Sum of Three Biquadatics a Multiple of a n^{th} Power, $n = (2,3,4,5,6,7,8 \text{ \& } 9)$

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Abstract Consider the below mentioned equation: $x^4 + y^4 + z^4 = w * t^n$ (A). Historically Leonard Euler has given parametric solution for equation (A) when $w=1$ (Ref. no. 9) and degree ' n '=2. Also S. Realis has given parametric solution for equation (A) when ' w ' equals 1 and degree ' n '=3. More examples can be found in math literature (Ref. no.6). As is known that solving Diophantine equations for degree greater than four is difficult and the novelty of this paper is that we have done a systematic approach and has provided parametric solutions for degree's ' n ' = (2,3,4,5,6,7,8 & 9) for different values of " w ". The paper is divided into sections (A to H) for degrees (2 to 9) respectively. $x^4 + y^4 + z^4 = w * t^n - - -$ (A)

Keywords Quartic Equation, Diophantine Equations, Pure Math, Number Theory, Sums of Powers

1. Summary of Background

Whereas in math literature, we find many examples of sums of equal powers, meaning the degree is same on both sides of equation, in this paper we have demonstrated that it is possible to equate parametrically, unequal powers. Meaning the degree on the left hand side of the equation is different from the right hand side of the equations for $n = 2, 3, 5, 6, 7, 8 \text{ \& } 9$

2. Section (A)

Equation 1

We have the below mentioned equation,

$$x^4 + y^4 + z^4 = w * t^n$$

Let, $w=1$, degree $n=2$

$$x^4 + y^4 + z^4 = t^2$$

Let:

$$x = x_0p + 1$$

$$y = y_0p$$

$$z = z_0p$$

$$t = t_0p^2 + kp + 1$$

Where (x_0, y_0, z_0, t_0) are known solutions

$$(x_0p + 1)^4 + (y_0p)^4 + (z_0p)^4 = (t_0p^2 + kp + 1)^2 \quad (1)$$

We have solution given by Leonard Euler which is given below for,

$$a^4 + b^4 + c^4 = t^2$$

$$a = 2pq(p^2 - q^2), \quad b = (p^2 - q^2)(p^2 + q^2),$$

$$c = 2pq(p^2 + q^2), d = (p^8 + 14p^4q^4 + q^8)$$

We have, $x^4 + y^4 + z^4 = t^2$

$$\text{let } (x_0, y_0, z_0, t_0) = (2mn(m^2 - n^2), (m^2 - n^2)(m^2 + n^2), 2mn(m^2 + n^2), (m^8 + 14m^4n^4 + n^8))$$

Substituting in equation (1) above we get the values of (k, p) using maple soft

$$k = -16m^3n^3 \frac{(-3m^2n^4 + n^6 + 3m^4n^2 - m^6)}{(m^8 + 14m^4n^4 + n^8)}$$

$$p = \frac{-4(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})mn}{n^{16} - 8m^2n^{14} + 12m^4n^{12} + 8m^6n^{10} + 230m^8n^8 + 8m^{10}n^6 + 12m^{12}n^4 - 8m^{14}n^2 + m^{16}}$$

Substitute k and p to (1), hence we obtain above parametric solution.

For (x, y, z, t)

$$x = -4m^4n^{12} + 128m^6n^{10} + 6m^8n^8 + 128m^{10}n^6 - 4m^{12}n^4 + n^{16} + m^{16}$$

$$y = 4(-m^2 + n^2)(m^2 + n^2)(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})mn$$

$$z = -8m^2n^2(m^2 + n^2)(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})$$

$$t = (n^{32} + 120m^4n^{28} - 256m^6n^{26} + 2332m^8n^{24} + 768m^{10}n^{22} + 5960m^{12}n^{20} - 512m^{14}n^{18} + 48710m^{16}n^{16} - 512m^{18}n^{14} + 5960m^{20}n^{12} + 768m^{22}n^{10} + 2332m^{24}n^8 - 256m^{26}n^6 + 120m^{28}n^4 + m^{32})$$

numerical example is: $(12, 15, 20)^4 = (481)^2$

We next show the parametric solutions of $x^4 + y^4 + z^4 = w \cdot t^2$ for several "w".

In the case of $w=2$ & 3, parametric solutions and numerical examples are shown.

Identity by S. Realis: $w=3, n=2$

$$(5a^4 + 4a^3 + 9a^2 + 10a + 5)^4 + (5a^4 + 10a^3 + 9a^2 + 4a + 5)^4 + (5a^4 + 16a^3 + 27a^2 + 16a + 5)^4 = 3((a^2 + a + 1)(25a^6 + 75a^5 + 222a^4 + 319a^3 + 222a^2 + 75a + 25))^2$$

Search for smallest solutions of $x^4 + y^4 + z^4 = w \cdot t^2$ in the case of $w < 20$ are as follows.

w	x	y	z	t
1	20	15	12	481
2	2	1	1	3
3	1	1	1	1
9	60	45	36	1443
11	5	5	3	11
17	22	17	6	137
18	2	1	1	1
19	15	15	1	73

Above equation doesn't have a solution in the case of:

$$w = 5, 6, 7, 10, 13, 14, 15 \pmod{16}.$$

Equation (2)

$$w = 2, \text{ degree } n = 2,$$

$$\text{we have } x^4 + y^4 + z^4 = wt^2 \quad (\text{B})$$

We have known solution, $(p, q, r, s) = (2, 1, 1, 3)$

$$\text{Let the known solution be } p^4 + q^4 + r^4 = 2s^2$$

$$\text{let } t = sk^2 + gk + h, x = pk + a, y = qk + b, z = rk \quad (\text{C})$$

Substituting equation (B) above in eqn. (A) we get

$$(pk + a)^4 + (qk + b)^4 + (rk)^4 = 2(sk^2 + gk + h)^2$$

After some algebra we get,

$$\begin{aligned} g &= \frac{2(ap^3 + bq^3)}{sw} \\ h &= \left(\frac{1}{2}\right) * \frac{(6a^2p^2 + 6b^2q^2 - g^2w)}{sw} \\ k &= -\left(\frac{1}{2}\right) * \frac{(a^4 + b^4 - h^2w)}{(2a^3p + 2b^3q - ghw)} \end{aligned}$$

Hence equation (B) above has general solution for any 'w' & $n = 2$

We have known solution for $w=2$, $(p, q, r, s) = (2, 1, 1, 3)$

After substituting the above we get:

$$\begin{aligned} g &= \frac{8a + b}{3} \\ h &= \frac{22a^2 + 13b^2 - 8ab}{22} \\ k &= -\left(\frac{1}{72}\right) * \frac{(239a^3 - 465a^2b + 807ab^2 + 391b^3)}{(7a^2 - 14ab + 34b^2)} \end{aligned}$$

We have equations,

$$t = sk^2 + gk + h, \quad x = pk + a, \quad y = qk + b, \quad z = rk$$

After substituting for (g, h, k, p, q, r) we get

$$\begin{aligned} x &= 26a^3 - 78a^2b + 834ab^2 - 782b^3 \\ y &= 239a^3 - 969a^2b + 1815ab^2 - 2057b^3 \\ z &= 239a^3 - 465a^2b + 807ab^2 + 391b^3 \\ t &= 9(6347a^6 - 38082a^5b + 147945a^4b^2 - 337900a^3b^3 + 583689a^2b^4 - 593130ab^5 + 336107b^6) \end{aligned}$$

For $(a, b) = (1, 0)$ we get $(26, 239, 239)^4 = 2 * (57123)^2$

Equation 3

For $w=3$ degree $n=2$

We have known solution $(1, 1, 1)^4 = 3(1)^2$

Substituting $(p, q, r, s) = (1, 1, 1, 1)$ in equation (A) we get

$$t = gk + k^2 + h, \quad x = a + k, \quad y = b + k, \quad z = k$$

Hence we arrive at after substitution:

$$\begin{aligned} x &= 23a^4 - 46a^3b + 39a^2b^2 - 16ab^3 + 11b^4 \\ y &= 11a^4 - 16a^3b + 39a^2b^2 - 46ab^3 + 23b^4 \end{aligned}$$

$$z = 11a^4 - 28a^3b + 57a^2b^2 - 28ab^3 + 11b^4$$

$$t = 321a^8 - 1284a^7b + 2562a^6b^2 - 3192a^5b^3 + 3507a^4b^4 - 3192a^3b^5 + 2562a^2b^6 - 1284ab^7 + 321b^8$$

For (a,b)=(1,1) we get: $(23,11,11)^4 = 3 * (321)^4$

Numerical Examples

$$w = 2, n = 2$$

$$(a, b) = (1, -3)1444^4 + 3751^4 + 83^4 = 2 * 10057653^2$$

$$(a, b) = (1, -2)362^4 + 959^4 + 47^4 = 2 * 656883^2$$

$$w = 3, n = 2$$

$$(a, b) = (1, -3)367^4 + 703^4 + 451^4 = 3 * 318201^2$$

$$(a, b) = (1, -2)115^4 + 187^4 + 139^4 = 3 * 24297^2$$

$$\text{we have equation. } x^4 + y^4 + z^4 = wt^n$$

Taking, $n=3$ we get

$$x^4 + y^4 + z^4 = w * t^3$$

We show the parametric solutions of $x^4 + y^4 + z^4 = wt^3$ for $w = 1, 2$ & 3 .

In the case of $w = 1, 2$, and 3 , parametric solutions and numerical examples are shown below.

Smallest solutions for:

$$x^4 + y^4 + z^4 = wt^3$$

In the case of “w” for are as follows:

w	x	y	z	T
1	76	72	4	392
2	19	18	1	49
3	228	216	12	1176
5	380	360	20	1960
6	57	54	3	147
7	32	20	12	56
9	106	91	80	297

There are parametric solutions of $x^4 + y^4 + z^4 = w * t^3$,

$w = 1, 2$, and 3 .

Proof:

we prove the case of $w=2$, since this is the simplest case.

First, we use the identity $[x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2]$

To find (x, y) of $(x^2 + xy + y^2) = P^3$, we select 'u' such that $u^3 = 1$.

$$\text{Hence } (u^3 - 1) = 0 \text{ or } (u - 1)(u^2 + u + 1) = 0$$

$$\text{Hence } u^2 = -(u + 1), u^3 = 1, u^4 = u \text{ \& } u^5 = -(u + 1), u^6 = 1$$

$$\text{Let, } (x^2 + xy + y^2) = (x - yu)(x - yu^2) = \{(a - bu)(a - bu^2)\}^3.$$

$$\text{So, } (x - yu) = (a - bu)^3 = (-3a^2b - 3ab^2)u + a^3 - 3ab^2 - b^3$$

$$(x - yu^2) = (a - bu^2)^3$$

$$\text{Hence, we obtain } x = a^3 - 3ab^2 - b^3, y = 3a^2b + 3ab^2, z = x + y = a^3 - b^3 + 3a^2b.$$

3. Section (B)

Equation (4)

Equation (4), $w = 2, n = 3$, substituting above values below we get:

$$\text{Solution is, } x^4 + y^4 + z^4 = 2t^3$$

$$x = a^3 - 3ab^2 - b^3$$

$$y = 3a^2b + 3ab^2$$

$$z = a^3 - b^3 + 3a^2b$$

$$t = (a^2 + ab + b^2)^2$$

Similarly, we can obtain the parametric solutions of the case $w=1$ and $w=3$.

Equation (5)

We get, $w=1$, degree $n=3$

$$x^4 + y^4 + z^4 = w * t^3$$

Since $z = (x+y)$, we get $2(x^2 + xy + y^2)^2 = t^3$

$$\text{For } w = 1, \quad x^4 + y^4 + z^4 = t^3$$

$$x = -76a^6 + 1080a^4b^2 + 1520a^3b^3 + 60a^2b^4 - 432ab^5 - 76b^6 - 24a^5b$$

$$y = 4a^6 - 1140a^4b^2 - 80a^3b^3 + 1080a^2b^4 + 456ab^5 + 4b^6 - 432a^5b$$

$$z = -72a^6 - 60a^4b^2 + 1440a^3b^3 + 1140a^2b^4 + 24ab^5 - 72b^6 - 456b$$

$$t = 392(a^2 + ab + b^2)^4$$

$$\text{Equation (6): } w = 3, n = 3 \quad x^4 + y^4 + z^4 = 3t^3$$

Since $z = (x + y)$ we get, $2(x^2 + xy + y^2)^2 = 3t^3$

$$x = -228a^6 + 3240a^4b^2 + 4560a^3b^3 + 180a^2b^4 - 1296ab^5 - 228b^6 - 72a^5b$$

$$y = 12a^6 - 3420a^4b^2 - 240a^3b^3 + 3240a^2b^4 + 1368ab^5 + 12b^6 - 1296a^5b$$

$$z = -216a^6 - 180a^4b^2 + 4320a^3b^3 + 3420a^2b^4 + 72ab^5 - 216b^6 - 1368a^5b$$

$$t = 1176(a^2 + ab + b^2)^4$$

Numerical Examples

$$w = 1, \text{ degree } n = 3$$

$$(a, b)$$

$$(a, b) = (1, 0) \quad 76^4 + 4^4 + 72^4 = 392^3$$

$$\text{for } (2, 1) \quad 23108^4 + 27212^4 + 4104^4 = 941192^3$$

For, $w = 2, n = 3$

$$(a, b) \text{ for } (2, 1) \quad 1^4 + 18^4 + 19^4 = 2 * 49^3$$

$$w = 3, n=3$$

$$(a, b) = (1, 0), \quad 228^4 + 12^4 + 216^4 = 3 * 1176^3$$

$$(a, b) = \text{for } (2, 1) \quad 69324^4 + 81636^4 + 12312^4 = 3 * 2823576^3$$

4. Section (C)

$$x^4 + y^4 + z^4 = w * t^n$$

$$\text{degree } n=4$$

$$x^4 + y^4 + z^4 = w * t^4.$$

For degree 'n'

Consideration of $x^4 + y^4 + z^4 = wt^4 \pmod{16}$. In the case of $(n \bmod 16) = 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$, above equation has no solutions.

Consideration of $x^4 + y^4 + z^4 = wt^4 \pmod{25}$

In the case of $(n \bmod 25) = 4, 5, 9, 10, 14, 15, 19, 20, 24$ above equation has no solutions.

For $w=2$

There are infinitely many solutions.

Using identity $x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2$, we can get a parametric solution.

One of the solution for $x^2 + xy + y^2 = 1$ is $(x, y) = (1, 0)$, so we obtain a following parameter solution.

$$(-1+k^2)^4 + (k^2+2k)^4 + (1+2k)^4 = 2(1+k+k^2)^4$$

Numerical solutions for above for $w=2$ are given below:

k	x	y	z	t
2	8	3	5	7
3	15	8	7	13
5	35	24	11	31
6	48	35	13	43
7	21	16	5	19
8	80	63	17	73
9	99	80	19	91

2. For, $w=18$

There are infinitely many solutions.

In the same way as $w=2$, we can get a parameter solution.

$$x^2 + xy + y^2 = 3 \text{ is } (x, y) = (1, 1)$$

$$(-1+2k+2k^2)^4 + (-2+2k+k^2)^4 + (1+4k+k^2)^4 = 18(1+k+k^2)^4$$

3. For, $w=98$

There are infinitely many solutions.

In the same way as $n=2$, we can get a parameter solution.

$$x^2 + xy + y^2 = 7 \text{ is } (x, y) = (2, 1)$$

$$(-3-2k+2k^2)^4 + (-1+4k+3k^2)^4 + (2+6k+k^2)^4 = 98(1+k+k^2)^4$$

5. Section (D)

$w=1$ & degree $n=5$

Equation (7)

$$x^4 + y^4 + z^4 = t^5$$

$$x = 5580a^8b^2 + 35760a^7b^3 - 31248a^5b^5 + 36540a^6b^4 - 20880a^3b^7 - 62580a^4b^6 \\ + 5580a^2b^8 - 1740a^9b + 2980b^9a - 298a^{10} + 174b^{10}$$

$$y = -13410a^8b^2 - 20880a^7b^3 + 75096a^5b^5 + 26040a^6b^4 - 14880a^3b^7 + 36540a^4b^6 \\ - 13410a^2b^8 - 1240a^9b - 1740b^9a + 174a^{10} + 124b^{10}$$

$$z = -7830a^8b^2 + 14880a^7b^3 + 43848a^5b^5 + 62580a^6b^4 - 35760a^3b^7 - 26040a^4b^6 \\ - 7830a^2b^8 - 2980a^9b + 1240b^9a - 124a^{10} + 298b^{10}$$

$$w = 98(a^2 + ab + b^2)^4$$

Numerical solution is;

$$(a, b) = (1, 0) \quad 298^4 + 174^4 + 124^4 = 98^5$$

$$(a, b) = (2, 1) \quad 5008486^4 + 2084068^4 + 2924418^4 = 235298^5$$

6. Section (E)

$$x^4 + y^4 + z^4 = wt^n$$

$$w=2 \text{ \& degree } n=6$$

Equation (8)

$$X^4 + y^4 + z^4 = 2t^6$$

We have the known identity given below,

$$(a^3 + 3a^2b - b^3)^4 + (-a^3 + 3ab^2 + b^3)^4 + (3a^2b + 3ab^2)^4 = 2 * (a^2 + ab + b^2)^6 \quad (2)$$

We have known solution $(1, 19, 18)^4 = 2 * (7)^6$

Since $(a^2 + ab + b^2) = 7$ when $(a, b) = (2, 1)$

Hence we let $a = 2 + m$ and $b = (1 + km)$ in equation (1) above

We get $m = -(4k + 5)/(k^2 + k + 1)$

After substituting this in equation (2) we get the below mentioned parametrization:

$$x = m^6 - 108m^5 - 285m^4 - 20m^3 + 270m^2 + 114m + 1$$

$$y = 19m^6 + 6m^5 - 270m^4 - 380m^3 - 15m^2 + 108m + 19$$

$$z = 18m^6 + 114m^5 + 15m^4 - 360m^3 - 285m^2 - 6m + 1$$

$$t = 7(m^4 + 2m^3 + 3m^2 + 2m + 1)$$

Hence we get the new Identity:

$$\begin{aligned} & (m^6 - 108m^5 - 285m^4 - 20m^3 + 270m^2 + 114m + 1)^4 + \\ & (19m^6 + 6m^5 - 270m^4 - 380m^3 - 15m^2 + 108m + 19)^4 + \\ & (18m^6 + 114m^5 + 15m^4 - 360m^3 - 285m^2 - 6m + 1)^4 \\ & = 2[7(m^4 + 2m^3 + 3m^2 + 2m + 1)]^6 \end{aligned} \quad (B)$$

For $m=2$ in above eqn. we get: $(1026, 6803, 5777)^4 = 2 * (343)^6$

$$x^4 + y^4 + z^4 = wt^7$$

$w=1$ & degree $n=7$

7. Section (F)

Equation (9)

$$\text{For } w = 1, \quad x^4 + y^4 + z^4 = t^7$$

$$x = -272832a^{13}b - 272832ab^{13} + 13888b^{14} - 33376a^{14} - 100228128a^8b^6$$

$$-27803776a^9b^5 + 12148864a^{11}b^3 + 1263808a^{12}b^2 + 41705664a^6b^8$$

$$-66882816a^7b^7 + 19507488a^{10}b^4 + 66818752a^5b^9 - 5055232a^3b^{11} + 19507488a^4b^{10} - 3037216a^2b^{12}$$

$$y = -194432a^{13}b - 194432ab^{13} - 33376b^{14} + 19488a^{14} + 58522464a^8b^6$$

$$+66818752a^9b^5 - 7093632a^{11}b^3 - 3037216a^{12}b^2 - 100228128a^6b^8$$

$$-47663616a^7b^7 + 13901888a^{10}b^4 - 39014976a^5b^9 + 12148864a^3b^{11}$$

$$+13901888a^4b^{10} + 1773408a^2b^{12}$$

$$\begin{aligned}
 z = & -467264a^{13}b - 467264ab^{13} - 19488b^{14} - 13888a^{14} - 41705664a^8b^6 \\
 & + 39014976a^9b^5 + 5055232a^{11}b^3 - 1773408a^{12}b^2 - 58522464a^6b^8 \\
 & - 114546432a^7b^7 + 33409376a^{10}b^4 + 27803776a^5b^9 + 7093632a^3b^{11} \\
 & + 33409376a^4b^{10} - 1263808a^{2b^{12}} \\
 t = & 392 * (a^2 + ab + b^2)^4
 \end{aligned}$$

Numerical solution for above is:

$$(a,b) = (1,0) 33376^4 + 19488^4 + 13888^4 = 392^7$$

$$(a,b) = (2,1) \quad 3862651968^4 + 21749064736^4 + 25611716704^4 = (941192)^7$$

8. Section (G)

Degree n = 8

$$x^4 + y^4 + z^4 = wt^8$$

For w=2, n=8

Equation (10)

$$x^4 + y^4 + z^4 = 2t^8$$

$$\text{Let } X = a, \quad Y = b, \quad Z = a + b$$

We have the identity,

$$X^4 + Y^4 + Z^4 = 2 * (a^2 + ab + b^2)^2$$

(1)

$$\text{Take } a = m^4 - 6m^2n^2 - 4mn^3$$

$$b = -(-4m^3n + n^4 - 6m^2n^2)$$

then the right hand side of above equation (1) becomes after substitution of

$$(a,b) \quad a^2 + ab + b^2 = (m^2 + mn + n^2)^4.$$

We obtain a parametric solution as follows.

$$X = m^4 - 6m^2n^2 - 4mn^3$$

$$Y = 4m^3n - n^4 + 6m^2n^2$$

$$Z = m^4 - 4mn^3 + 4m^3n - n^4$$

$$t = m^2 + mn + n^2$$

For (m,n) = (2,1) we get numerical solution as (x,y,z,w) = (16,55,39,7)

9. Section (H)

Degree n=9

For w=2, n=9

Equation (11)

A parametric solution of

$$X^4 + Y^4 + Z^4 = 2t^9.$$

Let X=a, Y=b, Z=a+b

Hence

$$X^4 + Y^4 + Z^4 = 2(a^2 + ab + b^2)^2$$

$$\text{Hence } t^9 = (a^2 + ab + b^2)^2$$

$$\text{Set } a = m^9 - 36m^7n^2 + 126m^4n^5 - 9mn^8 - 84m^6n^3 + 84m^3n^6 - n^9$$

$$\text{and, } b = 9m^8n - 126m^5n^4 + 36m^2n^7 + 36m^7n^2 - 126m^4n^5 + 9mn^8,$$

$$\text{Then } a^2 + ab + b^2 = (m^2 + mn + n^2)^9.$$

We obtain a parametric solution as follows.

$$X = m^9 - 36m^7n^2 + 126m^4n^5 - 9mn^8 - 84m^6n^3 + 84m^3n^6 - n^9$$

$$Y = 9m^8n - 126m^5n^4 + 36m^2n^7 + 36m^7n^2 - 126m^4n^5 + 9mn^8$$

$$Z = m^9 - 84m^6n^3 + 84m^3n^6 - n^9 + 9m^8n - 126m^5n^4 + 36m^2n^7$$

$$W = (m^4 + 2m^3n + 3m^2n + 2n^3m + n^4)$$

Numerical solution for $(m, n) = (1, 1)$ we get: $(x, y, z)^4 = (81, 81, 162)^4 = 2(9)^9$
 $w=1, n=9$

Equation (12)

$$\text{A parametric solution of } X^4 + Y^4 + Z^4 = t^9$$

Let $X=p, Y=q, Z=p+q$

Hence

$$X^4 + Y^4 + Z^4 = 2(p^2 + pq + q^2)^2 = t^9$$

$$\begin{aligned} X = & -4163436a^{16}b^2 + 233152416a^{13}b^5 - 1190742696a^{10}b^8 + 23108a^{18} + 23108b^{18} \\ & -35163072a^5b^{13} - 83268720a^4b^{14} + 865994688a^7b^{11} - 18856128a^{15}b^3 + 428976912a^{12}b^6 \\ & -1123510960a^9b^9 + 179582832a^8b^{10} + 12558240a^{14}b^4 - 130605696a^{11}b^7 + 627912a^2b^{16} \\ & + 489816ab^{17} + 428976912a^6b^{12} - 18856128a^3b^{15} - 73872a^{17}b \\ Y = & 36(-2b^2 + 2ab + 3a^2)(b^2 + 6ab + 2a^2)(-3b^2 - 4ab + a^2) \\ & * (b^6 - 108b^5a - 285b^4a^2 - 20b^3a^3 + 270b^2a^4 + 114ba^5 + a^6) * \\ & (19b^6 + 6b^5a - 270b^4a^2 - 380b^3a^3 - 15b^2a^4 + 108ba^5 + 19a^6) \end{aligned}$$

Then we arrive at:

$$p^2 + pq + q^2 = (2)^4 * (7)^9 * (a^4 + 2a^3b + 3a^2b + 2ab^3 + b^4)^9$$

We obtain a parametric solution as follows:

$$\begin{aligned} Z = & -627912a^{16}b^2 + 35163072a^{13}b^5 - 179582832a^{10}b^8 + 27212a^{18} + 27212 * b^{18} \\ & + 197989344a^5b^{13} - 12558240a^4b^{14} + 130605696a^7b^{11} - 22204992a^{15}b^3 + 505163568a^{12}b^6 \\ & -1323047440a^9b^9 - 1011159864a^8b^{10} - 70710480a^{14}b^4 + 735388992a^{11}b^7 - 3535524a^2b^{16} \\ & + 73872ab^{17} + 505163568a^6b^{12} - 22204992a^3b^{15} + 415944a^{17}b. \\ t = & 2 * 7^2 * (a^4 + 2a^3b + 3a^2b + 2ab^3 + b^4)^2 \end{aligned}$$

Numerical solution is: For $(a, b) = (1, 1)$

$$[x, y, z]^4 = [-454834764, -80779032, -535613796]^4 = [7938]^9$$

10. Conclusions

This paper has analyzed the equation. $\mathbf{x}^4 + \mathbf{y}^4 + \mathbf{z}^4 = \mathbf{w} * \mathbf{t}^n$, for $n = (2, 3, 4, 5, 6, 7, 8, 9)$. In the near future attempt can be made by others to find solution for degree $n > 9$ and for different values of integer 'w'.

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