Vague Additive Groups

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Abstract We introduce the concept of vague additive groups (VAG) by linking the vague set and group theory. Also we define the concept of vague normal additive group (VNAG) and vague ring (VR). The properties of VAG are investigated.

Keywords Vague Set, Groups, Normal Groups, Rings

1. Introduction

The concept of fuzzy set was introduced by Zadeh. Since then this idea has been applied to other algebraic structures such as groups, rings etc. With the development of fuzzy set, it is widely used in many fields meanwhile; the deficiency of fuzzy sets is also attract attention. Such as fuzzy set is single function, it cannot express the evidence of supporting and opposing. Based on this reason, the concept of vague set [3] introduced by Gau in 1993. Vague sets as a extension of fuzzy sets, the idea of vague sets is that the membership of every element can be divided into two aspects including supporting and opposing. The notion of fuzzy groups defined by Rosen field [7] is the first application fuzzy set theory in algebra. Ranjit Biswas [1] initiated the study of vague algebra by studying vague groups.

The object of this paper is to make a study of vague additive groups and its properties.

2. Preliminaries

In this section we collect important results which were already proved for our use in the next section.

Definition 2.1: [3] A vague set A in the universal of discourse X is characterized by two membership functions given by:

- (1) A truth membership function $t_A : X \rightarrow [0,1]$ and
- (2) A false membership function $f_A : X \rightarrow [0,1]$,

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the “evidence for x”, and $f_A(x)$ is a lower bound on the negation of x derived from the “evidence

against x” and $t_A(x) + f_A(x) \leq 1$. Thus the grade of membership of x in the vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of $[0,1]$. The vague set A is written as $A = \{ \langle x, [t_A(x), f_A(x)] \rangle / x \in X \}$.

Where the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in the vague set A and denoted by $V_A(x)$.

Definition 2.2: [3] A vague set A of a universe X with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in X$, is called the zero vague set of X.

Definition 2.3: [3] A vague set A of a universe X with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in X$, is called the unit vague set of X.

Notation [3]: Let $I [0, 1]$ denote the family of all closed subintervals of $[0, 1]$. If $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ are two elements of $I [0, 1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. We define the term imax to mean the maximum of two interval as $\text{imax} [I_1, I_2] = [\max \{a_1, a_2\}, \max \{b_1, b_2\}]$.

Similarly, we can define the term imin of any two intervals.

Throughout this paper X refers a additive group.

3. Vague Additive Groups

Definition 3.1:

Let $(X, +)$ be a group. A vague set A of X is called a vague additive (briefly VAG) group of X if the following conditions are satisfies: $\forall x, y \in X, V_A(x + y) \leq \text{imax} \{V_A(x), V_A(y)\}$,

i.e., (1). $t_A(x + y) \leq \text{imax} \{t_A(x), t_A(y)\}$, $1 - f_A(x + y) \leq \text{imax} \{1 - f_A(x), 1 - f_A(y)\}$ and

(2). $t_A(-x) \leq t_A(x)$ and $1 - f_A(-x) \leq 1 - f_A(x)$.

Example 3.2: Let $X = \{0, 1, \alpha, 1 + \alpha\}$ be a set.. Define addition modulo 2 on X as follows:

$+_2$	0	1	α	$1 + \alpha$
0	0	1	α	$1 + \alpha$
1	1	0	$1 + \alpha$	α
α	α	$1 + \alpha$	0	1
$1 + \alpha$	$1 + \alpha$	α	1	0

Clearly X is a group with respect to addition modulo 2

[6]. The vague set A of X defined by

$$A = \{ \langle 0, [0.5, 0.3] \rangle, \langle 1, [0.6, 0.2] \rangle, \langle \alpha, [0.7, 0.1] \rangle, \langle 1 + \alpha, [0.7, 0.1] \rangle \}$$

is a VAG of X.

The following propositions are obvious:

Proposition 3.3: If A is a VAG of a group X then for all $x \in X$, we have $V_A(-x) = V_A(x)$

Proposition 3.4: Zero vague set and unit vague set of a group X are trivial VAGs of X.

Theorem 3.5: Let A be a vague set of X. Then A is a VAG of X if and only if

$$V_A(x - y) \leq \max\{V_A(x), V_A(y)\} \text{ for } x, y \in X.$$

Proof: Let A be a VAG of X. Then we have $t_A(x - y) \leq \max\{t_A(x), t_A(-y)\}$

$$= \max\{t_A(x), t_A(y)\}$$

Similarly we can prove that $1 - f_A(x - y) \leq \max\{1 - f_A(x), 1 - f_A(y)\}$

It follows that $V_A(x - y) \leq \max\{V_A(x), V_A(y)\}$.

Conversely, suppose that A be a vague set of X of which e is the identity element.

Obviously $t_A(e) \leq t_A(y)$ then we have $t_A(e - y) \leq \max\{t_A(e), t_A(y)\}$

$$t_A(-y) \leq t_A(y).$$

$$\text{Also } t_A(x + y) \leq \max\{t_A(x), t_A(-y)\}$$

$$= \max\{t_A(x), t_A(y)\}$$

Similarly we can prove that $1 - f_A(-y) \leq 1 - f_A(y)$

$$1 - f_A(x + y) \leq \max\{1 - f_A(x), 1 - f_A(y)\}.$$

Hence A is a VAG of X.

Theorem 3.6: If A and B are two VAGs of X then $A \cap B$ is also VAG of X.

Proof: Let A and B are VAGs of X and $x, y \in X$

Then we have $t_{A \cap B}(x - y) = \min\{t_A(x - y), t_B(x - y)\}$

$$\leq \min\{\max\{t_A(x), t_A(y)\}, \max\{t_B(x), t_B(y)\}\}$$

$$= \max\{\min\{t_A(x), t_B(x)\}, \min\{t_A(y), t_B(y)\}\}$$

$$= \max\{t_{A \cap B}(x), t_{A \cap B}(y)\}$$

Similarly, we can prove that $1 - f_{A \cap B}(x - y) \leq \max\{1 - f_{A \cap B}(x), 1 - f_{A \cap B}(y)\}$.

Therefore $A \cap B$ is VAG of X.

Theorem 3.7: If A and B are VAG's of a additive group X then $A \cup B$ is also a VAG of X.

Proof: Let A and B are VAG's of a additive group X and $x, y \in X$.

Then we have $t_{A \cup B} = \max\{t_A(x - y), t_B(x - y)\}$

$$\leq \max\{\max\{t_A(x), t_A(y)\}, \max\{t_B(x), t_B(y)\}\}$$

$$= \max\{\max\{t_A(x), t_B(x)\}, \max\{t_A(y), t_B(y)\}\}$$

$$= \max\{t_{A \cup B}(x), t_{A \cup B}(y)\}.$$

Similarly, we can prove that $1 - f_{A \cup B}(x - y) \leq \max\{1 - f_{A \cup B}(x), 1 - f_{A \cup B}(y)\}$.

Hence $A \cup B$ is a VAG of X.

Example 3.8: Let $X = \{\bar{0}, \bar{1}, \bar{2}\}$ be a group with respect to addition modulo 3. Let A and B are two VAG of X defined as

$$A = \{\langle \bar{0}, [0.5, 0.2] \rangle, \langle \bar{1}, [0.6, 0.1] \rangle, \langle \bar{2}, [0.6, 0.1] \rangle\}$$

and $B = \{\langle \bar{0}, [0.6, 0.3] \rangle, \langle \bar{1}, [0.7, 0.2] \rangle, \langle \bar{2}, [0.7, 0.2] \rangle\}$. Then clearly $A \cup B$ is a VAG of X.

Theorem 3.9: Let X be a finite additive group and A be a VAG of X. Then the set H defined by

$$H = \{x \in X / V_A(x) = V_A(e)\}$$

is a subgroup of X. Proof: Let e is identity element of X and $H = \{x \in X / V_A(x) = V_A(e)\}$.

Let $x, y \in X$ and $x, y \in H$ then clearly, $V_A(x) = V_A(e)$, $V_A(y) = V_A(e)$

$$\text{We have } V_A(x - y) \leq \max\{V_A(x), V_A(y)\} = V_A(e)$$

Thus $x - y \in H$. Hence proved.

Definition 3.10: A be a VAG of a group X. Then A is called vague normal group (VNG) if for all $x, y \in X$, $V_A(x + y) = V_A(y + x)$.

Alternatively, we can say that a VAG is said to be a VNG of X if $V_A(x) = V_A(y + x + (-y))$, for all $x, y \in X$.

We use the notation $[x, y]$ to stand for the expression $(-x) + (-y) + x + y$.

Theorem 3.11: Let A be a VNG of a group X. Then for all $x, y \in X$, $V_A([x, y]) = V_A(e)$.

Proof: Since A be a VNG of a group X, we have $V_A(x + y) = V_A(y + x)$ for all $x, y \in X$.

$$\text{Then, } V_A(x) = V_A(y + x - y)$$

Replacing x by $(-y)$ and y by $(-x)$ we get $V_A(-y) = V_A(-x - y + x)$

$$V_A(-y + y) = V_A(-x - y + x + y)$$

$$V_A(e) = V_A(-x - y + x + y)$$

$$V_A(e) = V_A([x, y])$$

Hence proved.

Definition 3.12: Let f be a mapping from a set X into a set Y. Let B be a vague set in Y. Then the inverse image of B, $f^{-1}(B)$ is the vague set in X by $V_{f^{-1}(B)}(x) = V_B(f(x))$ for all $x \in X$.

Definition 3.13: Let f be a mapping from a set X into set Y. Let A be a vague set in X. Then the image of A, $f(A)$ is the vague set in Y by

$$V_{f(A)}(y) = \sup\{V_A(z) / z \in f^{-1}(y), \text{ if } f^{-1}(y) \neq \emptyset\}$$

$$= [0, 0] \text{ otherwise.}$$

Theorem 3.14: Let X and Y be two groups and f be a homomorphism from X into Y. Let B be a VAG of Y, then the inverse image $f^{-1}(B)$ of B is a VAG of X.

Proof: For all $x, y \in X$. $V_{f^{-1}(B)}(x - y) = V_B(f(x - y)) = V_B(f(x) - f(y)) \leq \max\{V_B(f(x)), V_B(f(y))\}$

$$= \max\{V_{f^{-1}(B)}(x), V_{f^{-1}(B)}(y)\}$$

Hence the set $f^{-1}(B)$ is a VAG of X.

Theorem 3.15: Let X and Y be two groups and f be a homomorphism from X into Y . Let A be a VAG of X , then the image $f(A)$ of A is a VAG of Y .

Proof: Let $u, v \in Y$ such that $V_{f(A)}(u) = \text{isup} \{ V_A(x) / x \in f^{-1}(u) \}$ and

$$\begin{aligned} V_{f(A)}(v) &= \text{isup} \{ V_A(y) / y \in f^{-1}(v) \} \\ V_{f(A)}(u - v) &= \text{isup} \{ V_A(r) / r \in f^{-1}(u - v) \} \\ &\leq \text{imax} \{ V_A(x), V_A(y) \} \\ &= \text{imax} \{ V_{f(A)}(u), V_{f(A)}(v) \} \end{aligned}$$

Thus the image $f(A)$ of A is a VAG of Y .

Definition 3.16: Let X be a ring and R be a vague set of X . Then R is a vague ring of X if the following conditions are satisfied:

- (1). $V_R(x + y) \leq \text{imax} \{ V_R(x), V_R(y) \}$,
- (2). $V_R(x) \leq V_R(-x)$,
- (3). $V_R(xy) \geq \text{imin} \{ V_R(x), V_R(y) \}$ for all $x, y \in X$.

Example 3.17: Let $X = \{ \bar{0}, \bar{1}, \bar{2} \}$ be the ring under residue classes of addition and multiplication modulo 3.

A vague set R of X defined by $R = \{ \langle \bar{0}, [0.5, 0.2] \rangle$,

$\langle \bar{1}, [0.6, 0.1] \rangle, \langle \bar{2}, [0.6, 0.1] \rangle \}$ is a vague ring of X .

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