On the Laplacian Coefficients of Bicyclic Graphs

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Abstract In this paper, we investigate how the Laplacian coefficients changed after some graph transformations. So, I express some results about Laplacian coefficients ordering of graphs, focusing our attention to the bicyclic graphs. Finally, as an application of these results, we discuss the ordering of graphs based on their Laplacian like energy.

Keywords Laplacian Coefficients, Bicyclic Graph, Laplacian-like Energy

1. Introduction

Let $G=(V, E)$ be a simple undirected graph with n vertices, $|E|$ edges and $L(G)=D(G)-A(G)$ be its Laplacian matrix. The Laplacian polynomial $\Phi(G, \lambda)$ of G is the characteristic polynomial of its Laplacian matrix. That is

$$\Phi(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^n (-1)^k c_k(G) \lambda^{n-k}.$$

It is well-known that $c_0(G)=1$, $c_1(G)=2|E|$, $c_n(G)=0$, and $c_{n-1}(G)=n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G (see [1]).

Recently, the study on the Laplacian coefficient attracts much attention. Some works on Laplacian coefficients can be found in [2-8]. In this paper, we determine the largest coefficient among all the bicyclic graphs of order n .

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. Let B_n be the set of all connected bicyclic graphs of order n . Obviously B_n consists of two types of graphs: one type, denoted by B_n^∞ , are those graphs each of which is an ∞ -graph with trees attached; the other type, denoted by B_n^θ , are those graphs each of which is an θ -graph with trees attached. Then $B_n = B_n^\infty \cup B_n^\theta$.

2. Transformations and Lemmas

Let F be a spanning forest of G with components $T_i, i=1, 2, \dots, k$ having n_i vertices, and set $\gamma(F) = \prod_{i=1}^k n_i$. The Laplacian coefficients $c_k(G)$ of a graph G can be expressed in terms of subtree structures of G by the following result of Kelmans and Chelnokov [6].

Figure 1. $B(p, l, q)$ and $P(l, p, q)$

Lemma 2.1([6]) The Laplacian coefficient $c_{n-k}(G)$ of a graph G is given by $c_{n-k}(G) = \sum_{F \in F_k} \gamma(F)$, where F_k is the set of all spanning forests of G with exactly k components.

Let $N_G(v)$ denote the adjacent vertex set of a vertex v of a graph G and let $d(v)$ denote the degree of v in G . Let C_p denote the cycle on n vertices and let $d_{C_p}(u, v)$ denote the distance between u, v on the cycle C_p .

Definition 2.2([3]) Let w be a vertex of G with degree $d(w) \geq 3$ and $P = wv_1v_2 \cdots v_p$ and $Q = wu_1u_2 \cdots u_q$ are distinct pendant paths of lengths $p \geq 1$ and $q \geq 1$, respectively, attached at w . Let $G' = G - wu_1 + v_pu_1 \triangleq \pi(G, w, P, Q)$. We say that G' is a π -transformation of G .

Lemma 2.3([3]) Let $G' = \pi(G, w, P, Q)$. Then for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G')$, with equality if and only if $k \in \{0, 1, n-1, n\}$.

Definition 2.4([3]) Let w be a vertex in a cycle C of a unicyclic graph G . Assume that G has pendant path $P = wv_1v_2 \cdots v_p$ attached to w . If u is one of two neighbors of w in C , let $G' = \gamma(G, w, P)$ be the graph obtained from G by removing edge uw and adding edge uv_p . We say that G' is a γ -transform of G .

Lemma 2.5([3]) Let $G' = \gamma(G, w, P)$. For $0 \leq k \leq n$ holds $c_k(G) \leq c_k(G')$, with equality if and only if $k \in \{0, 1, n\}$.

Definition 2.6([9]) Let v be a vertex of a tree T of degree $m+1$. Suppose that $P_1, P_2, \dots, P_m$ are pendent paths incident with v , with lengths $n_i \geq 1, i = 1, 2, \dots, m$. Let w be the neighbor of v distinct from the starting vertices of paths $v_1, v_2, \dots, v_m$, respectively. We form a tree $T' = \delta(T, v)$ by removing the edges $vv_1, vv_2, \dots, vv_{m-1}$ from T and adding $m-1$ new edges $wv_1, wv_2, \dots, wv_{m-1}$ incident with w . We say that T' is a δ -transform of T .

Lemma 2.7([9]) Let T be an arbitrary tree, rooted at the center vertex. Let vertex v be on the deepest level of tree T among all branching vertices with degree at least three. Then for the δ -transformation tree $T' = \delta(T, v)$ and $0 \leq k \leq n$ holds $c_k(T) \geq c_k(T')$.

Definition 2.8([11]) Let G be a simple connected graph with n vertices, and let $P_s = w_1w_2 \cdots w_s$ be a path of G

not contained in cycles of lengths s and $s+1$ such that $s \geq 2, d_G(w_1) \geq 2, d_G(w_2) = \cdots = d_G(w_{s-1}) = 2, d_G(w_s) \geq 2$, where $d_G(u)$ is the degree of u in G . Let

$N_G(w_1) = \{w_2, u_1, u_2, \dots, u_p\}$ be the set of vertices adjacent to w_1 in G , and put

$$G_{P_s} = G - \{w_1u_1, w_1u_2, \dots, w_1u_p\} + \{w_su_1, w_su_2, \dots, w_su_p\}.$$

The graph G_{P_s} is called a χ_s -transform of the graph G .

Lemma 2.9([11]) Let G and G_{P_s} be the n -vertex graphs given in Definition 2.8. Then $c_k(G) \geq c_k(G_{P_s}), k = 0, 1, \dots, n$, with equality if and only if $k \in \{0, 1, n-1, n\}$ if P_s is not contained in any cycle of G or $k \in \{0, 1, n\}$ otherwise.

Lemma 2.10([10]) Let G be an acyclic graph on n vertices, then $c_k(G) = m(S(G), k)$ for $0 \leq k \leq n$.

If v is a pendant vertex of the graph G , adjacent to the vertex u , then the matching numbers of G conform to the recurrence relation

$$m(G, k) = m(G - v, k) + m(G - u - v, k - 1).$$

Definition 2.11 Let $C_p = u_1u_2 \dots u_{p-1}v_1u_1$ be a cycle of a connected unicyclic graph G . Assume that G has pendant paths $P = v_1w_1w_2 \cdots w_s$ and Q attached to v_l . Let $G_\gamma = G - u_1v_1 + u_1w_r \triangleq \gamma(G, v_1, w_r)$, where $1 \leq r \leq s$. We say that G_γ is a γ -transform of G (see Fig. 2).

Lemma 2.12 Let G and G_γ be the unicyclic graphs as shown in Fig. 2, l, r and s are positive integers. Suppose that $1 \leq r \leq s$. Then for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_\gamma)$, with equality if and only if $k \in \{0, 1, n\}$.

Proof.

$$c_0(G) = c_0(G_\gamma) = 1, c_1(G) = c_1(G_\gamma) = 2|E|, c_n(G) = c_n(G_\gamma) = 0, c_{n-1}(G) < c_{n-1}(G_\gamma).$$

Let F_k and F_k^γ be the sets of spanning forests of G and G_γ with exactly k components, respectively. We distinguish the following two cases.

Case 1. $l = 1$. According to the definition of the spanning forest, the all edges in the cycle C_p can't exist in any forest F at the same time, where F is the arbitrary forests in F_k . We distinguish the following two cases.

Case 1.1. If we remove the edge u_1v_1 in G and remove the edge u_1w_r in G_γ , we can get the trees T and T_γ . It is easy to see that $T = T_\gamma$.

Figure 2. G and G_γ Figure 3. G'_γ

Case 1.2. If we remove the edges $u_i u_{i+1}$ ($i = 1, 2, \dots, p-2$), and $u_{p-1} v_1$ in G and G_γ , we can get the trees T_i and T'_i ($i = 1, 2, \dots, p-1$), respectively. According to Definition 2.6, we know that $T_i = \delta(T'_i, w_r)$ ($i = 1, 2, \dots, p-1$), by Lemma 2.7, it is easy to see $c_k(T_i) \leq c_k(T'_i)$. At the same time, we can get the number of trees obtained in G_γ more than that of trees obtained in G . By Lemma 2.1, it is easy to see that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_\gamma)$, with equality if and only if $k \in \{0, 1, n\}$.

Case 2. $l > 1$. Let $G'_\gamma = G_\gamma - u_1 w_r + u_1 v_l$ (see Fig. 3), it is easy to see that the relationship between G'_γ and G_γ is similar to the relationship between G and G_γ in Case 1. According to the proof of Case 1, we have that for every $k = 0, 1, \dots, n$, $c_k(G'_\gamma) \leq c_k(G_\gamma)$, with equality if and only if $k \in \{0, 1, n\}$. By Definition 2.8, G can be

obtained by G'_γ through χ_s -transform. According to Lemma 2.9, we can get that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G'_\gamma)$, with equality if and only if $k \in \{0, 1, n\}$. From the above, we have that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_\gamma)$, with equality if and only if $k \in \{0, 1, n\}$. Then the conclusion of Lemma 2.12 holds.

3. Main Results

3.1. Laplacian Coefficients of Bicyclic Graphs

Definition 3.1 Let w be a vertex of the cycle C_p (or C_q , C_{l+p} , C_{p+q}) in a connected bicyclic graph G . Assume that G has a pendant path $P = w z_1 z_2 \dots z_r$ attached at w . If u is one of two neighbors adjacent to w in C_p (or C_q , C_{l+p} , C_{p+q}). Then $G_1 = \eta_1(G, w, P)$

is a graph obtained from G by removing edge uw and adding edge uz_r . We say that G_1 is a η_1 -transformation of G .

Theorem 3.2 Let G be a bicyclic graph with n vertices, $G_1 = \eta_1(G, w, P)$. Then for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_1)$, with equality if and only if $k \in \{0, 1, n\}$.

Proof. Without loss of generality, we assume that w is a vertex of the cycle C_p .

$$c_0(G) = c_0(G_1) = 1, c_1(G) = c_1(G_1) = 2|E|, c_n(G) = c_n(G_1) = 0, c_{n-1}(G) < c_{n-1}(G_1).$$

Let F_k and F_k^1 be the sets of spanning forests of G and G_1 with exactly k components, respectively. If we remove one of the edges in C_q , say uv , we can get the unicyclic graphs U and U' in G and G_1 . By Definition 2.4 and Lemma 2.5, we have $U' = \gamma(U, w, P)$ and for every $0 \leq k \leq n$ holds $c_k(U) \leq c_k(U')$, with equality if and only if $k \in \{0, 1, n\}$. At the same time, we can get the same number of unicyclic graphs obtained from G and G_1 . According to Lemma 2.1, it is easy to see that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_1)$, with equality if and only if $k \in \{0, 1, n\}$. Then the conclusion of Theorem 3.2 holds.

Let $B(p, l, q) = \{B(p, l, q) : p \geq 3, q \geq 3, l \geq 1\}$ and $P(l, p, q) = \{P(l, p, q) : l, p, q \geq 1 \text{ and at most one of them is } 1\}$. Denote $B(n) = B(p, l, q) \cup P(l, p, q)$.

Theorem 3.3 Let G be a bicyclic graph. For $k = 0, 1, \dots, n$, there exists some graph $H \in B(n)$ to satisfy the following inequality $c_k(G) \leq c_k(H)$.

Proof. Let T_i be the tree attached at u_i . We can use Definition 2.2 to the vertex of T_i which is the largest distance from the root u_i , as long as T_i is not a path. By using Lemma 2.3, we increase Laplacian coefficients of G .

Assume that $G \in B_n^\infty$ has pendant path $P = a_i z_1 z_2 \dots z_r$ attached at a_i , ($2 \leq i \leq l-1$). Let G_1 be the graph obtained from G by removing the edge $a_i a_{i+1}$ and adding the edge $a_{i+1} z_r$. Obviously, we can have

$$c_0(G) = c_0(H) = 1, c_1(G) = c_1(H) = 2|E|, c_n(G) = c_n(H) = 0, c_{n-1}(G) < c_{n-1}(H).$$

Let F_k and F_k^H be the sets of spanning forests of G and H with exactly k components, respectively. Similar

to the above prove, we can get the tree $T'(T')$ on $G(H)$. By Lemma 2.10 and (2.1), it is easy to see that for every $0 \leq k \leq n$ holds $c_k(T) \leq c_k(T')$, with equality if and only if $k \in \{0, 1, n-1, n\}$. We can get the same number of trees obtained from G and H . According to Lemma 2.1, it is easy to see that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(H)$, with equality if and only if $k \in \{0, 1, n\}$.

Finally, we can apply η_1 -transformation to the graph as long as it is not in $B(n)$. According to Theorem 3.2 and the above proof, we further increase its Laplacian coefficients. This means that the maximum Laplacian coefficients must be obtained from $B(n)$.

Definition 3.4 Let $G \in B(p, l, q)$ and $p \geq 3, q \geq 3, l \geq 1$. Suppose that $u \in N_{C_p}(a_1)$,

$v \in V(C_q)$ and $d_{C_q}(a_l, v) = \left\lfloor \frac{q}{2} \right\rfloor$. Then

$G_2 = \eta_2(G, u, v)$ is a graph obtained by removing edge ua_1 and adding edge uv . We say that G_2 is a η_2 -transformation of G (see Fig. 4, if $l = 1$, see Fig. 5).

Theorem 3.5 Let $G \in B(p, l, q)$ and $p \geq 3, q \geq 3, l \geq 1$. If $G_2 = \eta_2(G, u, v)$, then for every $k = 0, 1, \dots, n$, we have $c_k(G) \leq c_k(G_2)$.

Proof. Obviously, $c_0(G) = c_0(G_2) = 1, c_1(G) = c_1(G_2) = 2|E|, c_n(G) = c_n(G_2) = 0$. If q is an even number,

4. Main Results

4.1. Laplacian Coefficients of Bicyclic Graphs

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Theorem 3.2 Let G be a bicyclic graph with n vertices, $G_1 = \eta_1(G, w, P)$. Then for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_1)$, with equality if and only if $k \in \{0, 1, n\}$.

Proof. Without loss of generality, we assume that w is a vertex of the cycle C_p .

$$c_0(G) = c_0(G_1) = 1, c_1(G) = c_1(G_1) = 2 \mid E \mid, c_n(G) = c_n(G_1) = 0, c_{n-1}(G) < c_{n-1}(G_1).$$

Let F_k and F_k^1 be the sets of spanning forests of G and G_1 with exactly k components, respectively. If we remove one of the edges in C_q , say uv , we can get the unicyclic graphs U and U' in G and G_1 . By Definition 2.4 and Lemma 2.5, we have $U' = \gamma(U, w, P)$ and for every $0 \leq k \leq n$ holds $c_k(U) \leq c_k(U')$, with equality if and only if $k \in \{0, 1, n\}$. At the same time, we can get the same number of unicyclic graphs obtained from G and G_1 . According to Lemma 2.1, it is easy to see that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(G_1)$, with equality if and only if $k \in \{0, 1, n\}$. Then the conclusion of Theorem 3.2 holds.

Let $B(p, l, q) = \{B(p, l, q) : p \geq 3, q \geq 3, l \geq 1\}$ and $P(l, p, q) = \{P(l, p, q) : l, p, q \geq 1 \text{ and at most one of them is } 1\}$. Denote $B(n) = B(p, l, q) \cup P(l, p, q)$.

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Assume that $G \in B_n^\infty$ has pendant path $P = a_i z_1 z_2 \cdots z_r$ attached at a_i , ($2 \leq i \leq l-1$). Let G_1 be the graph obtained from G by removing the edge $a_i a_{i+1}$ and adding the edge $a_{i+1} z_r$. Obviously, we can have

$$c_0(G) = c_0(H) = 1, c_1(G) = c_1(H) = 2 \mid E \mid, c_n(G) = c_n(H) = 0, c_{n-1}(G) = c_{n-1}(H).$$

Let F_k and F_k^H be the sets of spanning forests of G and H with exactly k components, respectively. Similar to the above prove, we can get the tree T (T') on G (H). By Lemma 2.10 and (2.1), it is easy to see that for every $0 \leq k \leq n$ holds $c_k(T) \leq c_k(T')$, with equality if and only if $k \in \{0, 1, n-1, n\}$. We can get the same number of trees obtained from G and H . According to Lemma 2.1, it is easy to see that for every $k = 0, 1, \dots, n$, $c_k(G) \leq c_k(H)$, with equality if and only if $k \in \{0, 1, n\}$.

Finally, we can apply η_1 -transformation to the graph as long as it is not in $B(n)$. According to Theorem 3.2 and the above proof, we further increase its Laplacian coefficients. This means that the maximum Laplacian coefficients must be obtained from $B(n)$.

Definition 3.4 Let $G \in B(p, l, q)$ and $p \geq 3, q \geq 3, l \geq 1$. Suppose that $u \in N_{C_p}(a_1)$, $v \in V(C_q)$ and $d_{C_q}(a_l, v) = \lfloor \frac{q}{2} \rfloor$.

Then $G_2 = \eta_2(G, u, v)$ is a graph obtained by removing edge ua_1 and adding edge uv . We say that G_2 is a η_2 -transformation of G (see Fig. 4, if $l = 1$, see Fig. 5).

Theorem 3.5 Let $G \in B(p, l, q)$ and $p \geq 3, q \geq 3, l \geq 1$. If $G_2 = \eta_2(G, u, v)$, then for every $k = 0, 1, \dots, n$, we have $c_k(G) \leq c_k(G_2)$.

Proof. Obviously, $c_0(G) = c_0(G_2) = 1, c_1(G) = c_1(G_2) = 2 \mid E \mid, c_n(G) = c_n(G_2) = 0$. If q is an even number,

Figure 4. η_2 -transformation for $l > 1$

$$c_{n-1}(G) = n\tau(G) = npq < n[(p+l-1)q + \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor] =$$

$$= n\tau(G_2) = c_{n-1}(G_2)$$

and if q is an odd number,

 Figure 5. η_2 -transformation for $l=1$

$$c_{n-1}(G) = n\tau(G) = npq < n[(p+l-1)q + \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor] =$$

$$= n\tau(G_2) = c_{n-1}(G_2)$$

Let F_k and F_k^2 be the sets of spanning forests of G and G_2 with exactly k components, respectively. We distinguish the following two cases.

Case 1. $l=1$. According to the same reason, if we remove one of the edges in the cycle C_q , say xy , we can get the unicyclic graphs U in G and U_2 in G_2 .

Obviously, by Definition 2.11, we know that $U_2 = \gamma(U)$, and by Lemma 2.12, we can verify that $c_k(U_2) \geq c_k(U)$. By the above equation and Lemma 2.1, we have that for every $k=0,1,2,\dots,n$, $c_k(G_2) \geq c_k(G)$, with equality if and only if $k \in \{0,1,n-1\}$.

Case 2. $l>1$. The next proof is the similar to that of Case 1. Then the conclusion of Theorem 3.5 holds.

According to the Theorem 3.5, we know that the graph with maximum Laplacian coefficients must be obtained from $P(l,p,q)$.

Definition 3.6 Suppose that $G \in P(l,p,q)$. Without loss of generality, let $q \leq l \leq p$. Then $G_3 = \eta_3(G)$ be a graph obtained by removing the edge vu_{l-1} and adding the edge $v_{p-j}u_{l-1}$, where $j = \left\lfloor \frac{p-q}{2} \right\rfloor$, i.e. $q+j = p-j$ or $q+j = p-j-1$.

We say that G_3 is a η_3 -transformation of G (see Fig. 6).

Theorem 3.7 Let $G_3 = \eta_3(G)$. Then for $0 \leq k \leq n$, $c_k(G) \leq c_k(G_3)$.

Proof. Obviously,

$$c_0(G) = c_0(G_3) = 1, c_1(G) = c_1(G_3) =$$

$$= 2|E|, c_n(G) = c_n(G_3) = 0$$

$$c_{n-1}(G) = n\tau(G) = n(l(p+q) + q - 1) <$$

$$< n(l(p+q) + q + j - 1) = n\tau(G_3) = c_{n-1}(G_3)$$

Case 1. If we remove the edge $u_{l-1}v$ in G and $u_{l-1}v_{p-j}$ in G_3 , we can get the unicyclic graphs U_1 and U'_1 , respectively. It is easy to see that $U_1 = U'_1$, therefore,

$$c_k(U_1) = c_k(U'_1) \quad (1)$$

Case 2. If we remove the edge uu_1 in G and G_3 , we can get the unicyclic graphs U_2 and U'_2 , respectively. We have

$$c_k(U_2) = c_k(U'_2) \quad (2)$$

Case 3. If we remove the edge uv_1 in G and uv_1 in G_3 , we can get the unicyclic graphs U_3 and U'_3 . By using Definition 2.11, $U'_3 = \gamma(U_3, v, v_{p-j})$, and by Lemma 2.12,

$$c_k(U_3) \leq c_k(U'_3) \quad (3)$$

 Figure 6. η_3 -transformation

Case 4. If we remove the edge $v_{p-1}v_{p-1}$ in G and $v_{p-1}v_{p-1}$ in G_3 , we can get the unicyclic graphs U_4 and U'_4 . By Definition 2.11, $U'_4 = U_4 - u_{l-1}v + u_{l-1}v_{p-j}$
 $= \gamma(U_4, v, v_{p-j})$, and according to Lemma 2.12, we have

$$c_k(U_4) \leq c_k(U'_4) \quad (4)$$

Case 5. If we remove the edge $v_{p-i}v_{p-i-1}$, $(i=1, 2, \dots, j-1)$ in G and $v_{p-j+i-1}v_{p-j+i}$, $(i=1, 2, \dots, j-1)$ in G_3 , we can get the unicyclic graphs $U_5(i)$ and $U'_5(i)$. By Definition 2.11, there exists some vertex satisfied that $U'_5(i) = \gamma(U_5(i))$, and according to Lemma 2.12, we have

$$c_k(U_5(i)) \leq c_k(U'_5(i)) \quad (i=1, 2, \dots, j-1) \quad (5)$$

Case 6. If we remove the edge $v_i v_{i+1}$, $(i=1, 2, \dots, p-j-1)$ in G and $v_i v_{i+1}$, $(i=1, 2, \dots, p-j-1)$ in G_3 , we get the unicyclic graphs $U_6(i)$ and $U'_6(i)$.

$$c_k(U_6(i)) \leq c_k(U'_6(i)) \quad (i=1, 2, \dots, p-j-1) \quad (6)$$

Case 7. If we remove the edge $u_i u_{i+1}$, $(i=1, 2, \dots, l-2)$ in G and $u_i u_{i+1}$, $(i=1, 2, \dots, l-2)$ in G_3 , we get the unicyclic graphs $U_7(i)$ and $U'_7(i)$.

$$c_k(U_7(i)) \leq c_k(U'_7(i)) \quad (i=1, 2, \dots, l-2) \quad (7)$$

According to (1)-(7) and Lemma 2.1, we have that $c_k(G) \leq c_k(G_3)$ where $k=0, 1, \dots, n$. Then the conclusion of Theorem 3.7 holds.

Theorem 3.8 Let G be a connected bicyclic graph on n vertices and k be an integer, $0 \leq k \leq n$. Let $t = n+1(\text{mod } 3)$,

(1) If $t=0$, then

$$c_k(G) \leq c_k(P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil));$$

(2) If $t=1$, then

$$c_k(G) \leq c_k(P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor));$$

(3) If $t=2$, then

$$c_k(G) \leq c_k(P(\left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor)).$$

Proof. Applying to Theorems 3.2, 3.3, and 3.5, we only need to prove that when $G \in P(l, p, q)$, the result is

correct. Without loss of generality, let $q \leq l \leq p$.

When $t=0$, according to Theorem 3.7, we can apply η_3 -transformation as long as it is not the $P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil)$. Obviously, we increase its Laplacian coefficients. When $t=1$ and $t=2$, the proof is similar to $t=0$, we omit it.

4.2. Laplacian-like Energy of Bicyclic Graphs

Let G be a graph. The Laplacian-like energy (LEL) of graph G is defined as follows: $LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu(k)}$,

where $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ are the Laplacian eigenvalues of G . Stevanovic in [7] presented a connection between LEL and Laplacian coefficients.

Theorem 4.1. Let G and H be two graphs with n vertices. If $c_k(G) \leq c_k(H)$ for $k=1, 2, \dots, n-1$, then $LEL(G) \leq LEL(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $LEL(G) < LEL(H)$.

Corollary 4.2. Let G be a connected bicyclic graph with n vertices. Then let $t = n+1(\text{mod } 3)$,

- (1) If $t=0$ and $G \not\cong P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil)$, then $LEL(G) < LEL(P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil));$
- (2) If $t=1$ and $G \not\cong P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor)$, then $LEL(G) < LEL(P(\left\lceil \frac{n+1}{3} \right\rceil, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor));$
- (3) If $t=2$ and $G \not\cong P(\left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor)$, then $LEL(G) < LEL(P(\left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor, \left\lfloor \frac{n+1}{3} \right\rfloor)).$

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