On Bilateral Generating Functions of Konhauser Biorthogonal Polynomials

K.P. Samanta^{1,*}, B. Samanta²

¹Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, India

²Department of Mathematics, Shibpur D.B. Institution(College), India

Copyright © 2015 Horizon Research Publishing All rights reserved.

Abstract In this article, we have obtained some novel results on bilateral generating functions of the polynomials, $Y_{n+r}^{\alpha-nk}(x; k)$, a modified form of Konhauser biorthogonal polynomials, $Y_n^\alpha(x; k)$ by group-theoretic method. As special cases, we obtain the corresponding results on Laguerre polynomials, $L_n^{(\alpha)}(x)$. Some applications of our results are also discussed.

Keywords Laguerre Polynomials, Biorthogonal Polynomials, Generating Functions

AMS-2010 Subject Classification Code: 33C 45, 33C 47

1. Introduction

In 1965[1], Konhauser extended the notion of a particular pair of biorthogonal polynomial sets as introduced by Spencer and Fano [3] and established general properties of biorthogonal sets. In [2], Konhauser also introduced two sets of polynomials $\{Y_n^\alpha(x; k)\}$ and $\{Z_n^\alpha(x; k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, $\alpha > -1$, k is a positive integer. These polynomials satisfy the following condition:

$$\int_0^\infty x^\alpha \exp(-x) Y_i^\alpha(x; k) Z_j^\alpha(x; k) dx \begin{cases} = 0, & i \neq j, \\ \neq 0, & i = j; \end{cases} \quad i, j = 0, 1, 2, \dots$$

For $k = 1$, these polynomials reduce to the generalized Laguerre polynomials, $L_n^\alpha(x)$. For previous works on these polynomials one can see the works [7-14]. In the present paper we are interested only on $Y_n^\alpha(x; k)$. In [7], Carlitz gave an explicit representation for the polynomials $Y_n^\alpha(x; k)$ in the following form:

$$Y_n^\alpha(x; k) = \frac{1}{n!} \sum_{i=0}^n \frac{x^i}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \binom{j+\alpha+1}{k}_n,$$

where $(a)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, \quad a \neq 0 \\ a(a+1) \dots (a+n-1), & \forall n \in \{1, 2, 3, \dots\}. \end{cases}$$

The aim at presenting this paper is to obtain some novel theorems on bilateral generating relations for the polynomials, $Y_{n+r}^{\alpha-nk}(x; k)$, a modified form of Konhauser biorthogonal polynomials, $Y_n^\alpha(x; k)$ by the group-theoretic method. As special cases, we obtain the corresponding results on Laguerre polynomials, $L_n^{(\alpha)}(x)$. The main results of our investigation are stated in the form of the following theorems:

Theorem 1. If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^\infty a_n Y_{n+r}^{\alpha-nk}(x; k) w^n \quad (1)$$

then

$$(1+kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, \frac{vt}{1+kt}\right) = \sum_{n=0}^\infty Y_{n+r}^{\alpha-nk}(x; k) \sigma_n(v) t^n, \quad (2)$$

where

$$\sigma_n(v) = \sum_{p=0}^n a_p k^{n-p} \binom{n+r}{p+r} v^p.$$

Theorem 2. If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^\infty a_n Y_{n+r}^\alpha(x; k) w^n \quad (3)$$

then

$$(1+kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, vt\right) = \sum_{n=0}^\infty \sigma_n(x, v) t^n, \quad (4)$$

where

$$\sigma_n(x, v) = \sum_{p=0}^n a_p k^{n-p} \binom{n+r}{p+r} Y_{n+r}^{\alpha-nk+pk}(x; k) v^p.$$

Finally, we like to point it out that some applications of our theorems are also given in the paper.

2. Operator and Extended Form of the Group

At first, we seek a linear partial differential operator R of the form:

$$R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_0,$$

where each $A_i (i = 0, 1, 2, 3)$ is a function of x and y which is independent of n such that

$$R(Y_{n+r}^{\alpha-nk}(x; k)y^n) = \Omega_n Y_{n+r+1}^{\alpha-nk-k}(x; k)y^{n+1}, \quad (5)$$

where Ω_n is a function of n and is independent of x and y .

Using (5) and with the help of the differential recurrence relation:

$$\begin{aligned} x \frac{d}{dx} [Y_{n+r}^{\alpha-nk}(x; k)] \\ = k(n+r+1)Y_{n+r+1}^{\alpha-nk-k}(x; k) + \\ + (x+k+nk-\alpha-1)Y_{n+r}^{\alpha-nk}(x; k) \end{aligned} \quad (6)$$

we easily obtain the following linear partial differential operator:

$$R = x y \frac{\partial}{\partial x} - k y^2 \frac{\partial}{\partial y} - (x+k-\alpha-1)y$$

such that

$$R(Y_{n+r}^{\alpha-nk}(x; k)y^n) = k(n+r+1)Y_{n+r+1}^{\alpha-nk-k}(x; k)y^{n+1}. \quad (7)$$

The extended form of the group generated by R is given by

$$\begin{aligned} & e^{wR} f(x, y) \\ & = (1+kwy)^{\frac{1+\alpha-k}{k}} \exp \left\{ x - x(1+kwy)^{\frac{1}{k}} \right\} \\ & \quad \times f \left(x(1+kwy)^{\frac{1}{k}}, \frac{y}{1+kwy} \right), \end{aligned} \quad (8)$$

where $f(x, y)$ is an arbitrary function and w is an arbitrary constant.

3. Derivation of Generating Function

Now writing $f(x, y) = Y_{n+r}^{\alpha-nk}(x; k)y^n$ in (8), we get

$$\begin{aligned} & e^{wR} (Y_{n+r}^{\alpha-nk}(x; k)y^n) \\ & = (1+kwy)^{\frac{1+\alpha-k}{k}} \exp \left\{ x - x(1+kwy)^{\frac{1}{k}} \right\} \\ & \quad \times Y_{n+r}^{\alpha-nk} \left(x(1+kwy)^{\frac{1}{k}}; k \right) y^n. \end{aligned} \quad (9)$$

Again, using (7), we obtain

$$\begin{aligned} & e^{wR} (Y_{n+r}^{\alpha-nk}(x; k)y^n) = \\ & \sum_{m=0}^{\infty} k^m \frac{w^m}{m!} (n+r+1)_m Y_{n+r+m}^{\alpha-nk-mk}(x; k) y^{n+m}. \end{aligned} \quad (10)$$

Equating (9) and (10) and then substituting $wy = t$, we get

$$\begin{aligned} & (1+kt)^{\frac{1+\alpha-nk-k}{k}} \exp \left\{ x - x(1+kt)^{\frac{1}{k}} \right\} Y_{n+r}^{\alpha-nk} \left(x(1+kt)^{\frac{1}{k}}; k \right) \\ & = \sum_{m=0}^{\infty} k^m \binom{n+r+m}{m} Y_{n+r+m}^{\alpha-nk-mk}(x; k) t^m, \end{aligned} \quad (11)$$

which does not seem to have appeared in the earlier works.

Corollary 1: Replacing α by $\alpha + nk$ in (11), we get

$$\begin{aligned} & (1+kt)^{\frac{\alpha+1-k}{k}} \exp \left\{ x - x(1+kt)^{\frac{1}{k}} \right\} Y_{n+r}^{\alpha-nk} \left(x(1+kt)^{\frac{1}{k}}; k \right) \\ & = \sum_{m=0}^{\infty} k^m \binom{n+r+m}{m} Y_{n+r+m}^{\alpha-mk}(x; k) t^m. \end{aligned} \quad (12)$$

Corollary 2: Again, putting $n = 0$ in (12), we get

$$\begin{aligned} & (1+kt)^{\frac{\alpha+1-k}{k}} \exp \left\{ x - x(1+kt)^{\frac{1}{k}} \right\} Y_r^{\alpha} \left(x(1+kt)^{\frac{1}{k}}; k \right) \\ & = \sum_{m=0}^{\infty} k^m \binom{r+m}{m} Y_{r+m}^{\alpha-mk}(x; k) t^m, \end{aligned} \quad (13)$$

which is found derived in [11, 17].

Corollary 3: Putting $r = 0$ in (13), we get

$$\begin{aligned} & (1+kt)^{\frac{\alpha+1-k}{k}} \exp \left\{ x - x(1+kt)^{\frac{1}{k}} \right\} = \\ & \sum_{m=0}^{\infty} k^m Y_m^{\alpha-mk}(x; k) t^m, \end{aligned} \quad (14)$$

which is found derived in [11, 13, 17] by different methods.

Special case 1: If we put $k = 1$, then $Y_n^{\alpha}(x; k)$ reduces to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. Thus Putting $k = 1$ in (11) - (14), we get the following generating relation on Laguerre polynomials:

$$\begin{aligned} & (1+t)^{\alpha-n} \exp(-xt) L_{n+r}^{\alpha-n}(x(1+t)) \\ & = \sum_{m=0}^{\infty} \binom{n+r+m}{m} L_{n+r+m}^{\alpha-n-m}(x) t^m. \end{aligned} \quad (15)$$

$$\begin{aligned} & (1+t)^{\alpha} \exp(-xt) L_{n+r}^{\alpha}(x(1+t)) \\ & = \sum_{m=0}^{\infty} \binom{n+r+m}{m} L_{n+r+m}^{\alpha-m}(x) t^m. \end{aligned} \quad (16)$$

$$\begin{aligned} & (1+t)^{\alpha} \exp(-xt) L_r^{\alpha}(x(1+t)) \\ & = \sum_{m=0}^{\infty} \binom{r+m}{m} L_{r+m}^{\alpha-m}(x) t^m, \end{aligned} \quad (17)$$

which is found derived in [15, 16, 18, 19, 20, 22].

$$(1+t)^{\alpha} \exp(-xt) = \sum_{m=0}^{\infty} L_m^{\alpha-m}(x) t^m, \quad (18)$$

which is found derived in [23].

Special case 2: Using the relation

$L_n^{\alpha-n}(x) = \frac{(-x)^n}{n!} C_n(\alpha; x)$, [6, pp.68-69] and from the above generating relations we get the following generating relations on Charlie rpolynomials [4, p. 226]:

$$\begin{aligned} & \left(1 - \frac{y}{x} \right)^{\alpha+r} \exp(y) C_{n+r}(\alpha+r; x-y) \\ & = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_{n+r+m}(\alpha+r; x) \end{aligned} \quad (19)$$

$$\begin{aligned} & \left(1 - \frac{y}{x}\right)^\alpha \exp(y) C_n(\alpha; x - y) \\ &= \sum_{m=0}^{\infty} \frac{y^m}{m!} C_{n+m}(\alpha; x), \end{aligned} \quad (20)$$

which is found derived in [6, pp. 70] and equivalent to Doetsch's formula given in [5, pp. 88(26)].

$$\left(1 - \frac{y}{x}\right)^\alpha \exp(y) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_m(\alpha; x). \quad (21)$$

Now we proceed to prove the Theorem 1.

4. Proof of Theorem 1

Let us consider the generating relation of the form:

$$G(x, w) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{\alpha-nk}(x; k) w^n. \quad (22)$$

Replacing w by wvy in the both sides of (22), we have

$$G(x, wvy) = \sum_{n=0}^{\infty} a_n (Y_{n+r}^{\alpha-nk}(x; k) y^n) (wv)^n. \quad (23)$$

Operating e^{wR} on both sides of (23), we get

$$\begin{aligned} & e^{wR} (G(x, wvy)) \\ &= e^{wR} \left(\sum_{n=0}^{\infty} a_n (Y_{n+r}^{\alpha-nk}(x; k) y^n) (wv)^n \right). \end{aligned} \quad (24)$$

Now the left member of (24), with the help of (8), reduces to

$$\begin{aligned} & (1 + kwy)^{\frac{1+\alpha-k}{k}} \exp\left(x - x(1 + kwy)^{\frac{1}{k}}\right) G\left(x(1 + kwy)^{\frac{1}{k}}, \frac{wvy}{1 + kwy}\right). \end{aligned} \quad (25)$$

The right member of (24), with the help of (7), becomes

$$= \sum_{n=0}^{\infty} \sum_{p=0}^n a_{n-p} \frac{w^p}{p!} k^p (n + r - p + 1)_p Y_{n+r}^{\alpha-nk}(x; k) y^n (wv)^{n-p}.$$

Now equating (25) and (26) and then substituting $wy = t$, we get

$$\begin{aligned} & (1 + kt)^{\frac{1+\alpha-k}{k}} \exp\left(x - x(1 + kt)^{\frac{1}{k}}\right) G\left(x(1 + kt)^{\frac{1}{k}}, \frac{vt}{1 + kt}\right) \\ &= \sum_{n=0}^{\infty} Y_{n+r}^{\alpha-nk}(x; k) \sigma_n(v) t^n, \end{aligned} \quad (27)$$

where

$$\sigma_n(v) = \sum_{p=0}^n a_p k^{n-p} \binom{n+r}{p+r} v^p.$$

This completes the proof the theorem.

Special case 3: Now putting $k = 1$ in our Theorem 1 we get the following result on generalised Laguerre polynomials:

Theorem 3. If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) w^n \quad (28)$$

then

$$\begin{aligned} & (1 + t)^\alpha \exp(-xt) G\left(x(1 + t), \frac{vt}{1 + t}\right) = \\ & \sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x) \sigma_n(v) t^n, \end{aligned} \quad (29)$$

where

$$\sigma_n(v) = \sum_{p=0}^n a_p \binom{n+r}{p+r} v^p.$$

which is found derived in [18].

Corollary 4: Putting $r = 0$ in Theorem 3, we get the theorem found derived in [18, 19].

5. Proof of Theorem 2

$$\begin{aligned} R.H.S. &= \sum_{n=0}^{\infty} \sigma_n(x, v) t^n \\ &= \sum_{p=0}^{\infty} a_p (vt)^p \sum_{n=0}^{\infty} \binom{p+r+n}{n} Y_{p+r+n}^{\alpha-nk}(x; k) (kt)^n \\ &= \sum_{p=0}^{\infty} a_p (vt)^p (1 + kt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1 + kt)^{\frac{1}{k}}\right\} Y_{p+r}^{\alpha}(x(1 + kt)^{\frac{1}{k}}; k) \quad [\text{using (12)}] \\ &= (1 + kt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1 + kt)^{\frac{1}{k}}\right\} \sum_{p=0}^{\infty} a_p Y_{p+r}^{\alpha}\left(x(1 + kt)^{\frac{1}{k}}; k\right) (vt)^p \\ &= (1 + kt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1 + kt)^{\frac{1}{k}}\right\} G\left(x(1 + kt)^{\frac{1}{k}}, vt\right), \quad [\text{using (3)}] \\ &= L.H.S., \end{aligned}$$

which is Theorem 2.

Corollary 5: If we put $r = 0$ in Theorem 2, then we get the following Theorem:

$$(1+t)^\alpha \exp(-xt) G(x(1+t), vt) = \sum_{n=0}^{\infty} \sigma_n(x, v) t^n, \quad (33)$$

Theorem 4. If

$$G(x, w) = \sum_{n=0}^{\infty} a_n Y_n^\alpha(x; k) w^n \quad (30)$$

where

$$\sigma_n(x, v) = \sum_{p=0}^n a_p \binom{n+r}{p+r} L_{n+r}^{(\alpha-n+p)}(x) v^p,$$

then

$$(1+kt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, vt\right) = \sum_{n=0}^{\infty} \sigma_n(x, v) t^n, \quad (31)$$

which is found derived in [18].

Theorem 6. If

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) w^n \quad (34)$$

then

$$(1+t)^\alpha \exp(-xt) G(x(1+t), vt) = \sum_{n=0}^{\infty} \sigma_n(x, v) t^n, \quad (35)$$

where

$$\sigma_n(x, v) = \sum_{p=0}^n a_p k^{n-p} \binom{n}{p} Y_n^{\alpha-nk+pk}(x; k) v^p.$$

Special case 4. Now putting $k = 1$ in our Theorem 2 and Theorem 4, we get the following results on generalised Laguerre polynomials:

Theorem 5. If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha)}(x) w^n \quad (32)$$

where

$$\sigma_n(x, v) = \sum_{p=0}^n a_p \binom{n}{p} L_n^{(\alpha-n+p)}(x) v^p,$$

which is found derived in [18, 19, 21, 22].

then

Below we have given some applications of our results.

6. Applications

A1. As an application of our Theorem 1, we consider the following generating relation [11, 12]:

$$\sum_{n=0}^{\infty} \binom{n+m}{n} Y_{n+m}^{\alpha-nk}(x; k) t^n = (1+t)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+t)^{\frac{1}{k}}\right\} Y_m^\alpha\left(x(1+t)^{\frac{1}{k}}; k\right). \quad (36)$$

If in our theorem, we take $a_n = \binom{n+m}{n}$,

then

$$G(x, w) = (1+w)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+w)^{\frac{1}{k}}\right\} Y_m^\alpha\left(x(1+w)^{\frac{1}{k}}; k\right).$$

Therefore by the application of our Theorem 1, we get the following generalization of the result (36):

$$(1+kt+vt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+kt+vt)^{\frac{1}{k}}\right\} Y_m^\alpha\left(x(1+kt+vt)^{\frac{1}{k}}; k\right) = \sum_{n=0}^{\infty} Y_{n+m}^{\alpha-nk}(x; k) \sigma_n(v) t^n, \quad (37)$$

where

$$\sigma_n(v) = \sum_{p=0}^n a_p k^{n-p} \binom{n+r}{p+r} v^p.$$

A2. As an application of Theorem 2, we consider the following generating relation [9,11,12]:

$$\sum_{n=0}^{\infty} \binom{n+r}{n} Y_{n+r}^\alpha(x; k) t^n = (1-t)^{\frac{-1-\alpha-rk}{k}} \exp\left\{x - x(1-t)^{-\frac{1}{k}}\right\} \times$$

$$\times Y_r^\alpha \left(x(1-t)^{-\frac{1}{k}}; k \right). \quad (38)$$

If in our theorem, we take $a_n = \binom{n+r}{n}$,
then

$$G(x, w) = (1-w)^{\frac{-1-\alpha-rk}{k}} \exp \left\{ x - x(1-w)^{-\frac{1}{k}} \right\} Y_r^\alpha \left(x(1-w)^{-\frac{1}{k}}; k \right).$$

Therefore by the application of our Theorem 2 we get the following generalization of (38):

$$\begin{aligned} & (1+kt)^{\frac{1+\alpha-k}{k}} (1-vt)^{\frac{-1-\alpha-rk}{k}} \exp \left\{ x - x(1+kt)^{\frac{1}{k}} (1-vt)^{-\frac{1}{k}} \right\} \\ & \times Y_r^\alpha \left(x(1+kt)^{\frac{1}{k}} (1-vt)^{-\frac{1}{k}}; k \right) \\ & = \sum_{n=0}^{\infty} t^n \sum_{p=0}^n a_p k^{n-p} \binom{n+r}{p+r} Y_{n+r}^{\alpha-nk+pk}(x; k) v^p. \end{aligned} \quad (39)$$

For $k = 1$, we get the corresponding generalization of the result involving generalized Laguerre polynomials, $L_n^{(\alpha)}(x)$.

7. Conclusions

From the above analysis, it is clear that whenever one knows a generating relations of the form (1, 3) then the corresponding bilateral generating relations can at once be written down from (2, 4). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1, 3).

Acknowledgements

The authors are greatly indebted to Prof. A.K. Chongdar of IEST, Shibpur for his constant encouragement and guidance.

REFERENCES

- [1] Konhauser, J. D. E. , Some properties of biorthogonal polynomials, J. Math. Anal. Appl., 11 (1965), 242-260.
- [2] Konhauser, J. D. E. , Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math., 21 (1967), 303-314.
- [3] Spencer, L. and Fano, U., Penetration and diffusion of X-rays, calculation of spatial distribution by polynomial expansion, J. Res. Nat. Bur. Standards 46(1951), 446-461.
- [4] Erdelyi, Arthur, with Magnus, W., Oberhettinger, F., Tricomi, F.G., et al., Higher transcendental function, vol. 2. New York: McGraw-Hill 1953.
- [5] Truesdell, C., An essay toward a unified theory of special functions, Princeton university press, 1948.
- [6] McBride, E.B.: Obtaining Generating Functions, Springer Verlag, Berlin, 1972.
- [7] Carlitz, L., A note on certain biorthogonal polynomials, Pacific J. Math., 24 (1968), 425-430.
- [8] Prabhakar, T.R., On a set of polynomials suggested by the Laguerre polynomials, Pacific J. Math., 35 (1970), 213-219.
- [9] Prabhakar, T.R., On the other set of biorthogonal polynomials suggested by Laguerre polynomials, Pacific J. Math., 37 (1971), 801-804.
- [10] Patil, K.R. and Thakare, N.K., Multilinear generating function for the Konhauser biorthogonal polynomial sets, SIAM J. Math. Anal., 9(1978), 921-923.
- [11] Srivastava, H.M., Some Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math., 98(1) (1982), 235-250.
- [12] Srivastava, H.M., A note on the Konhauser sets of Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math., 90(1) (1980), 197-200.
- [13] Srivastava, A.N. and Singh, S.N., On the Konhauser polynomials $Y_n^\alpha(x; k)$, Indian J. pure appl. Math., 10(1979), 1121-1126.
- [14] Shreshtha, R.M. and Bajracharya, S., Group theoretical study of a bilateral generating function, Int. Trans. and Spec. func., 5(1-2) (1997), 147-152.
- [15] Carlitz, L., A note on the Laguerre polynomials, Michigan Math. J., 7(3) (1960), 219-223.
- [16] Al-Salam, W.A., Operational representations for the Laguerre and other polynomials, Duke Math. Jour., 31(1964), 127-142.
- [17] Calvez, L.C. et Genin, R., Applications des relations entre les Fonctions génératrices et les formules de type Rodrigues, C.R. Acad. Sci. Paris Ser. A-B, 270(1970), A41-A44.
- [18] Alam, S. and Chongdar, A.K., On generating functions of modified Laguerre polynomials, Rev. Real Academia de Ciencias Zaragoza, 62(2007), 91-98.
- [19] Das, S. and Chatterjea, S.K., On a partial differential operator for Laguerre polynomials, Pure Math. Manuscript, 4(1985), 187-193.
- [20] Chongdar, A.K., Some generating functions involving Laguerre polynomials, Bull. Cal. Math. Soc., 76(1984), 262-269.
- [21] Majumder, A.B., Some generating functions of Laguerre polynomials, J. Ramanujan Math. Soc., 10(2) (1995),

195-199.

- [22] Desale, B.S. and Qashash, G.A., A general class of generating functions of Laguerre polynomials, Jour. Ineq. Special fun., 2(2011), 1-7.
- [23] Chongdar, A.K., Pittaluga, G. and Sacripante, L., On generating functions for certain special functions by Weisner's group theoretic method, Math. Bal., 12(1998), new series, 369-381.