Parametric Solutions to (six) n^{th} Powers Equal to Another (six) n^{th} Powers for Degree 'n' = 2,3,4,5,6,7,8,& 9

Seiji Tomita¹, Oliver Couto^{2,*}

¹Computer Engineer, Tokyo software co. Inc., Japan

²University of Waterloo, Waterloo, Ontario, Canada

Copyright©2016 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract Consider the below mentioned equation: $[a^n + b^n + c^n + d^n + e^n + f^n] = [p^n + q^n + r^n + s^n + t^n + u^n]$ ---- (A). Historically in math literature there are instances where solutions have been arrived at by different authors for equation (A) above. Ref.no. (1) by A. Bremner & J. Delorme and Ref. no. (10) by Tito Piezas. The difference is that this article has done systematic analysis of equation (A) for $n=2,3,4,5,6,7,8$ & 9. While numerical solutions for equation (A) is available on “Wolfram math” website, search for parametric solutions to equation (A) in various publications for all $n=2,3,4,5,6,7,8$ & 9 did not yield much success. The authors of this paper have selected six terms on each side of equation (A) since the difficulty of the problem increases every time a term is deleted on each side of equation (A). The authors have provided parametric solutions for equation (A) for $n=2, 3, 4, 5$ & 6 and for $n=7, 8$ & 9 solutions using elliptical curve theory has been provided. Also we would like to mention that solutions for $n=7, 8$ & 9 have infinite numerical solutions.

Keywords Pure Math, Diophantine Equations, Equal Sums, Parametric Solutions

1. Degree Two, $n=2$

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = p^2 + q^2 + r^2 + s^2 + t^2 + u^2 \quad (1)$$

We have the below mentioned numerical solution:

$$(1,30,31,36,7,17)^2 = (3,4,19,27,34,35)^2$$

In the above let $a=1$, $b=30$ & $c=31$, Since $1+30=31$ or $(a+b=c)$

Hence we have $(a^2 + b^2 + c^2) = 2(a^2 + ab + b^2)$ ---(2)

Let $a = (30+t)$ and $b = (1+kt)$. Substituting for (a,b) & solving for 't' in (a^2+ab+b^2) in right hand side of equation (2) above, we get parameterization of $(a \& b)$ given below.

$$a = (30k^2 - 2k - 31)/(k^2 + k + 1)$$

$$b = (-31k^2 - 60k + 1)/(k^2 + k + 1)$$

$$c = a + b = (-k^2 - 62k - 30)/(k^2 + k + 1)$$

Also since $(a, b, c)^2 = (p, q, r, s, t, u)^2 - (d, e, f)^2$, after substituting above values of (a,b,c) we get the below mentioned parameterization of equation (A) for degree two.

$$\begin{aligned} (30k^2 - 2k - 31)^2 + (-31k^2 - 60k + 1)^2 + (-k^2 - 62k - 30)^2 + (36^2 + 7^2 + 17^2) * (k^2 + k + 1)^2 \\ = (3^2 + 4^2 + 19^2 + 27^2 + 34^2 + 35^2) * (k^2 + k + 1)^2 \end{aligned}$$

For $k=2$ we get the below mentioned new numerical solution:

$$(85,158,243,252,49,119)^2 = (21,28,133,189,238,245)^2$$

2. Degree Three, $n=3$

First method:

$$a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = p^3 + q^3 + r^3 + s^3 + t^3 + u^3 \quad (3)$$

Let $(a,b,c,d,e,f) = [(Ax+1), (Bx+1), (Cx+1), (Dx+1), (Ex+1), (Fx+1)]$

and

$$(p,q,r,s,t,u) = [(Px+1), (Qx+1), (Rx+1), (Sx+1), (Tx+1), (Ux+1)]$$

Let $(A, B, C, D, E, F)^3 = (P, Q, R, S, T, U)^3 - \dots - (4)$ be known solution

We have numerical solution, $(1,2,4,8,9,12)^3 = (3,5,6,7,10,11)^3$

Substituting values of (a,b,c,d,e,f) & (p,q,r,s,t,u) in equation (3) above we get after simplification the below mentioned condition,

$$x = - \frac{[(A+B+C+D+E+F)-(P+Q+R+S+T+U)]}{W} \quad (5)$$

Where, $W = (P^2 + Q^2 + R^2 + S^2 + T^2 + U^2) - (A^2 + B^2 + C^2 + D^2 + E^2 + F^2)$

After substituting values of (A, B, C, D, E, F) and (P, Q, R, S, T, U)

in equation (5) we get the solution $x = - (1/5)$

Hence substituting values of (A, B, C, D, E, F) and (P, Q, R, S, T, U) in equation (3) above we get for $x = (-1/5)$,

$$\begin{aligned} & [(x+1), (2x+1), (4x+1), (8x+1), (9x+1), (12x+1)]^3 \\ &= [(3x+1), (5x+1), (6x+1), (7x+1), (10x+1), (11x+1)]^3 \end{aligned}$$

Substituting $x = (-1/5)$ we get,

$$(-4, -3, -1, 3, 4, 7)^3 = (-2, 0, 1, 2, 5, 6)^3$$

Similarly for different numerical solutions for equation (4) we can get another value of 'x' from equation (5) and thus more numerical solutions.

Second Method:

$$a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = p^3 + q^3 + r^3 + s^3 + t^3 + u^3 \quad (3)$$

Let $(a,b,c,d,e,f) = [(Ax+1), (Bx-1), (Cx+1), (Dx-1), (Ex+1), (Fx-1)]$

& $(p,q,r,s,t,u) = [(Ax-1), (Bx+1), (Cx-1), (Dx+1), (Ex-1), (Fx+1)]$

Substituting values of (a,b,c,d,e,f) & (p,q,r,s,t,u) in equation (3) above we get after simplification the below mentioned condition,

$$(A^2 + C^2 + E^2) = (B^2 + D^2 + F^2) \quad (4)$$

Equation (4) has the numerical solution, $(A,C,E) = (2,10,21)$ & $(B,D,F) = (5,6,22)$

Hence substituting values of (A, B, C, D, E, F) we get the parametric form given below,

$$\begin{aligned} & [(2x+1), (5x-1), (10x+1), (6x-1), (21x+1), (22x-1)]^3 \\ &= [(2x-1), (5x+1), (10x-1), (6x+1), (21x-1), (22x+1)]^3 \end{aligned}$$

For $x=1$ we get,

$$(3,4,5,11,22,21)^3 = (1,6,9,20,7,23)^3$$

3. Degree Four, n=4

$$a^4 + b^4 + c^4 + d^4 + e^4 + f^4 = p^4 + q^4 + r^4 + s^4 + t^4 + u^4 \quad (5)$$

We have numerical solution given below,

$$(16,480,496,532,798,1330)^4 = (342,336,224,560,950,1292)^4$$

In the above let $a=16$, $b=480$, $c=496$. Since $(16+480=496)$ or $(a+b=c)$

$$\text{So we have } (a^4 + b^4 + c^4) = a^4 + b^4 + (a+b)^4 = 2(a^2 + ab + b^2)^2 \quad (6)$$

Let $a=480+t$ and $b=16+kt$

Hence parameterization $(a^2 + ab + b^2)$ on the right hand side of equation (6) we get

$$a = (496k^2 + 960k - 16)/(k^2 + k + 1)$$

$$b = (-16k^2 - 992k - 480)/(k^2 + k + 1)$$

$$c = (480k^2 - 32k - 496)/(k^2 + k + 1)$$

$$\text{Since } (a, b, c)^4 = (16, 480, 496)^4 = (342, 336, 224, 560, 950, 1292)^4 - (532, 798, 1330)^4$$

After substituting for (a,b,c) we get the below mentioned parameterization,

$$[(496k^2 + 960k - 16)^4 + (-16k^2 - 992k - 480)^4 + (480k^2 - 32k - 496)^4 + (532^4 + 798^4 + 1330^4) * (k^2 + k + 1)^4 = (342^4 + 336^4 + 224^4 + 560^4 + 950^4 + 1292^4) * (k^2 + k + 1)^4$$

For k=2 we get,

$$(1944, 1264, 680, 1862, 2793, 4655)^4 = (1197, 3325, 1176, 4522, 784, 1960)^4$$

4. Degree Five, n=5

$$a^5 + b^5 + c^5 + d^5 + e^5 + f^5 = p^5 + q^5 + r^5 + s^5 + t^5 + u^5 \quad (7)$$

$$\text{Consider equation } a^5 + b^5 + c^5 + d^5 + e^5 + f^5 = 2(t)^5 \quad (8)$$

Let,

$$a = Am^2 + Um + 3B$$

$$b = Bm^2 - Um + 3A$$

$$c = Cm^2 + Vm + 3D$$

$$d = Dm^2 - Vm + 3C$$

$$e = Em^2 + Wm + 3F$$

$$f = Fm^2 - Wm + 3E$$

$$t = Tm^2 + 3T$$

Substituting the values of (a,b,c,d,e,f) in equation (8) above we get,

$$(Am^2 + um + 3B)^5 + (Bm^2 - um + 3A)^5 + (Cm^2 + um + 3D)^5 + (Dm^2 - vm + 3C)^5 + (Em^2 + wm + 3F)^5 + (Fm^2 - wm + 3E)^5 = 2(T)^5 * (m^2 + 3)^5 \quad (9)$$

We have known solution given below,

$$(A, B, C, D, E, F)^5 = (91, 7, -21, 119, 161, -63)^5 = 2 * (147)^5 = 2 * (T)^5$$

Since in equation (9) we have the parameter 'm', so when we substitute values of ((A,B,C,D,E,F) in equation (9), the only unknowns are (u,v,w), after simplification of equation (9) we get the below conditions,

$$(U+V+W) = 4(D-F) \text{ \& } U=2(D-F)$$

$$V=2(2D-3B+F)$$

$$W=2(-D+3B-2F)$$

Since (D,B,F) = (119,7,-63) we get (U,V,W) = (364,308,56)

Substituting the values of (A,B,C,D,E,F,U,V,W,T) in equation (9) above we get the below mentioned parameterization ,

$$(91m^2 + 364m + 21)^5 + (7m^2 - 364m + 273)^5 + (-21m^2 + 308m + 357)^5 + (119m^2 - 308m - 63)^5 + (161m^2 + 56m - 189)^5 + (-63m^2 - 56m + 483)^5 = 2(147)^5 * (m^2 + 3)^5 \quad (10)$$

We also have the below mentioned numerical solution,

$$(91, 7, -21, 119, 161, -63)^5 = 2 * (147)^5 \text{ and } (159, -61, 127, -29, 81, 17)^5 = 2 * (147)^5$$

Using the new values (159,-61,127,-29,81,17) in place of (A,B,C,D,E,F) in equation (9)

we get another set of values for (U,V,W) = (-92,284,-376)

Substituting the above values in equation (9) we get another parametric equation given below,

$$(159m^2 - 92m - 183)^5 + (-61m^2 + 92m + 477)^5 + (127m^2 + 284m - 87)^5 + (-29m^2 - 284m + 381)^5 + (81m^2 - 376m + 51)^5 + (17m^2 + 376m + 243)^5 = 2(147)^5 * (m^2 + 3)^5 \quad (11)$$

Since the right hand sides of equations (10) & (11) are equal, hence we can equate their left hand sides and we get the

below mentioned (5-6-6) equation for degree $n=5$.

$$\begin{aligned} & (91m^2 + 364m + 21)^5 + (7m^2 - 364m + 273)^5 + (-21m^2 + 308m + 357)^5 + (119m^2 - 308m - 63)^5 \\ & + (161m^2 + 56m - 189)^5 + (-63m^2 - 56m + 483)^5 \\ & = (159m^2 - 92m - 183)^5 + (-61m^2 + 92m + 477)^5 + (127m^2 + 284m - 87)^5 \\ & + (-29m^2 - 284m + 381)^5 + (81m^2 - 376m + 51)^5 + (17m^2 + 376m + 243)^5 \end{aligned}$$

For $m=2$ we get,

$$(1113, 377, 889, 303, 567, 119)^5 = (269, 417, 989, 203, 427, 1063)^5$$

5. Degree Six, $n=6$

There are parameter solutions to,

$$A_1^6 + A_2^6 + A_3^6 + A_4^6 + A_5^6 + A_6^6 = B_1^6 + B_2^6 + B_3^6 + B_4^6 + B_5^6 + B_6^6 \quad (2)$$

Let,

$$A_1 = a_1a + b_1b - c_1$$

$$A_2 = a_3a + c_3$$

$$A_3 = a_4a + c_4$$

$$A_4 = a_1a - b_2b + c_2$$

$$A_5 = a_2a - b_1b + c_2$$

$$A_6 = a_2a + b_2b + c_2$$

$$B_1 = a_1a + b_1b + c_1$$

$$B_2 = a_3a - c_3$$

$$B_3 = a_4a - c_4$$

$$B_4 = a_1a - b_2b - c_2$$

$$B_5 = a_2a - b_1b - c_2$$

$$B_6 = a_2a + b_2b - c_2$$

$$a_2 = ma_1, a_3 = na_1, a_4 = pa_1, c_1 = qc_2, c_3 = rc_2, c_4 = tc_2$$

As for this equation (2), the factorization is done as follows.

$$(m, n, p, q, r, t) = (3, 1, -1, -2, -6, 3)$$

$$240a_1ac_2(19c_2^2 + b_2^2b^2 + 2a_1b_2ba + 4a^2a_1^2) * (21c_2^2 + b_2^2b^2 + 2a_1b_2ba + 6a^2a_1^2)$$

$$(m, n, p, q, r, t) = (3, 1, -1, -1, -5, 3)$$

$$240a_1ac_2(14c_2^2 + 6a_1^2a^2 + 2b_2ba_1a + b_2^2b^2) * (-12c_2^2 + 4a_1^2a^2 + 2b_2ba_1a + b_2^2b^2)$$

$$(m, n, p, q, r, t) = (3, 1, -1, 2, -2, 3)$$

$$240a_1ac_2(5c_2^2 + b_2^2b^2 + 2a_1ab_2b + 6a_1^2a^2) * (-3c_2^2 + b_2^2b^2 + 2a_1ab_2b + 4a_1^2a^2)$$

$$(m, n, p, q, r, t) = (3, 1, -1, 3, -3, 1)$$

$$240a_1ac_2(6c_2^2 + b_2^2b^2 + 2a_1ab_2b + 6a_1^2a^2) * (-4c_2^2 + b_2^2b^2 + 2a_1ab_2b + 4a_1^2a^2)$$

$$(m, n, p, q, r, t) = (3, 2, -4, 3, 2, 2)$$

$$-240a_1ac_2(5c_2^2 + 15a_1^2a^2 + 2b_2ab_2b + b_2^2b^2) * (3c_2^2 + 5a_1^2a^2 - 2b_2ab_2b - b_2^2b^2)$$

$$(m, n, p, q, r, t) = (3, 3, -5, 3, 2, 2)$$

$$-240a_1ac_2(3c_2^2 - b_2^2b^2 + 2a_1ab_2b + 12a_1^2a^2) * (5c_2^2 + b_2^2b^2 + 2a_1ab_2b + 22a_1^2a^2)$$

$$(m, n, p, q, r, t) = (3, 4, -6, 3, 2, 2)$$

$$-240a_1ac_2(3c_2^2 + 21a_1^2a^2 - 2b_2ab_2b - b_2^2b^2) * (5c_2^2 + 31a_1^2a^2 + 2b_2ab_2b + b_2^2b^2)$$

Take $c_2=1$

$$\text{Case 1. } 4a_1^2 + 2a_1b_2 + b_2^2 = 19 \quad (3)$$

We can find infinitely many rational solutions of (3), then we obtain infinitely parameter solutions of (2).

$$\text{Take } x=2a_1, y=b_2$$

$$\text{Then (3) becomes to } x^2 + xy + y^2 = 19 \quad (4)$$

$(x,y) = (3,2)$ is a solution of (4).

We obtain parameter solution of (3) by using $(a,b) = (3/(2a_1), 2/b_2)$. And 'k' is parameter.

$$\begin{aligned} A_1 &= 7k^2b_2^2 - 4a_1^2 \\ A_2 &= -9k^2b_2^2 - 32kb_2a_1 - 68a_1^2 \\ A_3 &= 3k^2b_2^2 + 20kb_2a_1 + 44a_1^2 \\ A_4 &= 15k^2b_2^2 + 20kb_2a_1 - 28a_1^2 \\ A_5 &= 11k^2b_2^2 - 20kb_2a_1 - 52a_1^2 \\ A_6 &= k^2b_2^2 - 44kb_2a_1 - 36a_1^2 \\ B_1 &= -k^2b_2^2 - 16kb_2a_1 - 36a_1^2 \\ B_2 &= 15k^2b_2^2 + 16kb_2a_1 + 28a_1^2 \\ B_3 &= -9k^2b_2^2 - 4kb_2a_1 - 4a_1^2 \\ B_4 &= 11k^2b_2^2 + 12kb_2a_1 - 44a_1^2 \\ B_5 &= 7k^2b_2^2 - 28kb_2a_1 - 68a_1^2 \\ B_6 &= -3k^2b_2^2 - 52kb_2a_1 - 52a_1^2 \end{aligned}$$

Example,

$$(a_1, b_2) = (1, 1) \text{ and } (k=1)$$

$$\begin{aligned} \text{a) } 3^6 + 109^6 + 67^6 + 7^6 + 61^6 + 79^6 &= 53^6 + 59^6 + 17^6 + 21^6 + 89^6 + 107^6 \\ \text{b) } 59^6 + 245^6 + 131^6 + 167^6 + 13^6 + 159^6 &= 93^6 + 211^6 + 97^6 + 91^6 + 89^6 + 235^6 \\ \text{c) } 27^6 + 85^6 + 43^6 + 73^6 + 11^6 + 49^6 &= 29^6 + 83^6 + 41^6 + 45^6 + 17^6 + 77^6 \end{aligned}$$

6. Degree Seven, n=7

Let,

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k + x_6^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k + y_6^k \quad (1)$$

We show that,

for $k=1, 3, 5, 7$ eqn. (1) has infinitely many integer solutions.

We used the below identity and Theorem.

Identity (Tito Piezas) (Ref.no.10).

$$(c + bp + aq)^k + (c - bp - aq)^k + (d + ap - bq)^k + (d - ap + bq)^k = (c + bp - aq)^k + (c - bp + aq)^k + (d + ap + bq)^k + (d - ap - bq)^k,$$

for $k=2,4,6$,

$$\text{where } (4q^2 - p^2)b^2 + (4p^2 - q^2)a^2 = 15c^2$$

$$\text{and } (4p^2 - q^2)b^2 + (4q^2 - p^2)a^2 = 15d^2.$$

Theorem,

If $a_1^k + a_2^k + \dots + a_m^k = b_1^k + b_2^k + \dots + b_m^k$, for $k=2,4,\dots,2n$,

Then (using t) we get,

$$\begin{aligned} &(t + a_1)^k + (t + a_2)^k + \dots + (t + a_m)^k, (t - a_1)^k + (t - a_2)^k + \dots + (t - a_m)^k \\ &= (t + b_1)^k + (t + b_2)^k + \dots + (t + b_m)^k, (t - b_1)^k + (t - b_2)^k + \dots + (t - b_m)^k, \end{aligned}$$

for $k=1,2,3,\dots,2n+1$, where t is arbitrary integer.

where $(4q^2 - p^2)b^2 + (4p^2 - q^2)a^2 = 15c^2$

and $(4p^2 - q^2)b^2 + (4q^2 - p^2)a^2 = 15d^2$

Above for $k=1,3,5,7$ has infinite many solutions,

$$x_1 = -c+d+ap-bq$$

$$y_1 = -c+d+ap+bq$$

$$x_2 = -c+d-ap+bq$$

$$y_2 = -c+d-ap-bq$$

$$x_3 = -2c-bp-aq$$

$$y_3 = -2c-bp+aq$$

$$x_4 = -2c+bp+aq$$

$$y_4 = -2c+bp-aq$$

$$x_5 = -c-d-ap+bq$$

$$y_5 = -c-d-ap-bq$$

$$x_6 = -c-d+ap-bq$$

$$y_6 = -c-d+ap+bq$$

We have,

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k + x_6^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k + y_6^k \quad (1)$$

Let $[a_1, a_2, \dots, a_m] = [b_1 b_2, \dots, b_m]$ ($k=1,2,\dots,n$) denote

$$a_1^k + a_2^k + \dots + a_m^k = b_1^k + b_2^k + \dots + b_m^k \quad (k=1,2,\dots,n).$$

First we apply Tito Piezas's identity to above theorem, we obtain (using t)

$$[t + c + bp + aq, t + c - bp - aq, t + d + ap - bq, t + d - ap + bq, t - c - bp - aq, t - c + bp + aq, t - d - ap + bq, t - d + ap - bq]^k =$$

$$[t + c + bp - aq, t + c - bp + aq, t + d + ap + bq, t + d - ap - bq, t - c - bp + aq, t - c + bp - aq, t - d - ap - bq, t - d + ap + bq]^k,$$

where $(4q^2 - p^2)b^2 + (4p^2 - q^2)a^2 = 15c^2$

and $(4p^2 - q^2)b^2 + (4q^2 - p^2)a^2 = 15d^2$

for $k=1,3,5,7$,

Here, set $t = (-c)$, we can reduce four terms and obtain

$$[-c + d + ap - bq, -c + d - ap + bq, -2c - bp - aq, -2c + bp + aq, -c - d - ap + bq, -c - d + ap - bq]^k =$$

$$[-c + d + ap + bq, -c + d - ap - bq, -2c - bp + aq, -2c + bp - aq, -c - d - ap - bq, -c - d + ap + bq]^k,$$

for $k=1, 3, 5, 7$

where $(4q^2 - p^2)b^2 + (4p^2 - q^2)a^2 = 15c^2$

and $(4p^2 - q^2)b^2 + (4q^2 - p^2)a^2 = 15d^2$

By transform simultaneous equation $\{(4q^2 - p^2)b^2 + (4p^2 - q^2)a^2 = 15c^2, (4p^2 - q^2)b^2 + (4q^2 - p^2)a^2 = 15d^2\}$ to an elliptic curve,

we can prove the infinity of solutions to this equation.

Since Tito Piezas's identity has infinitely many integer solutions, therefore this identity also has infinitely many integer solutions.

Example,

$$(p, q) = (3, 2)$$

$$[-c + d + 3a - 2b, -c + d - 3a + 2b, -2c - 3b - 2a, -2c + 3b + 2a, -c - d - 3a + 2b, -c - d + 3a - 2b]^k =$$

$$[-c + d + 3a + 2b, -c + d - 3a - 2b, -2c - 3b + 2a, -2c + 3b - 2a, -c - d - 3a - 2b, -c - d + 3a + 2b]^k$$

for $k=1, 3, 5, 7$

where, $7b^2 + 32a^2 = 15c^2$

and $32b^2 + 7a^2 = 15d^2$.

Numerical example,

$$\begin{aligned} \{a, b, c, d\} &= \{1, 13, 9, 19\}: \\ [-13, 33, -59, 23, -5, -51]^k &= [39, -19, -55, 19, -57, 1]^k \\ \{a, b, c, d\} &= \{466, 607, 797, 942\}: \\ [329, -39, -4347, 1159, -1923, -1555]^k &= [2757, -2467, -2483, -705, -4351, 873]^k \\ \{a, b, c, d\} &= \{607, 466, 942, 797\}: \\ [372, -517, -2248, 364, -1314, -425]^k &= [1304, -1449, -1034, -850, -2246, 507]^k \\ (p, q) &= (4, 1) \\ [-c + d + 4a - b, -c + d - 4a + b, -2c - 4b - a, -2c + 4b + a, -c - d - 4a + b, -c - d + 4a - b]^k &= \\ [-c + d + 4a + b, -c + d - 4a - b, -2c - 4b + a, -2c + 4b - a, -c - d - 4a - b, -c - d + 4a + b]^k & \end{aligned}$$

for $k=1, 3, 5, 7$

where, $(-12b^2 + 63a^2) = 15c^2$

and $63b^2 - 12a^2 = 15d^2$.

One Solution is,

$$\begin{aligned} \{a, b, c, d\} &= \{89, 82, 167, 148\}: \\ x_1^k + x_2^k + x_3^k + x_4^k + x_5^k + x_6^k &= \\ y_1^k + y_2^k + y_3^k + y_4^k + y_5^k + y_6^k & \end{aligned}$$

For $k=7$, we get:

$$[255, 457, 573, 83, 95, 753]^7 = [419, 293, 751, 589, 41, 123]^7$$

Second solution is,

$$\begin{aligned} \{a, b, c, d\} &= \{82, 89, 148, 167\}: \\ [129, 199, 285, 71, 11, 366]^7 &= [218, 110, 367, 277, 38, 51]^7 \end{aligned}$$

7. Degree Eight, $n=8$

$$a^8 + b^8 + c^8 + d^8 + e^8 + f^8 = p^8 + q^8 + r^8 + s^8 + t^8 + u^8$$

There are many solutions for $A_1^8 + A_2^8 + A_3^8 + A_4^8 + A_5^8 + A_6^8 = B_1^8 + B_2^8 + B_3^8 + B_4^8 + B_5^8 + B_6^8$.

We show that there are infinitely many solutions for 8.6.6 by Sinha's Theorem.

By Sinha's Theorem,

$$A_1^8 + A_2^8 + A_3^8 + A_4^8 + A_5^8 + A_6^8 + A_7^8 = B_1^8 + B_2^8 + B_3^8 + B_4^8 + B_5^8 + B_6^8 + B_7^8.$$

But, if we set $(A_1 + B_1 = 0)$, we can obtain the solution for (8.6.6) equation.

Theorem

There are infinitely many solutions for $A_1^8 + A_2^8 + A_3^8 + A_4^8 + A_5^8 + A_6^8 = B_1^8 + B_2^8 + B_3^8 + B_4^8 + B_5^8 + B_6^8$.

1. Solving for $a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2$.

$$\text{Set } a_1 = a(x) + s_1, a_2 = b(x) + s_2, a_3 = b(x) + s_2 - 3(ax + s_1),$$

$$b_1 = a(x) - s_2, b_2 = b(x) - s_1, b_3 = (b - 3a)x - s_2 + 3s_1$$

(1)

Take $s_1 = 5a - 3b, s_2 = 19a - 5b$ then

$$a_1^2 + a_2^2 + a_3^2 - (b_1^2 + b_2^2 + b_3^2) = 0$$

2. Solving for $a_1^4 + a_2^4 + a_3^4 = b_1^4 + b_2^4 + b_3^4$.

$$a_1^4 + a_2^4 + a_3^4 - (b_1^4 + b_2^4 + b_3^4) = -32x(a+b)(3a-b)*f$$

$$f = (8x^2 + 21x - 275)*a^2 + (-5x^2 - 24x + 170)*ab + (3x - 3)b^2$$

We must find rational value (a, b) for above equation.

Discriminant,

$$25x^4 + 144x^3 - 1280x^2 - 4608x + 25600 = y^2 \quad (2)$$

So, we must find rational numbers x, y .

$$U=x \text{ and } V=y$$

$$V^2 = 25U^4 + 144U^3 - 1280U^2 - 4608U + 25600 \quad (3)$$

Using elliptic curve theory, and using 'APECS' program by Ian Connell, Weierstrass form is,

$$Y^2 = X^3 + X^2 - 920X + 10404 \quad (4)$$

$$U = (200X - 3248) / (5Y + 9X - 162)$$

$$V = (25344Y - 194880X^2 + 3550080X - 23468800 + 4000X^3) / (5Y + 9X - 162)^2 \quad (5)$$

$$X = (5V + 800 - 72U - 7U^2) / U^2$$

$$Y = (200V + 32000 - 4320U - 800U^2 - 9VU + 45U^3) / U^3 \quad (6)$$

Point $P = (0, 160)$ solution for (3).

Rational point $Q(X, Y)$ on the curve (4) corresponding to the values $U=0, V=160$ is

$$X=406/25, Y=-396/125.$$

So, we get the relation of the curve (3) and the curve (4).

Point, $(P) = (0, 160)$ on the curve (3), Point, $(Q) = (406/25, -396/125)$ on the curve (4)

We obtain $2Q = (4939/25, -344112/125)$ on the curve (4) using APECS.

As this point on the curve (4) does not have integer coordinates, there are infinitely many rational points on the curve (4) by Nagell-Lutz theorem.

Point, $(2P) = (-200/67, 725280/4489)$ is given by $2Q$ using (5).

We can obtain infinitely many integer solutions for (2) by applying the group law.

By Sinha's Theorem,

$$A_1^8 + A_2^8 + A_3^8 + A_4^8 + A_5^8 + A_6^8 + A_7^8 = B_1^8 + B_2^8 + B_3^8 + B_4^8 + B_5^8 + B_6^8 + B_7^8 \quad (1)$$

$$A_1 = 2a_1$$

$$A_2 = 2a_2$$

$$A_3 = b_1 + b_2 + b_3$$

$$A_4 = 2a_3$$

$$A_5 = b_1 - b_2 + b_3$$

$$A_6 = -b_1 + b_2 + b_3$$

$$A_7 = b_1 + b_2 - b_3$$

$$B_1 = a_1 - a_2 + a_3$$

$$B_2 = -a_1 + a_2 + a_3$$

$$B_3 = 2b_3$$

$$B_4 = a_1 + a_2 + a_3$$

$$B_5 = 2b_1$$

$$B_6 = 2b_2$$

$$B_7 = a_1 + a_2 - a_3$$

By Sinha's Theorem, substitute (a, b, x) to (1), then we obtain infinitely many solutions of (1).

Example,

$$(x, a, b) : (8x^2 + 21x - 275)a^2 + (-5x^2 - 24x + 170)ab + (3x - 3)b^2 = 0$$

Since, $a_1 = ax + s_1, a_2 = bx + s_2, a_3 = bx + s_2 - 3(ax + s_1), b_1 = ax - s_2, b_2 = bx - s_3,$

$$b_3 = (b - 3a)x - s_2 + 3s_1 \quad (2)$$

and, $s_1=5a-3b$, $s_2=19a-5b$ then

For, $(x, a, b) = (1, 47, 82)$ we get after substituting values of $(a_1, a_2, a_3, b_1, b_2, b_3)$ in equation (2) and equation (1) above,

$$565^8 + 459^8 + 457^8 + 552^8 + 23^8 + 116^8 = 493^8 + 575^8 + 529^8 + 436^8 + 93^8 + 72^8$$

$$(x, a, b) = (-6, 21, 113)$$

$$211^8 + 155^8 + 59^8 + 44^8 + 165^8 + 54^8 = 31^8 + 209^8 + 121^8 + 10^8 + 111^8 + 180^8$$

$$(x, a, b) = (6, 15, 139)$$

$$106^8 + 203^8 + 295^8 + 91^8 + 78^8 + 216^8 =$$

$$232^8 + 13^8 + 169^8 + 125^8 + 294^8 + 126^8$$

$$(x, a, b) = (-14, 5, 9)$$

$$19^8 + 27^8 + 35^8 + 4^8 + 3^8 + 34^8 =$$

$$17^8 + 7^8 + 1^8 + 30^8 + 31^8 + 36^8$$

$$(x, a, b) = (-14, 3, -37)$$

$$190^8 + 111^8 + 127^8 + 13^8 + 182^8 + 84^8 =$$

$$148^8 + 195^8 + 169^8 + 71^8 + 98^8 + 42^8$$

8. Degree Nine, $n=9$

$$x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 =$$

$$y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9$$

Andrew Bremner and J. Delorme (Ref. no. 1) showed that,

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k + x_6^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k + y_6^k$$

for $k=1, 2, 3, 9$ has infinitely many integer solutions. ([1])

By different way, Tito Piezas [10] showed that above equation has infinitely many solutions.

A. Choudhry (Ref. no. 11) showed that $x_1^k + x_2^k + x_3^k = y_1^k + y_2^k + y_3^k$

for $k=1, 2, 6$ has infinitely many integer solutions ([3])

We show that there are infinity many solutions for above (k.6.6) equation.

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k + x_6^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k + y_6^k \quad (2)$$

for $k=1, 2, 3, 9$ has infinitely many solutions.

Set variables as following [equation (3)],

$$x_1 = 2(a+b)m + (a-b+t)n + w$$

$$y_1 = 2(a+b)m + (a-b+t)n - w$$

$$x_2 = -2am + (a+b+t)n + w$$

$$y_2 = -2am + (a+b+t)n - w$$

$$x_3 = -2bm - (a+b-t)n + w$$

$$y_3 = -2bm - (a+b-t)n - w$$

$$x_4 = -2(a+b)m + (a-b+t)n - w$$

$$y_4 = -2(a+b)m + (a-b+t)n + w$$

$$x_5 = 2am + (a+b+t)n - w$$

$$y_5 = 2am + (a+b+t)n + w$$

$$x_6 = 2bm - (a+b-t)n - w$$

$$y_6 = 2bm - (a+b-t)n + w$$

By using Ajai Choudhry's [Ref. no.11], equation (2) is always equals to zero for degree $k=1, 2, 3$.

And for $k=9$ we have :

$$x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 = y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9$$

Substituting values of above from equation (3) and after simplification we get,

Expression = $18432 \cdot a \cdot b \cdot m \cdot n^2(m-n)(m+n)(a+b)(a-b+3t)$

$$\begin{aligned} & * (-5m^2b^3 - n^2b^3 + 7m^2b^2t - 4m^2b^2a + 7n^2b^2t + 2n^2b^2a + 7am^2tb + 4m^2ba^2 - 14n^2bt^2 - 2n^2ba^2 - 7an^2bt \\ & + 5m^2a^3 + 7m^2a^2t + 7n^2a^2t + 14n^2at^2 + n^2a^3 + 14n^2t^3) \end{aligned}$$

$$* (2m^2b^2 + 2m^2ba + 2m^2a^2 + 3n^2a^2 + 14n^2ta + 21n^2t^2 - 4bn^2a - 14bn^2t + 3b^2n^2)$$

So, we have to find the rational solution (m, n) of:

$$\begin{aligned} & (-14bt^2 - 7bat + a^3 + 2ab^2 + 14at^2 - 2a^2b - b^3 + 7a^2t + 7b^2t + 14t^3)n^2 + (-4ab^2 - 5b^3 + 4a^2b + 5a^3 + \\ & 7b^2t + 7a^2t + 7bat)m^2 = 0 \end{aligned} \quad (3)$$

So that there are rational solution,

$$\begin{aligned} & -(-14bt^2 - 7bat + a^3 + 2ab^2 + 14at^2 - 2a^2b - b^3 + 7a^2t + 7b^2t + 14t^3) \\ & * (-4ab^2 - 5b^3 + 4a^2b + 5a^3 + 7b^2t + 7a^2t + 7bat) \end{aligned}$$

must be square number (s^2), then we have to find rational solution (a, b, t, s) of

$$\begin{aligned} s^2 = & (-98a^2 - 98ab - 98b^2)t^4 + (56ab^2 - 56a^2b + 168b^3 - 168a^3)t^3 + (63a^2b^2 - 119a^4 + 14ab^3 - 119b^4 + \\ & 14a^3b)t^2 + (14a^3b^2 - 14a^2b^3 - 42a^5 + 42b^5 - 14ab^4 + 14a^4b)t + 6ab^5 + 6a^5b - 5a^6 + 2a^4b^2 - 5b^6 - \\ & 6a^3b^3 + 2a^2b^4 \dots (4) \end{aligned}$$

By computer search, we found a solution (a, b, t) = (3, 4, 27/41).

Substitute (a, b, t) = (3, 4, 27/41) to (3), then (3) becomes to $160/68921(139n-164m)(139n+164m)=0$.

So, we get (m, n) = (139, 164).

Substitute (a, b, t, m, n) = (3, 4, 27/41, 139, 164) to (2),

then we get following solution,

$$[x_1, x_2, x_3, x_4, x_5, x_6] = [1025, 291, -996, -1081, 965, -44]$$

$$[y_1, y_2, y_3, y_4, y_5, y_6] = [865, 131, -1156, -921, 1125, 116].$$

Next, substitute,

(a, b) = (3, 4) to (4), then we get a quartic equation,

$$s^2 = -3626t^4 + 6888t^3 - 26831t^2 + 24570t - 3029 \quad (5)$$

Using elliptic curve theory, transform (5) to minimal Weierstrass form (6).

$$V^2 + UV + V = U^3 - 7166374 - 22875861928 \quad (6)$$

We get a point P (U, V) = (1026337/64, -1026359837/512).

As this point on the curve (6) does not have integer coordinates, there are infinitely many rational points on the curve (6) by Nagell-Lutz theorem.

By using point $2P = (t, s) = (3181201/12876603, 6408411316637440/165806904819609)$, we obtain a new solution.

$$[x_1, x_2, x_3, x_4, x_5, x_6] = [15677071397, 40208111671, -63297775068, -26458358421, 63560861593, -33396207172]$$

$$[y_1, y_2, y_3, y_4, y_5, y_6] = [19383367397, 43914407671, -59591479068, -30164654421, 59854565593, -37102503172]$$

Example,

$$[a, b, t] = [1, 3, 6/5],$$

$$[18, 13, 14, 23, 13, 1]^9 = [5, 10, 15, 21, 22, 9]^9$$

$$[a, b, t] = [4, 9, 13/3],$$

$$[453, 122, 331, 431, 150, 281]^9 = [429, 98, 307, 455, 174, 305]^9$$

Table (A) :

(six) n^{th} powers equal to another (six) n^{th} powers :

$$[a^n + b^n + c^n + d^n + e^n + f^n] = [p^n + q^n + r^n + s^n + t^n + u^n]$$

Numerical solutions: For degree's $n=2, 3, 4, 5, 6, 7, 8 \& 9$

n	a	b	c	d	e	f		p	q	r	s	t	u
2	1	7	17	30	31	36	=	3	4	19	27	34	35
3	11	22	4	3	21	5	=	20	7	6	23	9	1
4	16	480	496	532	798	1330	=	224	342	336	560	950	1292
5	87	233	264	396	496	540	=	90	206	309	366	522	523
6	61	3	109	67	7	79	=	21	17	53	59	89	107
7	129	199	285	71	11	366	=	218	110	367	277	38	51
8	3	6	8	10	15	23	=	5	9	12	9	20	22
9	1	13	14	13	18	23	=	5	9	10	15	21	22

REFERENCES

- [1] A, Bremner & J. Delorme, On equal sums of ninth powers, Math Computation, Vol.79, 201
- [2] Published math paper, Oliver Couto & Seiji Tomita,), Sum of three biquadratics a multiple of a n^{th} power, Universal Journal of Applied Mathematics , Vol. (4)(1),pages 22-31,2016, <http://www.hrpub.org>
- [3] Published math paper, Oliver Couto & Seiji Tomita, Generalized parametric solution to multiple sums of powers, Universal Journal of Applied Mathematics, Jan 2016, Volume 3(5), <http://www.hrpub.org>
- [4] Published math paper, Oliver Couto & Seiji Tomita, Solution of Polynomial Equation of any degree 'n' with Special Emphasis For $n = (2, 3, 4, 5 \text{ \& } 6)$, Journal "SOP transactions on applied mathematics" ISSN (Print):2373-8472, July 2015. <http://www.scipublish.com/journals/am>
- [5] Published math paper, Oliver Couto, Taxicab Equations for power (2, 3, 4 & 5), Journal, International Math Forum, Hikari ltd., Vol.9, Jan 2014.n0.12, pages 561-577. <http://www.m-hikari.com/imf.html>
- [6] Math paper, Seiji Tomita & Oliver Couto, Sum $(k+3)$ & $(k+5)$ bi-quadratics, Scheduled to be posted on-line, Universal Journal of applied mathematics , Vol. (4) (2), 2016, <http://www.hrpub.org>
- [7] Seiji Tomita, webpage <http://www.maroon.dti.ne.jp/fermat> Web site "Computation number theory"
- [8] Oliver Couto, webpage, <http://www.celebrating-mathematics.com> Web site "Number theory Math"
- [9] Ramanujan lost notebook, Narosa publishing house, New Delhi, India
- [10] Euler Leonhard, Opera Omnia books, 1984 publication.
- [11] Tito Piezas-Online collection algebraic identities, page 022, $k=2,3,4,5$, <http://sites.google.com/site/tpiezas>
- [12] Ajai Choudhry, Equal sums of three sixth powers, Rocky Mountain, Journal of mathematics, V30,2000 , Rocky mountain journal vol. 34, no.4, winter (2004), pages 1261-1298.
- [13] Jaroslaw Wroblewski , Tables of Numerical solutions for degree three, four, six, Seven & nine, website www.math.uni.wroc.pl/~jwr/eslp
- [14] L. E. Dickson, history of the theory of numbers, Vol. II, (Diophantine analysis), Chelsea publication, year 2000.