Fixed Point Theorems in S-Metric Spaces

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Abstract In this paper, we prove two fixed point theorems in S-metric spaces. Our results extend and improve some known results.

Keywords S-metric Space, Fixed Point

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1. Introduction and Preliminaries

In 2006, Z. Mustafa and B.I. Sims [7] introduced the concept of G-metric space which is a generalization of metric space, and proved some fixed point theorems in G-metric spaces. After that, many authors have proved some fixed point theorems in these G-metric spaces (see, e.g. [3], [8], [12]). In 1992, B. C. Dhage [4] introduced the notion of D-metric space and proved some fixed point theorems. In 2007, S. Sedghi, N.Shobe and H. Zhou [11] introduced D*-metric spaces which is a modification of D-metric spaces of [4] and proved some fixed point theorems in D*-metric spaces. Later on many authors have studied the fixed point theorems in generalized metric spaces (see, e.g. [1, 5, 6]). In 2012, S. Sedghi et al. [10] introduced the notion of S-metric space which is a generalization of a G-metric space of [4] and D*-metric space of [11] and obtained some fixed point theorems on S-metric spaces. Recently, S.Sedghi and N.V.Dung [9] have proved some generalized fixed point theorems in S-metric spaces which are generalization of [10]. In this paper, we proved two fixed point results in S-metric spaces. Our results extend and improve the results of [9].

Definition 1.1.[2] Let X be a nonempty set. A metric on X is a function d: X² → [0, ∞) if there exists a real number b ≥ 1 such that the following conditions holds for all x, y, z ∈ X.

(i) d(x, y) = 0 if and only if x = y.
(ii) d(x, y, z) ≤ b[d(x, y) + d(y, z)].

The pair (X, d) is called a B-metric space.

Definition 1.2. [10] Let X be a nonempty set. A function S: X³ → [0, ∞) satisfies the following conditions for all x, y, z, a ∈ X.

(i) S(x, y, z) = 0 if and only if x = y = z.
(ii) S(x, y, z) ≤ S(x, x, a) + S(y, y, a) + S(z, z, a).

The pair (X, S) is called an S-metric space.

Definition 1.3. [10] Let (X, S) be an S-metric space. For r > 0 and x ∈ X, we define the open ball Br(x, r) and the closed ball Br[x, r] with centre x and radius r as follows

Br(x, r) = \{y ∈ X: S(y, y, x) < r\},
Br[x, r] = \{y ∈ X: S(y, y, x) ≤ r\}.

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

Definition 1.4. [10] Let (X, S) be an S-metric space.

(i) A sequence \{xn\} ⊂ X converges to x ∈ X if S(xn, xn, x) → 0 as n → ∞. That is, for each ε > 0, there exists n₀ ∈ ℕ such that for all n ≥ n₀ we have S(xn, xn, x) < ε. We write for xn → x.

(ii) A sequence \{xn\} ⊂ X is a Cauchy sequence if S(xn, xn, xm) → 0 as n, m → ∞. That is, for each ε > 0, there exists n₀ ∈ ℕ such that for all n, m ≥ n₀ we have S(xn, xn, xm) < ε.

(iii) The S-metric space (X, S) is complete if every Cauchy sequence is convergent.

Lemma 1.5.[10] In an S-metric space, we have S(x, x, y) = S(y, y, x) for all x, y ∈ X.

2. Main Results

In this section, we prove two fixed point theorems in S-metric spaces.

Let \mathcal{M} be the family of all continuous functions of five variables M: \mathbb{R}_{+}^5 → \mathbb{R}_{+}. For some k ∈ (0, 1), we consider the following conditions.

\begin{align*}
\text{(i)} \quad S(x, y, z) &= 0 \text{ if and only if } x = y = z. \\
\text{(ii)} \quad S(x, y, z) &\leq S(x, x, a) + S(y, y, a) + S(z, z, a) \\
\text{The pair } (X, S) \text{ is called an S-metric space.}
\end{align*}
Theorem 2.1 with $M(x, y, z, s, t) = h \max \{x, z, s\}$ for some $h$

Proof. First, we have,

\[ M(x, x, 0, z, y) = h \max \{x, 0, z\} \]

Moreover, for any $x_0 \in X$ and the fixed point $x$, we have

\[ S(Tx_0, Tx_0, x) \leq 2k^5 \]

for all $x_0 \in X$ and some $M \in \mathcal{M}$. Then we have

(i) If $M$ satisfies the condition $(C_1)$, then $T$ has a fixed point.

Moreover, for any $x_0 \in X$ and the fixed point $x$, we have

\[ S(Tx_0, Tx_0, x) \leq 2k^5 / 1-k S(x_0, x_0, Tx_0) \]

(ii) If $M$ satisfies the condition $(C_2)$ and $T$ has a fixed point, then the fixed point is unique.

(iii) If $M$ satisfies the condition $(C_3)$ and $T$ has a fixed point, then $T$ is continuous at $x$.

Theorem 2.2. Let $T$ be a self-map on a complete S-metric space $(X, S)$ and

\[ S(Tx, Tx, Ty) \leq h \max \{S(x, x, y), S(Tx, Tx, y), S(Ty, Ty, x)\} \]

for some $h \in [0,1/2)$ and for all $x, y \in X$. Then $T$ has a unique fixed point in $X$. Moreover, $T$ is continuous at the fixed point.

Proof. The following ascertain is by using the above Theorem 2.1 with $M(x, y, z, s, t) = h \max \{x, z, s\}$ for some $h \in [0,1/2)$ and for all $x, y, z, s, t \in \mathbb{R}$. Indeed, $M$ is continuous.

First, we have,

\[ M(x, x, 0, z, y) = h \max \{x, 0, z\} \]

So, if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then

\[ y \leq hx \text{ or } y \leq hy \]

Therefore, $T$ satisfies the condition $(C_1)$.

Next, if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then

\[ y \leq hx \text{ or } y \leq hy \]

Therefore, $T$ satisfies the condition $(C_2)$.

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

\[ M(x_1, x_2, x_3, x_4, x_5) = h \max \{x_1, x_2, x_3\} \]

Therefore, $T$ satisfies the condition $(C_3)$.

REFERENCES


