Minkowski Sum of a Voronoi Parallelotope and a Segment

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Abstract By a Voronoi parallelotope \( P(a) \) we mean a parallelotope determined by linear in normal vectors \( p \) inequalities with a non-negative quadratic form \( a(p) \) as right hand side. For a positive form \( a \), it was studied by Voronoi in his famous memoir. For a set of vectors \( \mathcal{P} \), we call its dual a set of vectors \( \mathcal{P}^* \) such that \( (p, q) \in \{0, \pm 1\} \) for all \( p \in \mathcal{P} \) and \( q \in \mathcal{P}^* \). We prove that Minkowski sum of an irreducible Voronoi parallelotope \( P(a) \) and a segment \( z(u) \) is a Voronoi parallelotope if and only if \( u = w_e \), where \( w > 0 \) and \( e \) is a vector of the dual of the set of normal vectors of all facets of \( P(a) \). Then the segment \( z(u) \) is described by the same set of inequalities with \( w_e a(p) = w(e, p)^2 \), as right hand side and \( P(a) + z(u) = P(a + w_e e) \). A similar assertion is true for Minkowski sum of a reducible Voronoi parallelotope with a segment.

Keywords Parallelotope, Voronoi Parallelotope, Minkowski Sum, Dual Set

1 Introduction

1.1 Polytopes

Consider a \( d \)-dimensional polytope \( P(a) \) given by the following system of inequalities

\[
P(a, \mathcal{P}) = \{ x \in \mathbb{R}^d : (p, x) \leq a(p) \text{ for all } p \in \mathcal{P} \},
\]

where \( (p, x) \) is a scalar product of vectors \( p, x \in \mathbb{R}^d \). Here \( \mathcal{P} \subseteq \mathbb{R}^d \) is a set of vectors containing a set of normal vectors \( \mathcal{P}_n \) of all facets of \( P(a) \). If the set \( \mathcal{P} \) spans the whole space \( \mathbb{R}^d \), then \( P(a, \mathcal{P}) \) is a bounded polytope. The function \( a : \mathcal{P} \rightarrow \mathbb{R} \) is an arbitrary function.

A polytope \( P(a, \mathcal{P}) \) is called reducible if it is a direct sum

\[
P(a, \mathcal{P}) = P(a, \mathcal{P}_1) \oplus P(a, \mathcal{P}_2) \oplus \ldots \oplus P(a, \mathcal{P}_k)
\]

of polytopes \( P(a, \mathcal{P}_i) \) such that the sets \( \mathcal{P}_i \) span \( d_i \)-dimensional spaces \( \mathbb{R}(\mathcal{P}_i) \), \( 1 \leq i \leq k \), that intersect in a point. We denote by \( \mathcal{P}_{n, \mathcal{P}} \) a set of normal vectors of all facets of \( P(a, \mathcal{P}) \). A polytope is called irreducible if it is not reducible.

For a vector \( p \in \mathbb{R}^d \) and a number \( a(p) \in \mathbb{R} \), we define the following affine hyperplane

\[
H_p(a) = \{ x \in \mathbb{R}^d : (p, x) = a(p) \}
\]

Call a face \( F \) of the polytope \( P(a, \mathcal{P}) \) contact face and denote if \( F(p) \) if \( F = H_p(a) \cap P(a, \mathcal{P}) \), i.e., if the hyperplane \( H_p(a) \) supports \( P(a, \mathcal{P}) \) at the face \( F \). Hence, we call the corresponding vector \( p \in P(a) \) contact vector and denote by \( P(a) \) the set of all contact vectors.

A special case of a polytope is a segment \( z(e) \) of a line \( l(e) \) spanned by a vector \( e \in \mathbb{R}^d \), where

\[
z(e) = \{ x \in \mathbb{R}^d : x = \lambda e, \quad 1 \leq \lambda \leq 1 \}.
\]

The segment \( z(e) \) is symmetric with respect to origin \( 0 \). We show below that \( z(e) = P(f_e, \mathcal{P}) \) for some function \( f_e(p) \) if the set \( \mathcal{P} \) is good, i.e., it satisfies a special property (See Section “Segments”).

1.2 Parallelotopes

We call the above \( d \)-dimensional polytope \( P(a, \mathcal{P}) \) a Voronoi parallelotope if the following conditions hold:

(i) the function \( a(p) = (p, Ap) \) is a non-negative quadratic form;

(ii) the set \( \mathcal{P} \) is symmetric, and the set \( \mathcal{P}_n(a) \) of normal vectors generates integrally a \( d \)-dimensional lattice \( L \) containing \( \mathcal{P} \);

(iii) if \( \ker A \neq \emptyset \), then \( \dim(\ker A \cap \mathcal{P}) = \dim(\ker A) \).

Here symmetric means that if \( p \in \mathcal{P} \) then \( -p \in \mathcal{P} \), too, and \( \ker A \) denote kernel of the matrix \( A \).

Recall that a parallelotope is a polytope whose parallel translations form a tiling, i.e. they fill its space without interstices (gaps) and intersections by inner points. Voronoi proved in Voronoi [1] that if the above conditions (i) with \( a(p) \) positive and (ii) hold, then \( P(a) \) is a parallelotope. Besides, if \( a \) is positive, the parallelotope \( P(a, \mathcal{P}) \) is a Dirichlet-Voronoi cell of zero point 0 of the lattice \( 2AL \) with respect to the metric form \( a^*(q) = \frac{1}{2}(q, A^{-1}q), \) i.e.

\[
P(a, \mathcal{P}) = P(a^*) = \{ x \in \mathbb{R}^d : a^*(x) \leq a^*(x - q) \text{ for all } q \in 2AL \}.
\]

In fact, since \( a^*(x - q) = a^*(x) - 2\frac{1}{2}(x, A^{-1}q) + a^*(q) \) and \( q = 2Ap \in 2AL \), the above inequalities are equivalent to
\( x, A^{-1}2Ap \leq \frac{1}{2}(2Ap, A^{-1}2Ap) \), i.e. \( \langle p, x \rangle \leq a(p) \) for all \( p \in L \).

One can prove that, for any Voronoi parallelotope \( P(a, P) \), the set \( P \) can be enlarged up to a set \( P(a) \subset L \) of minimal (with respect to the form \( a \)) vectors of each parity class of \( L \). Moreover, the whole lattice \( L \) may be taken as the set \( P \).

Note that only for \( p \in P(a) \) the hyperplane \( H_k(p) \) supports the Voronoi parallelotope \( P(a, P) \) at a face \( F(p) \). For a parallelotope, each of its contact faces is an intersection of two parallelotopes. Dolbilin call in Dolbilin [2] faces with this property by standard faces.

For each \( p \in P(a) \), the vector \( 2Ap \) is called commensurate (with the parallelotope \( P(a) \)). The commensurate vector \( 2Ap \) connects the center of the parallelotope \( P(a) \) with the center of a parallelotope that is adjacent to \( P(a) \) by the contact face \( F(p) \). Commensurate vectors generate the lattice \( 2AL \).

In general, lattices \( L \) and \( 2AL \) are distinct. But, if \( a(p) = \frac{1}{2}p^2 \), then the lattices \( L \) and \( 2AL \) coincide. In this case the Voronoi parallelotope is called usually Voronoi polytope, or Dirichlet-Voronoi cell.

Recall that the dual of a lattice \( L \) is the lattice

\[
L^* = \{ q \in \mathbb{R}^d : \langle q, p \rangle \in \mathbb{Z} \text{ for all } p \in L \}.
\]

Since the set \( P(a) \) generates the lattice \( L \), we can change \( L \) by \( P(a) \) in the above definition of \( L^* \). Define the following important subset \( P^*_n(a) \subset L^* \) as follows

\[
P^*_n(a) = \{ e \in \mathbb{R}^d : \langle e, p \rangle \in \{0, \pm 1\} \text{ for all } p \in P_n(a) \}.
\]

We call this set dual of \( P_n(a) \).

For a vector \( e \), define the following quadratic form of rank \( 1 \)

\[
a_e(p) = \langle p, e \rangle^2,
\]

We prove below, that \( z(e) = P(ae, P) \) is a parallelotope for a set \( P \) that is “very good” for \( e \).

In this paper we prove the following

**Theorem 1.** Let \( P(a, P) \) be a Voronoi parallelotope, defined in (1), where \( P \supseteq P(a) \supseteq P_n(a) \). Let \( P(a) = \bigcup_{k=1} P(a, P_k) \), where sum is direct and \( P(a, P_k) \) is an irreducible parallelotope for each \( k \). Let \( u \in \mathbb{R}^d \) be a vector. Then the following assertions are equivalent:

(i) Minkowski sum \( P(a) + z(u) \) is a Voronoi parallelotope;

(ii) a projection of the vector \( u \) on the space \( \mathbb{R}(P_k) \) is parallel to some vector \( e_k \in P^*_n(a_k) \), i.e. \( u = \sum_{k=1}^{k} w_k e_k \).

For \( u = \sum_{k=1}^{k} w_k e_k \), we have

\[
P(a, P) + z(u) = P(a, P) + \sum_{k=1}^{k} w_k P(ae_k, P) = P(a + \sum_{k=1}^{k} w_k a_e, P).
\]

The implication (ii) \( \Rightarrow \) (i) was proved in Grishukhin [3].

Magazinov proved in Magazinov [4] that (in our terms) \( P(a) + bze(e) \) is a Voronoi parallelotope if this sum is a parallelotope. It seems to us that our proof is simpler.

Note that the quadratic form \( a_e(p) = \langle p, e \rangle^2 \) of rank 1 is not positive. If we consider \( e \) as a column vector, then \( A_e = ee^T \) is Gram matrix of the form \( a_e \). It transforms vectors \( p \) of the lattice \( L \) generated by \( P \) into \( Ap = e(p) \) that is projection of \( p \) onto the line \( e(p) \) spanned by \( e \). The kernel \( \ker A_e \) is the hyperplane \( H_e(0) \). By condition (iii), \( P(ae, P) \) is a parallelotope if the intersection \( P \cap H_e(0) \) generates a \( (d-1) \)-dimensional lattice. This lattice determines a partition of \( L \) into layers.

Hence \( A_e L \) is projection of \( L \) onto \( l(e) \). If \( L = \sum_{i=1}^{k} L_i \), then projection \( A_e L_i \) is a lattice if and only if, for all \( p \in L_i \), \( (e, p) = w_i k(p) \), where \( k(p) \in \mathbb{Z} \) is an integer, and \( w_i \in \mathbb{R} \) does not depend on \( p \). We see that this condition holds if condition (ii) of Theorem 5 holds.

### 2 Minkowski sum of polytopes

For a fixed set \( P \) of vectors, the polytopes \( P(a, P) \) defined in (1) have the following simple property

**Lemma 1.** For any functions \( a_1(p) \) and \( a_2(p) \), the following inclusion holds:

\[
P(a_1, P) + P(a_2, P) \subseteq P(a_1 + a_2, P).
\]

**Proof.** If \( p \in P \) is a contact vector of both the polytopes \( P(a_1, P) \) and \( P(a_2, P) \), then \( p \) is a contact vector of the sum \( P(a_1, P) + P(a_2, P) \) and of the polytope \( P(a_1 + a_2, P) \).

**Lemma 2.** Suppose that each facet of the sum \( P(a_1, P) + P(a_2, P) \) is determined by a contact vector \( p \in P \). Then

\[
P(a_1, P) + P(a_2, P) = P(a_1 + a_2, P)
\]

**Proof.** The condition of this Lemma means that each facet \( F \) of the sum \( P(a_1, P) + P(a_2, P) \) has the form \( F = F_1(p) + F_2(p) \), where \( F_k(p) \) is a contact face of \( P(a_k, P) \) for both \( K = 1, 2 \). Let \( P_n \subseteq P \) be a set of normal vectors of all facets of the sum \( P(a_1, P) + P(a_2, P) \). Then the following system of inequalities

\[
(p, x) \leq a_1(p) + a_2(p) \text{ for all } p \in P_n
\]

describes the sum \( P(a_1, P) + P(a_2, P) \). But this is equivalent to the equality

\[
P(a_1, P) + P(a_2, P) = P(a_1 + a_2, P_n).
\]

By Lemma 1, inequalities \( (p, x) \leq a_1(p) + a_2(p) \) are feasible for the sum \( P(a_1, P) + P(a_2, P) \) for all \( p \in P \). This implies that \( P(a_1 + a_2, P_n) = P(a_1 + a_2, P) \).

We show below that the equality \( P(a_1, P) + P(a_2, P) = P(a_1 + a_2, P) \) holds when \( P(a_2) \) is a segment \( z(e) \) and \( a_2 = e \), where the function \( f(e) \) is defined below in (6).

For \( i = 1, 2 \), let \( a_i \) be a non-negative quadratic form and \( P(a_i) \) be the corresponding Voronoi parallelotope described in (1). It is also a problem to find conditions when the sum \( P(a_1) + P(a_2) \) is a parallelotope, and, in particular, it is a Voronoi parallelotope.
It is shown in Ryshkov at al [5] that the equality $P(a_1, P) + P(a_2, P) = P(a_1 + a_2, P)$ holds if $a_1$ and $a_2$ belong to closure of an L-type domain. We show below that this equality holds if $a_2(p) = a_2(p)$, where the quadratic form $a_2(p) = (p, e)^2$ of rank 1 relates to the segment $(e)$, and then the sum $P(a_1, P) + P(a_2, P)$ is a parallelepiped.

3 Segments

Let $e, p \in \mathbb{R}^d$. Consider the affine hyperplane $H_p(f_e)$ defined in (2), where

$$f_e(p) = \frac{(p, e)^2}{(p, e)} \quad (6)$$

It is natural to suppose that $f_e(p) = 0$ if $(p, e) = 0$.

Lemma 3. For any vector $p \in \mathbb{R}^d$, the hyperplane $H_p(f_e)$ supports the segment $(e)$.

Proof. Note that end-vertices of the segment $(e)$ are points $\pm e$. If $(p, e) > 0$, then the end-vertex $e$ lies on $H_p(f_e)$. If $(p, e) < 0$, then the end-vertex $-e$ lies on $H_p(f_e)$. If $(p, e) = 0$, then the whole segment $(e)$ lines on $H_p(f_e)$.

We say that a set of vectors $P$ is good for a vector $e$, if the following conditions hold:

(i) scalar products $(p, e)$ have all the three signs +, − and 0, for all $p \in P$;

(ii) the polytope $P(0, P(e)), P(0, e)$ is a line spanned by the vector $e$.

Lemma 3 implies the following fact.

Lemma 4. Let the function $f_e(p)$ is defined in (6), $P(f_e, P)$ is $P(a, P)$ for $a = f_e, P(a, P)$ is given by (1) and $P$ is good for $e$. Then

$$z(e) = P(f_e, P) .$$

Proof. Note that $f_e(p) = 0$ for $p \in P(0, e)$. Since $P(0, P(e))$ is a line spanned by the vector $e$, the conditions of this Lemma and Lemma 3 imply that $z(e) = P(f_e, P)$.

4 Minkowski sum of a polytope with a segment

At first, we consider the Minkowski sum $P(a, P) + z(e)$ of an arbitrary polytope $P = P(a, P)$ defined in (1) with the segment $(e)$ defined in (3). A.Horváth call in Horváth [6] the sum $P + z(e)$ by an extension $P^e$ of $P$. Recall that we call a face $F$ contact and denote it by $F(p)$ if $F = P(a, P) \cap H_p(a)$. The vector $p$ is called contact vector of the face $F = F(p)$.

For a face $F$ of a polytope $P = P(a, P)$, let $l_F(e)$ be a parallel shift of the line $l(e)$ (spanned by $e$) such that $l_F(e) \cap F \neq \emptyset$. Call the face $F$ transversal to $e$ if $l_F(e) \cap F$ is a point. Otherwise, call the face $F$ parallel to $e$ and denote this fact as $F \parallel e$.

If $F = F(p)$ is a contact face of $P(a, P)$, then $F(p) \parallel e$ implies $l_F(e) \subseteq H_p(a)$.

We say that a face $F$ belongs to a shadow boundary of $P$ in direction $e$ if $l_F(e) \cap F = l_F(e) \cap P$. Denote by $F_e(P)$ a set of all faces of $P$ that belong to the shadow boundary of $P$ in direction of $e$. It is worth to note that faces of the shadow boundary are considered as open faces. Let $F \in F_e(P)$ and $F' \subset F$ be a subface of $F$. If $l_F(e) = l_F(e)$ and $l_F(e) \cap$ $F' \neq l_F(e) \cap F$, then $F' \not\subset F_e(P)$.

A face $F$ is transformed into a face $F + z(e)$ in the extension $P^e = P + z(e)$. Denote dimension of $F$ by $dim F$. Lemma 5 below helps to understand how faces of $P^e$ change with respect to faces of $P$. Assertions of Lemma 5 are obvious.

Lemma 5. Let $F$ be a face of a polytope $P$. Consider the sum $P^e = P + z(e)$. There are the following three possibilities for the sum $F + z(e)$:

(i) if $F$ is parallel to $e$, then $F + z(e) = F^e$ is an extension of $F$, and $dim(F + z(e)) = dim F$;

(ii) if $F$ is transversal to $e$ and $F \not\subset F_e(P)$, then $F + z(e) = F + e$ is a parallel shift of $F$ by the vector $e$;

(iii) if $F$ is transversal to $e$ and $F \subset F_e(P)$, then $F + z(e) = F + z(e)$ is direct sum of $F$ and $z(e)$, and $dim(F + z(e)) = dim F + 1$.

Now consider Minkowski sum of a polytope $P(a, P)$ given in (1) and a segment $(e)$. Suppose that the set $P$ is good for $e$. Then, by Lemma 4, $z(e) = P(f_e, P)$. If we want to prove that $P(a, P) + P(f_e, P) = P(a + f_e, P)$, then, according to Lemma 2, we have to prove that each facet of the sum $P(a, P) + P(f_e, P)$ is determined by a contact vector $p \in P$.

Lemma 6. Let $P = P(a, P)$ be a polytope, where the set $P$ is good for a vector $e$. Suppose that each $(d-2)$-face $F(x) \not\subset F_a(P)$, which is transversal to $e$, is a contact face $F = F(p)$ such that $(p, e) = 0$. Then each facet of the sum $P(a, P) + P(f_e, P)$ is supported by the hyperplane $H_p(a + f_e)$ for some $p \in P$.

Proof. By Lemma 4, we have $z(e) = P(f_e, P)$. According to Lemma 5, each facet $F_x$ of the sum $P^e(a) = P(a) + z(e)$ has one of the following three types

(i) extension $F_x(p) = F^e(p)$ of a facet $F(p)$ of $P$;

(ii) a parallel shift $F_x(p) = F(p) + e$ of a facet $F(p)$ of $P$;

(iii) direct sum $F_x(p) = F(p) \oplus z(e)$ of a $(d-2)$-face $F(p)$ of $P$ and the segment $(e)$. Consider the three cases (i), (ii) and (iii).

Case (i). In this case, $F(p) \parallel e$. Since $F(p)$ lies on the hyperplane $H_p(a)$, we have $e \parallel H_p(a)$. This implies that $(p, e) = 0$, and therefore $f_e(p) = 0$. Hence the facet $F_e(p) = (F(p))^{e}$ lies in the hyperplane $H_p(a) = H_p(a + f_e)$.

Case (ii). Let the facet $F(p)$ of $P$ be transversal to $e$ and $F(p) \not\subset F_a(P)$. Then $(p, e) \neq 0$. Hence the sum $F(p) + e$ is a shift of $F(p)$ obtained as follows. Let $x \in F(p)$. Then the point

$$x + \frac{(p, e)}{(p, e)^2}$$

belongs to $F(p) + e$. Here the multiple $\frac{(p, e)}{(p, e)}$ describes direction of the shift. Since $F(p)$ is a facet of $P = P(a, P)$, we have $F(x) = a(p)$ and therefore

$$x + \frac{(p, e)}{(p, e)^2} = a(p) + f_e(p) = (a + f_e)(p).$$

Since $x$ is an arbitrary point of $F(p)$, this implies that the facet $F_x(p) = F(p) + e$ of $P + z(e)$ is supported by $H_p(a + f_e)$.

Case (iii). Let $F(p)$ be a contact $(d-2)$-face of $P$ that is transversal to $e$ and $F(p) \not\subset F_a(P)$. The face $F(p)$ is transformed into the facet $F_x(p) = F(p) \oplus z(e)$ of $P + z(e)$. Since $(p, e) = 0$, the hyperplane $H_p(a) = H_p(a + f_e)$ supports the face $F_x(p) = F(p) \oplus z(e)$.

Now Lemma 6 and Lemma 2 imply the following
Theorem 2. Let $P = P(a, P_a)$ be a polytope and $z(e) = P(f_e, P)$ be a segment, where the set $P$ is good for $e$ and $P_a \subseteq P$. Let each $(d-2)$-face $F \in \mathcal{F}(P)$ that is transversal to $e$ be a contact face $F(p) = P(a, P_a)$ for $p \in P_a$ such that $(p,e) = 0$. Then

$$P + z(e) = P(a, P_a) + P(f_e, P) = P(a + f_e, P_a).$$

Proof. If $P_a$ is good for $e$, without loss of generality, we can set $P = P_a$. If $P_a$ is not good for $e$, we can enlarge the set $P_a$ up to the set $P$ such that right-hand sides $a(p)$ for new vectors $p \in P - P_a$ are chosen such that the halfspaces $\{x \in \mathbb{R}^d : (x, p) \leq a(p)\}$ contain $P(a, P_a)$, $(p,e) = 0$ and the hyperplane $H_2(a)$ supports $P$. Hence $P(a, P_a) = P(a, P)$. Now Lemmas 6 and 2 imply the following equality

$$P + z(e) = P(a, P) + P(f_e, P) = P(a + f_e, P).$$

Note that the hyperplanes $H_2(a) = H_2(a + f_e)$ for $p \in P - P_a$ support neither facets nor $(d-2)$-faces of $P = P(a, P_a)$ that are transversal to $e$. Hence $P(a + f_e, P) = P(a, P_a)$. So, we have the wanted equality.

Remark. It is worth to note that Theorem 2 demands only that $P_a$ is contained in the set $P$ that is good for $e$. But the set $P_a$ can be contained in $P$ strictly.

5 Minkowski sum of a Voronoi parallelootope with a segment

Now we consider Minkowski sum of a Voronoi parallelootope $P(a, P)$ and the segment $z(u)$ for some vector $u$. We suppose that $P$ contains the set of contact vectors of all contact faces of $P(a)$. At first, we suppose that $P(a, P)$ is irreducible.

Obviously, each segment is a parallelootope, and moreover it is a Voronoi parallelootope $P(a, P)$. In fact, we can choose lengths of vectors $p \in P$ such that $(p, u) \in [0, 1]$. If $P$ is good for $u$, the function $f_u(p)$ is transformed in the quadratic form $a_u(p)$ defined in (5), and $a_u(a, P)$ is a Voronoi parallelootope.

But when we consider a sum of a parallelootope $P(a, P)$ with a segment $z(u)$, we cannot change lengths of vectors $p \in P$. We can represent $z(u)$ in the form $P(f_u, P)$ only if the set $P$ is good for $u$. In the case of a parallelootope $P(a, P)$, we change the notion “good” by notion “very good” as follows.

We say that a symmetric set of vectors $P(e)$ is very good for a vector $e$, if the following conditions hold:

(i) scalar products $(p,e)$ takes all the three values $\pm 1$ and 0, for all $p \in P(e)$;

(ii) the set $P_0(e) = \{p \in P(e) : (p, e) = 0\}$ spans a hyperplane $H_2(0)$.

The conditions (ii) of notions “good” and “very good” are equivalent, since in the last case the set $P(e)$ is symmetric. Hence the notion “very good” is a strengthening of the notion “good”.

It is worth to note that (ii) above is equivalent to item (iii) of definition (see Introduction) of the Voronoi parallelootope $P(a, P)$.

Let $P_n(a) \subseteq P(a)$ be a set of normal vectors of facets of $P(a, P)$. If, for the vector $u \in \mathbb{R}^d$, the inclusions $(p, u) \in \{0, \pm 1\}$ hold for all $p \in P_n(a)$, then one can set $u = u_e$ for $e$ such that $(p,e) \in \{0, \pm 1\}$ for all $p \in P_n(a)$. Hence we will consider vectors $e \in P_n(a)$, where the dual $P_n(a)$ is defined in (4). Recall that the set $P_n(a)$ generates a $d$-dimensional lattice $L$.

Of course, there may be another vectors $p \in P$ with $(p,e) \in \{0, \pm 1\}$. Hence we introduce the following set

$$P_e(a) = \{p \in P(a) : (p, e) \in \{0, \pm 1\}\}.$$  \hspace{1cm} (7)

Obviously, $P_n(a) \subseteq P_e(a) \subseteq P(e)$, where, recall, $P(e)$ is a very good for $e$ set.

Let a parallelootope $P(a, P)$ be given by (1). Then, for $w > 0$, we have

$$wP(a, P) = P(wa, P).$$  \hspace{1cm} (8)

Lemma 7. Let $u = we$, where $w > 0$ and $e \in P_n(a)$. Let $p \in P(e)$. Then $z(u) = wz(e), f_e(p) = (p, e)^2 = a_e(p)$, and $z(u) = wP(a_e, P(e)) = P(wea, P(e))$ where the function $f_e(p)$ is defined in (6).

Proof. It is easy to see that $f_e(p) = a_e(p)$ for all $p \in P(e)$.

By Lemma 4, $z(e) = P(a_e, P(e))$. Using (8), we obtain last equalities.

Lemma 8. Let $P_n(a)$ be a set of normal vectors of a Voronoi parallelootope $P = P(a, P)$. Let $e \in P_n(a)$, and let $F \in \mathcal{F}(P)$ be a $(d-2)$-face of $P$ that is transversal to $e$. Then $F = P(f_e)$ is a contact face for a contact vector $p$ such that $(p, e) = 0$.

Proof. Suppose to the contrary that $F$ generates a 6-belt $B$. Let $p_1, p_2, p_3 \in P_n(a)$ be normal vectors of the 6-belt $B$. Let $F = F(p_1) \cap F(p_2)$. Note that $F(p_1), F(p_2) \notin F(P)$, since $F$ is transversal to $e$ and $F \notin F(P)$. Hence, for $i = 1, 2$, $(p_1, e) \neq 0$, and therefore $(p_i, e) \in \{0, \pm 1\}$. Without loss of generality, we can suppose that $(p_1, e) = 1$ and $(p_2, e) = -1$. Let $F(p_3) \neq F(p_2)$ be the other facet of the 6-belt $B$ that is adjacent to $F(p_1)$. Since $P = P(a, P)$ is a Voronoi parallelootope, the equality $p_3 = p_1 - p_2$ holds. This equality implies the equality $(p_3, e) = (p_1, e) - (p_2, e) = 2$ that contradicts to $(p_3, e) \in \{0, \pm 1\}$. Hence $F$ cannot generate a 6-belt. Therefore $F = F(p)$ is a contact face.

Obviously, $F(p) = F(p_1) \cap F(p_2)$, where $F(p_1), F(p_2)$ are facets of $P(a, P)$. Recall that $2A_{p_1}$ and $2A_{p_2}$ are commensurate vectors of facets $F(p_1)$ and $F(p_2)$. Hence the vector $2A_{(p_1 + p_2)}$ is a commensurate vector of the contact face $F(p)$. Therefore $p = p_1 + p_2$. Since $e \in P_n(a_1)(p_1, e), (p_2, e) \in \{0, \pm 1\}$. If $(p_1, e) = (p_2, e) = 0$, then $(p, e) = 0$. Otherwise, without loss of generality, we can suppose that $(p_1, e) = 1, (p_2, e) = -1$. This implies $(p, e) = 0$.

Note that Lemma 8 implies that $p \in P_e(a)$ for all contact vectors of $(d-2)$-faces $F(p) \in \mathcal{F}(P)$. Recall that $P_n(a) \subseteq P_e(a)$.

Theorem 3. For $P \supseteq P(a)$, let $P = P(a, P)$ be a Voronoi parallelootope defined in (1), where $P_n(a) \subseteq P_e(a) \subseteq P(e)$, and $P(e)$ is a very good for $e$ set. Then

$$P(a, P) + z(e) = P(a, P_e(a)) + P(a_e, P(e)) = P(a + ae, P).$$

Proof. Let $F(p) \in \mathcal{F}(P)$ be a $(d-2)$-face that is transversal to $e$. Then, by Lemma 8, $F(p) = F(p)$ is a contact face of $P = P(a, P)$ and $(p, e) = 0$. Since $a_e(p) = f_e(p)$ for all $p \in P(e)$, we can apply Theorem 2, where $P_0 = P_e(a)$ and $P = P(e)$. We obtain $P(a, P_e(a)) + z(e) = P(a + ae, P_e(a))$. Note that new normal vectors of the sum $P + z(e)$ are obtained from above contact vectors of $(d-2)$-faces $F \in \mathcal{F}(P)$. Hence the set
such that $\alpha(e) = (e, p)^2$. Now we will prove the following

**Theorem 4.** If the Voronoi parallelootope $P(a, \mathcal{P})$ is irreducible and the sum $P(a, \mathcal{P}) + z(u)$ is a parallelootope, then $u = w$ for some $e \in P_n^a(a)$ and $P(a, \mathcal{P}) + z(u) = P(a + \omega a, \mathcal{P})$.

A proof of Theorem 4 is based on two Lemmas below. Let

$$P_0(u) = \{p \in P_n(a) : (p, u) = 0\}$$
and

$$P_1(u) = \{p \in P_n(a) : (p, u) > 0\}.$$

Note that the sets $2AP_0(u)$ and $\pm 2AP_1(u)$ are sets of facet vectors of facets $F$ of $P = P(a, \mathcal{P})$ that belong and do not belong to shadow boundary $F_2(P)$, respectively. Let $L_0(u)$ be a lattice integrally generated by vectors of the set $P_0(u)$.

Obviously, the parallelootope $P^u = P(a, \mathcal{P}) + z(u)$ has a nonzero width in the direction of the line $l(u)$. Let $C(P^u)$ be the set of commensurate (facet) vectors of facets of $P^u$ that belong to shadow boundary of $P(a, \mathcal{P})$ in direction of $l(u)$. Facet vectors of $C(P^u)$ are either facet vectors of $P(a, \mathcal{P})$ of facets that belong to shadow boundary, or commensurate vectors of contact $(d - 2)$-faces of that shadow boundary. B. Venkov proved in Venkov [7] that vectors of $C(P^u)$ generate a $(d - 1)$-dimensional lattice $L_0(u)$. Obviously, $2AL_0(u) \subseteq L_0(u)$. Hence $\dim L_0(u) \leq d - 1$. Lemma 9 below shows that the equality $\dim L_0(u) = d - 1$ holds if $P(a, \mathcal{P})$ is irreducible.

**Lemma 9.** If the Voronoi parallelootope $P(a, \mathcal{P})$ is irreducible, then dimension of the lattice $L_0(u) = d - 1$.

**Proof.** It is proved in Grishukhin [3] and Dutour at al [9] that the set $P_0(u)$ intersects triples of normal vectors of all 6-belt of $P(a, \mathcal{P})$ if the sum $P(a, \mathcal{P}) + z(u)$ is a parallelootope. Consider the space $H_0 = \cap_{p \in P_0(u)} H_0(p)$. A.Magazinov called the space $H_0$ perfect. For each vector $v \in H_0$, the sum $P(a, \mathcal{P}) + z(v)$ is a parallelootope. It is obvious that $u \in H_0$. A.Magazinov proved in Magazinov [4], Theorem 4.4, that the parallelootope $P(a, \mathcal{P})$ is reducible if dimension of the space $H_0$ is greater than 1. This implies that, for the irreducible Voronoi parallelootope $P(a, \mathcal{P})$, the space $H_0$ is the line $l(u)$, and the set $P_0(u)$ generates a hyperplane $H(u)$ that is orthogonal to the vector $u$. The lattice $L_0(u) \subseteq L \cap H(u)$ has dimension $d - 1$, since $P_0(u)$ generates it.

Let $L_1(u)$ be a lattice integrally generated by differences $p - p'$ of vectors $p, p' \in P_1(u)$.

**Lemma 10.** $L_1(u) \subseteq L_0(u)$.

**Proof.** Suppose to the contrary, that there are $p, p' \in P_1(u)$ such that $p - p' \not\in L_0(u)$. Then the sublattice $L_1(u)$ generated by $p - p'$ and by $L_0(u)$ has dimension $d$.

Consider the corresponding facet (commensurate) vectors $2Ap$ and $2Ap'$. Since $p, p' \in P_1(u)$, the facets $P(p)$ and $P(p')$ do not belong to the shadow boundary. Hence in the sum $P(a, \mathcal{P}) + z(u)$, these vectors are transformed into vectors $2Ap + u$ and $2Ap' + u$. The difference of these vectors $2Ap - 2Ap'$ does not depend on the length of the vector $u$.

The lattice $2AL$ contains the layers $2AL_0(u)$, $2AL_0(u) + p$ for $p \in P_1(u)$ and $2AL_0(u) + (p - p')$. The spacing between layers $2AL_0(u)$ and $2AL_0(u) + 2A(p - p')$ does not depend on the length of $u$. Call the last layer stationary. But the spacing between layers $2AL_0(u)$ and $2AL_0(u) + p$ for $p \in P_1(u)$ do depend on the length of $u$. Chose a particular layer that moves and compare it with a layer that is stationary. Compare vertical heights of these layers by taking a ratio. The length of $u$ can be adjusted so that this ratio is irrational, in which case these layers do not generate a lattice. This is a contradiction.

**Remark.** Another proof of Lemma 10 can be found in Magazinov [4]. Referee asserts that Lemma 10 can be proved by using Theorem 6 of V'egh [8]. But Theorem 6 is not correct as it is stated.

Lemma 10 has the following important

**Corollary 1.** Let the Voronoi parallelootope be reducible. Then, for any $p \in P_1(u)$, the scalar product $(p, u) = w > 0$ does not depend on $p$. In other words, $u = w$ for some $e \in P_n^a(a)$, since $(p, u) = 0$ for $p \in P_0(u)$ and $P_0(u) = P_0(u) \cup (\pm P_1(u)$).

Corollary 1 implies Theorem 4.

Now Theorems 3 and 4 imply

**Theorem 5.** Let $P(a, \mathcal{P})$ be an irreducible Voronoi parallelootope, defined in (1). Let $u \in \mathbb{R}^d$ be a vector. Then the following assertions are equivalent:

(i) Minkowski sum $P(a, \mathcal{P}) + z(u)$ is a Voronoi parallelootope;
(ii) $u = w$ for some vector $e \in P_n^a(a)$, $z(u) = wP(ae, \mathcal{P})$.

Both the above conditions imply

$$P(a, \mathcal{P}) + z(u) = P(a, \mathcal{P}) + wP(ae, \mathcal{P}) = P(a + \omega a, \mathcal{P}).$$

Theorem 5 is a generalization of results for Voronoi polytopes of root lattices $\Lambda_6$, $E_8$ and $F_4$ obtained in papers Grishukhin [10], Dutour at all [9] and Grishukhin [11], respectively.

Theorem 5 has the following important Corollary.

**Corollary 2.** If $P_n^a(a) = \emptyset$, then $P(a, \mathcal{P}) + z(u)$ is not a parallelootope for any vector $u$.

Examples of $P(a, \mathcal{P})$ with $P_n^a(a) = \emptyset$ are Voronoi parallelotopes of lattices $E_6$ and $E_7$ that are dual of the root lattices $E_6$ and $E_7$ (see Dutour at al [9] and Grishukhin [11]).

Note that if the Voronoi parallelootope is reducible, then one can apply Theorem 5 to each component separately. This gives the following

**Theorem 6.** Let $P(a, \mathcal{P})$ be a Voronoi parallelootope, defined in (1). Let $P(a, \mathcal{P}) = \sum_{i=1}^k P(a, P_i)$, where sum is direct and $P(a, P_i)$ is an irreducible parallelootope for each $i$. Let $P_{in}(u)$ be a set of normal vectors of the parallelootope $P(a, P_i)$ for all $1 \leq i \leq k$. Let $u \in \mathbb{R}^d$ be a vector. Then the following assertions are equivalent:

(i) Minkowski sum $P(a, \mathcal{P}) + z(u)$ is a Voronoi parallelootope;
(ii) $u = w$ for some vector $e \in P_n^a(a)$, and $z(u) = \sum_{i=1}^k w_i P(ae_i, \mathcal{P})$.

Both the above conditions imply

$$P(a, \mathcal{P}) + z(u) = P(a, \mathcal{P}) + w_i P(ae_i, \mathcal{P}) = P(a + \sum_{i=1}^k w_i a e_i, \mathcal{P}).$$

Let $P_{00} = \{p \in P_n : (p, u)\}$ and let $L_0(u)$ be the lattice generated by vectors of $P_{00}$. Let $d_i$ be dimension of $P(a, P_i)$. Mathematics and Statistics 3(6): 151-156, 2015 155
By Lemma 9, \( \dim L_{i0}(u) = d_i - 1 \). The set \( \bigcup_i P_{i0}(u) \) generates the lattice \( \sum_i L_{i0}(u) \) of dimension \( \sum_{i=1}^k (d_i - 1) = d - k \). This shows that dimension of the perfect space is less than \( d - 1 \) if the parallelotope \( P(\alpha, P) \) is reducible.

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