Universal Journal of Computational Mathematics 3(2): 19-21, 2015 http://www.hrpub.org
DOI: 10.13189/ujcmj.2015.030201

Fixed Point Theorems in S-Metric Spaces

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Abstract In this paper, we prove two fixed point theorems in S-metric spaces. Our results extend and improve some known results.

Keywords S-metric Space, Fixed Point

2010 AMS Subject Classification: 54H25, 54E99

1. Introduction and Preliminaries

In 2006, Z. Mustafa and B.I. Sims [7] introduced the concept of G-metric space which is a generalization of metric space, and proved some fixed point theorems in G-metric spaces. After that, many authors have proved some fixed point theorems in these G- metric spaces (see, e.g. [3], [8], [12]). In 1992, B. C. Dhage [4] introduced the notion of D-metric space and proved some fixed point theorems. In 2007, S. Sedghi, N.Shobe and H. Zhou [11] introduced D^*-metric spaces which is a modification of D-metric spaces of [4] and proved some fixed point theorems in D^*-metric spaces. Later on many authors have studied the fixed point theorems in generalized metric spaces (see, e.g. [1, 5, 6]). In 2012, S. Sedghi et al. [10] introduced the notion of S-metric space which is a generalization of a G-metric space of [4] and D^*- metric space of [11] and obtained some fixed point theorems on S-metric spaces. Recently, S.Sedghi and N.V.Dung [9] have proved some generalized fixed point theorems in S-metric spaces which are generalization of [10]. In this paper, we proved two fixed point results in S-metric spaces. Our results extend and improve the results of [9].

Definition 1.1.[2] Let X be a nonempty set. A metric on X is a function d: $X^2 \to [0,\infty)$ if there exists a real number $b \geq 1$ such that the following conditions holds for all x, y, z ∈ X.
(i) d(x, y) = 0 if and only if x = y.
(ii) d(x, y) = d(y, x).
(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.
The pair (X, d) is called a B-metric space.

Definition 1.2. [10] Let X be a nonempty set. An S-metric on X is a function S: $X^3 \to [0, \infty)$ that satisfies the following conditions holds for all x, y, z, a ∈ X.
(i) S(x, y, z) = 0 if and only if x = y = z.
(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called an S-metric space.

Definition 1.3. [10] Let (X, S) be an S-metric space. For r>0 and x∈X, we define the open ball $B_S(x, r)$ and the closed ball

$B_S[x, r]$ with centre x and radius r as follows

$B_S(x, r) = \{y\in X: S(y, y, x)<r\}$,

$B_S[x, r] = \{y\in X: S(y, y, x)\leq r\}$.

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

Definition 1.4. [10] Let (X, S) be an S-metric space.

(i)A sequence $\{x_n\}\subset X$ converges to $x \in X$ if $S(x_n, x_n, x)\to 0$ as $n\to\infty$. That is, for each $\varepsilon> 0$, there exists $n_0 \in \mathbb{N}$ such that

for all $n \geq n_0$ we have $S(x_n, x_n, x)<\varepsilon$. We write for $x_n \to x$.

(ii)A sequence $\{x_n\}\subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m)\to 0$ as $n, m \to\infty$. That is, for each $\varepsilon> 0$, there exists $n_0 \in \mathbb{N}$

such that for all $n,m \geq n_0$ we have $S(x_n, x_n, x_m)<\varepsilon$.

(iii) The S-metric space (X, S)is complete if every Cauchy sequence is a convergent.

Lemma 1.5.[10] In an S-metric space, we have S(x, x, y) = S(y, y, x) for all x, y∈X.

Lemma 1.6.[10] Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n)\to S(x, x, y)$.

2. Main Results

In this section, we prove two fixed point theorems in S-metric spaces.

S. Sedghi, N.V. Dung [9] introduced an implicit relation to investigate some fixed point theorems on S-metric spaces.

Let $\mathcal{M}$ be the family of all continuous functions of five variables $M:\mathbb{R}_+^5 \to\mathbb{R}_+$. For some k∈[0,1), we consider the following conditions.

(C_1) For all x, y, z $\in \mathbb{R}_+$, if $y \le M(x, x, 0, z, y)$ with $z \le 2x + y$, then $y \le kx$.

(C_2) For all $y \in \mathbb{R}_+$, if $y \le M(y, 0, y, y, 0)$, then $y = 0$.

(C_3) If $x_i \le y_i + z_i$ for all $x_i, y_i, z_i \in \mathbb{R}_+$, $i \le 5$, then $M(x_1, x_2, x_3, x_4, x_5) \le M(y_1, y_2, y_3, y_4, y_5) + M(z_1, z_2, z_3, z_4, z_5)$.

Moreover, for all $y \in X$, $M(0, 0, 0, y, 2y) \le ky$.

Note: The coefficient k in conditions (C_1) and (C_3) may be different, for example, $k = k_1$ and k_3 respectively. But we assume that they are equal by putting $k = \max\{k_1, k_3\}$.

The following Theorem is proved in [9] (Theorem 2.6 [9]).

Theorem 2.1. (Theorem 2.6 [9]). Let T be a self-map on a complete S-metric space (X, S) and

$$S(Tx, Tx, Ty) \le M(S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y))$$

for all x, y,z $\in$ X and some $M \in \mathcal{M}$. Then we have

(i) If M satisfies the condition (C_1), then T has a fixed point. Moreover, for any $x_0 \in X$ and the fixed point x, we have

$$S(Tx_n, Tx_n, x) \le 2k^n / 1\text{-}k \; S(x_0, x_0, Tx_0).$$

(ii) If M satisfies the condition (C_2) and T has a fixed point, then the fixed point is unique.

(iii) If M satisfies the condition (C_3) and T has a fixed point, then T is continuous at x.

Theorem 2.2. Let T be a self-map on a complete S-metric space (X, S) and

$S(Tx, Tx, Ty) \le h \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y)\}$

for some $h \in [0,1)$ and for all x, y $\in$ X. Then T has a unique fixed point in X. Moreover, if $h \in [0,1/2)$, then T is continuous at the fixed point.

Proof. The following ascertain is by using the above Theorem 2.1 with$M(x, y, z, s, t) = h \max\{x, y, t\}$ for some $h \in [0,1)$ and for all x, y, z, s, t$\in \mathbb{R}_+$. Indeed, M is continuous. First, we have,

$$M(x, x, 0, z, y) = h \max\{x, x, y\}.$$

So, if $y \le M(x, x, 0, z, y)$ with $z \le 2x + y$, then

$$y \le hx \text{ or } y \le hy.$$

Therefore, $y \le hx$.

Therefore, T satisfies the condition (C_1).

If $y \le M(y, 0, y, y, 0) = h \max\{y, 0, 0\} = hy$, then $y = 0$. Since, $h<1/2$.

Therefore, T satisfies the condition (C_2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1, x_2, x_3, x_4, x_5) &= h \max\{x_1, x_2, x_5\} \\ &= h \max\{y_1 + z_1, y_2 + z_2, y_5 + z_5\} \\ &\le h \max\{y_1, y_2, y_5\} + h \max\{z_1, z_2, z_5\} \\ &= M(y_1, y_2, y_3, y_4, y_5) + M(z_1, z_2, z_3, z_4, z_5). \end{aligned}$$

Moreover, $h \in [0,1/2)$, then $2h<1$ and

$$M(0, 0, 0, y, 2y) = h \max\{0, 2y\} = 2hy, \text{ where } 2h<1.$$

Therefore, T satisfies the condition (C_3).

Theorem 2.3. Let T be a self-map on a complete S-metric space (X, S) and

$S(Tx, Tx, Ty) \le h \max\{S(x, x, y), S(Tx, Tx, y), S(Ty, Ty, x)\}$

for some $h \in [0,1/3)$ and for all x, y $\in$ X. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The following ascertain is by using the above Theorem 2.1 with$M(x, y, z, s, t) = h \max\{x, z, s\}$ for some $h \in [0,1/3)$ and for all x,y,z, s, t$\in \mathbb{R}_+$. Indeed, M is continuous. First, we have,

$$M(x, x, 0, z, y) = h \max\{x, 0, z\}.$$

So, if $y \le M(x, x, 0, z, y)$ with $z \le 2x + y$, then

$y \le hx$ or$y \le 2hx+hy$. Then $y \le kx$ with

$$k = \max\{h, 2h/1\text{-}h\}<1.$$

Therefore, T satisfies the condition (C_1).

Next, if $y \le M(y, 0, y, y, 0) = h.y$, then $y = 0$ since, $h<1/3$.

Therefore, T satisfies the condition (C_2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1, x_2, x_3, x_4, x_5) &= h \max\{x_1, x_3, x_4\} \\ &= h \max\{y_1 + z_1, y_3 + z_3, y_4 + z_4\} \\ &\le h \max\{y_1, y_3, y_4\} + h \max\{z_1, z_3, z_4\} \\ &= M(y_1, y_2, y_3, y_4, y_5) + M(z_1, z_2, z_3, z_4, z_5). \end{aligned}$$

Moreover, $h \in [0,1/2)$, then $2h<1$ and

$$M(0, 0, 0, y, 2y) = h \max\{0, 0, y\} = hy, \text{ where } h<1.$$

Therefore, T satisfies the condition (C_3).

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