitz tails

Nakamura [86, 87] proved an upper bound for the Lifshitz tails for 2D lattice Schrödinger operators with random magnetic field.

7.8 Random Landau Hamiltonians

The Landau Hamiltonian (originally introduced in physics by Landau in 1930) describes a particular, and physically interesting, exactly solvable model of an electrically charged quantum particle in a constant magnetic field.

7.8.1 Wegner estimate

Combes *et al.* [23] studied the 2D Landau Hamiltonian perturbed by a random alloy-type potential and proved Hölder continuity of the IDS.

7.8.2 Lifshitz tails

Klopp and Raikov [76] established three types of Lifshitz asymptotics for a class of random Landau Hamiltonians.

7.9 Anderson–Delone type models

The ergodicity plays an important role in the proof of the Lifshitz asymptotics at the band edge(s), so the main technical challenge of the Delone–Anderson type models is the possible lack of ergodicity. See the works by Rojas-Molina and Veselić [91], Klein [64], Elgart and Klein [29], and Germinet *et al.* [50].

7.10 Random displacements models

Here the potential has a modified alloy structure,

$$V(x, \omega) = \sum_{a \in \mathbb{Z}^d} u(x - a - X(a, \omega)),$$

where the scatterer potential u has compact support, $\text{supp } u \subset \{x \in \mathbb{R}^d : |x| \leq r\}$, with $0 < r < 1/4$, and $X : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}^d$ is an IID random field with vector values in $\mathbb{R}^d$; more precisely, X takes values in some cube $[-r_*, r_*]^d \subset \mathbb{R}^d$, with $r_* = \frac{1}{2} - r$.

7.10.1 Lifshitz tails

Klopp *et al.*, using some previously obtained results [72] on Lifshitz tails for the alloy-type models in $\mathbb{R}^d$, proved a finite-volume eigenvalue concentration bound at the lower edge of the spectrum for the random displacement model; cf. [71, Corollary 3.4]. Their techniques also imply an asymptotic upper bound on the IDS; cf. [71, Theorem 3.5].

7.10.2 Wegner estimate

Klopp *et al.* [71] proved the Hölder regularity of the IDS for the random displacement model, which, combined with the ILS stemming from the Lifshitz tails asymptotics at low energies, provides a sufficient input for the exponential scaling limit. Their Wegner bound [71, Theorem 4.1] is optimal both in the volume and in the size of the energy interval.

7.11 Randomly shaped waveguides

In this class of quasi-one-dimensional models, randomness is introduced through a random geometry of the media where evolves a quantum particle. Specifically, having in mind possible applications to the realistic physical systems, one can either randomly modulate the width of the waveguide (planar or tubular), or embed it into $\mathbb{R}^3$ along a randomly twisted curve. See the works by Kleespies and Stollmann [63], Najjar [82, 83], and Najjar and Raissi [85].

7.12 Matrix-valued Anderson model in $\mathbb{R}^d$

Boumaza and Najjar [8] proved the Lifshitz tails asymptotics for a matrix-valued continuous Anderson model, with Hamiltonian acting in the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^D$, $d, D \in \mathbb{N}$, of functions in a D -dimensional complex "spin" space.

7.13 Quantum graphs

7.13.1 Analytic description of the model

See the definition of the model and results on localization in Exner *et al.* [35],

The original Stollmann's approach was later improved so as to provide an optimal volume dependence. See [100], and also [92] for a discussion, some general results on eigenvalue concentration in an abstract functional-analytic setting, and references therein. Recall that the exponent in the power of

$|\Lambda|$ has a quantitative impact on the scale-free localization criterion, of the Germinet–Klein type; in particular, in applications to the acoustic localization, where $|\Lambda|$ enters in the Wegner estimate with power $b = 2$, according to [39], Germinet and Klein [47] modify accordingly the GF decay rate in the scale-free criterion.

7.13.2 Random edge lengths

Klopp and Pankrashkin [75] studied the random lengths model with the help of the MSA techniques. Compared to the construction given in Section 7.13.1, the disorder come not from the operator (the potential on the edges is constant) but from the metric graph itself which is no longer assumed isometrically embedded into $\mathbb{R}^d$, or any other specific metric space (albeit the combinatorial vertex graph is $\mathbb{Z}^d$ in [75]), so the edges can have arbitrary (nonzero) lengths in the internal metric. Specifically, the edge lengths are IID random variables $\{l_e(\omega), e \in \mathcal{E}(\mathcal{G})\}$ with common probability distribution admitting a Lipschitz continuous compactly supported probability density ρ , $\text{supp } \rho = [l_{\min}, l_{\max}] \subset (0, +\infty)$.

The models with the coupling constant α equal (or close) to 0 cannot be treated; in fact their properties for α small resemble those of the acoustic operators where Najjar [84] established non-Lifshitz behaviour of the IDS at the lower spectral edge.

A Wegner estimate is the subject of the following

Proposition 7.1 ([75, Theorem 4]). (a) Let $I \subset (0, +\infty)$ be an interval such that $\bar{I} \cap \Delta = \emptyset$, then there exists a constant $C = C(I) > 0$ such that for any interval $J \subset I$, and for any cube $\Lambda = \Lambda_n$ there holds

$$\mathbb{P} \{ \Sigma(H_\Lambda(\alpha, \omega)) \cap J \neq \emptyset \} \leq C |\Lambda| |J|. \quad (30)$$

(b) There exists $\alpha > 1$ such that for $\alpha \in (-\infty, -\alpha) \cup (-\alpha^{-1}, 0)$ the Wegner estimate also holds at negative energies near the bottom of the spectrum, i.e. there exists an interval I with $I \ni \inf \Sigma(\alpha)$ and $C = C(I) > 0$ such that for any interval $J \subset I$, and any cube $\Lambda = \Lambda_n$ the estimate (30) holds.

Since the Wegner estimate is established for Lipschitz continuous marginal densities, the stochastic regularization method [17] applies and provides an extension to arbitrary continuous marginal distributions.

The Lifshitz tails asymptotics in [75] is not encapsulated in a separate statement (see, however, Eqn. [75, 21]) but is organically incorporated in the initial length scale estimate [75, Theorem 5]. For this reason, we do not quote the latter; an interested reader can easily extract the main arguments to adapt them to a different setting, if necessary.

Along with the general SLI for the quantum graphs and the ILS obtained via the Lifshitz tails finite-volume asymptotics, the Wegner estimate from Proposition 7.1 gives the required input for the proof of spectral and dynamical localization. This also provides all necessary ingredients for the proof of exponential scaling limit, in the parameter zone where the MSA was carried out in [75].

Lenz *et al.* [79] considered a more general class of models; in particular, the basic combinatorial vertex graph can have a more general structure, with uniformly bounded coordination numbers. The authors of [79] go a great length and make special efforts in order to make the class of admissible vertex graphs as rich as possible, so it would be unrealistic to reproduce here the entire discussion of what is admissible for their methods and what is not, but suffices to say that usually it is assumed that an amenable group acts on the configuration spaces involved. Besides, having in mind applications to the MSA, one should not hope (at least, today) to be able to cover the case of Cayley trees and other graphs with exponential growth of balls.

An optimal Wegner estimate for the random lengths model is established in [79, Theorem 2.9], under the assumption of existence of (common) marginal probability density for the edge lengths, $\rho \in \mathcal{C}^1([l_{\min}, l_{\max}])$, $l_{\min} > 0$. See also [51, 52, 53] and references therein.

7.13.3 Random vertex couplings

Apparently, the first or one of the first mathematical treatments of the quantum graphs with disorder introduced not through the underlying metric graphs but rather the random coefficients of the Hamiltonian thereupon, was done by Chen *et al.* [15] who proved the existence of point spectrum in this model, using a nice reduction of the spectral problem for the original Hamiltonian to its counterpart for an auxiliary Hamiltonian on the underlying combinatorial graph.

Pictorially, in this model the potential is given not by a *bona fide* function, distributed over the edges, but rather by a generalized function supported by the vertex set, similar to the so-called Kronig–Penney models (cf., e.g., [13, 34]).

Klopp and Pankrashkin [74], further developing the analytic ideas from [15] and building on techniques from [67, 70, 89, 12] (referring to the notion of boundary triples, see the references in [89, 12]), proved in this model strong dynamical localization with the help of reduction to the FMM on the underlying discrete graph. The latter method, as is well-known, does not make direct use of the Wegner estimate.

Sabri [92] recently proved a new Wegner estimate for the random vertex couplings model. See also [74] where the case of a constant edge length is considered.

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