On Bilateral Generating Functions of Konhauser Biorthogonal Polynomials

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Abstract In this article, we have obtained some novel results on bilateral generating functions of the polynomials, \( Y_{n+r}^{\alpha-nk}(x;k) \), a modified form of Konhauser biorthogonal polynomials, \( Y_n^\alpha(x;k) \) by group-theoretic method. As special cases, we obtain the corresponding results on Laguerre polynomials, \( Y_n^\alpha(x) \). Some applications of our results are also discussed.

Keywords Laguerre Polynomials, Biorthogonal Polynomials, Generating Functions

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1. Introduction

In 1965[1], Konhauser extended the notion of a particular pair of biorthogonal polynomial sets as introduced by Spencer and Fano [3] and established general properties of biorthogonal sets. In [2], Konhauser also introduced two sets of polynomials \{\( Y_n^\alpha(x;k) \)\} and \{\( Z_n^\alpha(x;k) \)\}, which are biorthogonal with respect to the weight function \( x^\alpha e^{-x} \) over the interval \((0, \infty)\), \( \alpha > -1, k \) is a positive integer. These polynomials satisfy the following condition:

\[
\int_0^\infty x^i \exp(-x) Y_i^\alpha(x;k) Z_j^\alpha(x;k) dx = \begin{cases} 0, & i \neq j, \\ \delta_{ij}, & i = j; \end{cases}
\]

For \( k = 1 \), these polynomials reduce to the generalized Laguerre polynomials, \( L_n^\alpha(x) \). For previous works on these polynomials one can see the works [7-14]. In the present paper we are interested only on \( Y_n^\alpha(x;k) \). In [7], Carlitz gave an explicit representation for the polynomials \( Y_n^\alpha(x;k) \) in the following form:

\[
Y_n^\alpha(x;k) = \frac{1}{\alpha!} \sum_{i=0}^{\alpha} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{j+\alpha+1}{k},
\]

where \( (a)_n \) is the pochhammer symbol defined by

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, \\ (a+1) \ldots (a+n-1), & \forall \ n \in \{1,2,3 \ldots \}. \end{cases}
\]

Finally, we like to point it out that some applications of our theorems are also given in the paper.
2. Operator and Extended Form of the Group

At first, we seek a linear partial differential operator $R$ of the form:

$$R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_0,$$

where each $A_i (i = 0, 1, 2, 3)$ is a function of $x$ and $y$ which is independent of $n$ such that

$$R(Y_{a-n}^{a-nk}(x; k)y^n) = \Omega_n Y_{a-n}^{a-nk}(x; k)y^{n+1},$$

where $\Omega_n$ is a function of $n$ and is independent of $x$ and $y$.

Using (5) and with the help of the differential recurrence relation:

$$x \frac{d}{dx}[y_{a-n}^{a-nk}(x; k)] = k(n + r + 1)y_{a-n}^{a-nk}(x; k) + (x + k + nk - \alpha - 1)y_{a-n}^{a-nk}(x; k)$$

we easily obtain the following linear partial differential operator:

$$R = xy \frac{\partial}{\partial x} - ky^2 \frac{\partial}{\partial y} - (x + k - \alpha - 1)y$$

such that

$$R(Y_{a-n}^{a-nk}(x; k)y^n) = k(n + r + 1)Y_{a-n}^{a-nk}(x; k)y^{n+1}.$$ \hspace{1cm} (7)

The extended form of the group generated by $R$ is given by

$$e^{wR}f(x, y) = (1 + kwy)^{a+1-k} \exp \left\{ x - x(1 + kwy)^{\frac{1}{2}} \right\} \times f \left( x(1 + kwy)^{\frac{1}{2}}, \frac{1}{1+kwy} \right),$$ \hspace{1cm} (8)

where $f(x, y)$ is an arbitrary function and $w$ is an arbitrary constant.

3. Derivation of Generating Function

Now writing $f(x, y) = Y_{a-n}^{a-nk}(x; k)y^n$ in (8), we get

$$e^{wR}(Y_{a-n}^{a-nk}(x; k)y^n) = (1 + kwy)^{a+1-k} \exp \left\{ x - x(1 + kwy)^{\frac{1}{2}} \right\} \times Y_{a-n}^{a-nk}(x(1 + kwy)^{\frac{1}{2}}; k) y^n.$$ \hspace{1cm} (9)

Again, using (7), we obtain

$$e^{wR}(Y_{a-n}^{a-nk}(x; k)y^n) = \sum_{m=0}^{\infty} k^m \frac{w^m}{m!} (n + r + 1)_m Y_{a-n}^{a-nk-mk}(x; k)y^{n+m}.$$ \hspace{1cm} (10)

Equating (9) and (10) and then substituting $wy = t$, we get

$$(1 + kt)^{a+1-k} \exp \left\{ x - x(1 + kt)^{\frac{1}{2}} \right\} Y_{a-n}^{a-nk}(x(1 + kt)^{\frac{1}{2}}; k)$$

$$= \sum_{m=0}^{\infty} k^m \frac{n + r + m}{m} Y_{a-n}^{a-nk-mk}(x; k)t^m,$$ \hspace{1cm} (11)

which does not seem to have appeared in the earlier works.

**Corollary 1:** Replacing $\alpha$ by $\alpha + nk$ in (11), we get

$$(1 + kt)^{a+1-k} \exp \left\{ x - x(1 + kt)^{\frac{1}{2}} \right\} Y_{a-n}^{a-nk}(x(1 + kt)^{\frac{1}{2}}; k)$$

$$= \sum_{m=0}^{\infty} k^m \frac{n + r + m}{m} Y_{a-n}^{a-nk-mk}(x; k)t^m.$$ \hspace{1cm} (12)

**Corollary 2:** Again, putting $n = 0$ in (12), we get

$$(1 + kt)^{a+1-k} \exp \left\{ x - x(1 + kt)^{\frac{1}{2}} \right\} Y_{a-n}^{a-nk}(x(1 + kt)^{\frac{1}{2}}; k)$$

$$= \sum_{m=0}^{\infty} k^m \frac{r + m}{m} Y_{a-n}^{a-nk-mk}(x; k)t^m,$$ \hspace{1cm} (13)

which is found derived in [11, 17].

**Corollary 3:** Putting $r = 0$ in (13), we get

$$(1 + kt)^{a+1-k} \exp \left\{ x - x(1 + kt)^{\frac{1}{2}} \right\} Y_{a-n}^{a-nk}(x(1 + kt)^{\frac{1}{2}}; k)$$

$$= \sum_{m=0}^{\infty} k^m \frac{n + m}{m} Y_{a-n}^{a-nk-mk}(x; k)t^m,$$ \hspace{1cm} (14)

which is found derived in [11, 13, 17] by different methods.

**Special case 1:** If we put $k = 1$, then $Y_n^a(x; k)$ reduces to the generalized Laguerre polynomials, $L_n^a(x)$. Thus Putting $k = 1$ in (11) - (14), we get the following generating relation on Laguerre polynomials:

$$(1 + t)^{a-n} \exp(-xt) L_n^a(x; 1 + t) = \sum_{m=0}^{\infty} \frac{n + r + m}{m} L_n^{a-m}(x)t^m.$$ \hspace{1cm} (15)

$$(1 + t)^{a} \exp(-xt) L_n^a(x; 1 + t) = \sum_{m=0}^{\infty} \frac{n + r + m}{m} L_n^{a-m}(x)t^m.$$ \hspace{1cm} (16)

$$(1 + t)^{a} \exp(-xt) L_n^a(x; 1 + t) = \sum_{m=0}^{\infty} \frac{r + m}{m} L_n^{a-m}(x)t^m,$$ \hspace{1cm} (17)

which is found derived in [15, 16, 18, 19, 20, 22].

$$(1 + t)^{a} \exp(-xt) = \sum_{m=0}^{\infty} L_n^{a-m}(x)t^m,$$ \hspace{1cm} (18)

which is found derived in [23].

**Special case 2:** Using the relation

$$L_n^{a-n}(x) = \frac{(-x)^n}{n!} C_n(a; x),$$

[6, pp.68-69] and from the above generating relations we get the following generating relations on Charlie polynomials [4, p. 226]:

$$(1 - \frac{x}{a})^{a+r} \exp(y) C_n(a + r; x - y) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_n(a + r + m; a + r; x).$$ \hspace{1cm} (19)
\[(1 - \frac{y}{x})^{\alpha} \exp(y)C_n(\alpha; x - y) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_{n+m}(\alpha; x), \quad (20)\]

which is found derived in [6, pp. 70] and equivalent to Doetsch’s formula given in [5, pp. 88(26)].

\[(1 - \frac{y}{x})^{\alpha} \exp(y) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_m(\alpha; x). \quad (21)\]

Now we proceed to prove the Theorem 1.

### 4. Proof of Theorem 1

Let us consider the generating relation of the form:

\[G(x, w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) w^n. \quad (22)\]

Replacing \(w\) by \(wv y\) in the both sides of (22), we have

\[G(x, wv y) = \sum_{n=0}^{\infty} a_n (L_{n+r}^{(\alpha-n)}(x)y^n) (wv y)^n. \quad (23)\]

Operating \(e^{wv y}\) on both sides of (23), we get

\[e^{wv y} \left( \sum_{n=0}^{\infty} a_n (L_{n+r}^{(\alpha-n)}(x)y^n) (wv y)^n \right). \quad (24)\]

Now the left member of (24), with the help of (8), reduces to

\[(1 + kw y)^{\frac{1+\alpha-k}{k}} \exp \left( x - x(1 + kw y) \frac{1}{k} \right) G \left( x(1 + kw y) \frac{1}{k} \right). \quad (25)\]

The right member of (24), with the help of (7), becomes

\[= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{w^p}{p!} y^n (n + r - p + 1)_p Y_{n+r}^{(\alpha-n)}(x; k) Y^n (wv y)^{n-p}. \quad (26)\]

Now equating (25) and (26) and then substituting \(wy = t\), we get

\[(1 + kt)^{\frac{1+\alpha-k}{k}} \exp \left( x - x(1 + kt) \frac{1}{k} \right) G \left( x(1 + kt) \frac{1}{k}, \frac{vt}{1 + kt} \right) = \sum_{n=0}^{\infty} V_{n+r}^{(a-n)}(x; k) \sigma_n (v) t^n, \quad (27)\]

where

\[\sigma_n (v) = \sum_{p=0}^{n} a_p k^{n-p} \left( \frac{n + r}{p + r} \right) v^p. \]

This completes the proof the theorem.

**Special case 3:** Putting \(k = 1\) in our Theorem 1 we get the following result on generalised Laguerre polynomials:

**Theorem 3.** If there exists a unilaterating generating relation of the form

\[G(x, w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) w^n \quad (28)\]

then

\[(1 + t)^{\alpha} \exp(-xt) G \left( x(1 + t), \frac{vt}{1 + t} \right) = \sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x) \sigma_n (v) t^n, \quad (29)\]

where

\[\sigma_n (v) = \sum_{p=0}^{n} a_p \left( \frac{n + r}{p + r} \right) v^p. \]

which is found derived in [18].

**Corollary 4:** Putting \(r = 0\) in Theorem 3, we get the theorem found derived in [18, 19].

### 5. Proof of Theorem 2

\[\text{R.H.S.} = \sum_{n=0}^{\infty} \sigma_n (x, v) t^n = \sum_{p=0}^{\infty} a_p (vt)^p \sum_{n=0}^{\infty} \left( \frac{p + r + n}{n} \right) Y_{n+r}^{(\alpha-n)}(x; k)(kt)^n \]

\[= \sum_{p=0}^{\infty} a_p (vt)^p (1 + k)^{\frac{1+\alpha-k}{k}} \exp \left( x - x(1 + k) \frac{1}{k} \right) Y_{n+r}^{(\alpha-n)}(x; k)(kt)^n \quad [\text{using (12)}] \]

\[= (1 + k)^{\frac{1+\alpha-k}{k}} \exp \left( x - x(1 + k) \frac{1}{k} \right) \sum_{p=0}^{\infty} a_p Y_{n+r}^{(\alpha-n)}(x; k)(kt)^p \]

\[= (1 + k)^{\frac{1+\alpha-k}{k}} \exp \left( x - x(1 + k) \frac{1}{k} \right) G \left( x(1 + k) \frac{1}{k}, vt \right), \quad [\text{using (3)}] \]

which is Theorem 2.
Corollary 5: If we put \( r = 0 \) in Theorem 2, then we get the following Theorem:

**Theorem 4.** If

\[
G(x, w) = \sum_{n=0}^{\infty} a_n Y_n^\alpha (x; k) w^n
\]  

then

\[
(1 + \alpha - k) \exp \left\{ x - x(1 + k t)^{1/k} \right\} G \left( x(1 + k t)^{1/k}, \, vt \right) = \sum_{n=0}^{\infty} \sigma_n (x, v) t^n,
\]

where \( \sigma_n (x, v) = \sum_{p=0}^{n} a_p \left( \frac{n + r}{p + r} \right) L_{n+r}^{(a-n+p)} (x) v^p \),

which is found derived in [18].

**Theorem 6.** If

\[
G(x, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)} (x) w^n
\]

then

\[
(1 + \alpha - k) \exp \left\{ x - x(1 + k t)^{1/k} \right\} G \left( x(1 + k t)^{1/k}, \, vt \right) = \sum_{n=0}^{\infty} \sigma_n (x, v) t^n,
\]

where \( \sigma_n (x, v) = \sum_{p=0}^{n} a_p \left( \frac{n + r}{p + r} \right) L_n^{(a-n+p)} (x) v^p \),

which is found derived in [18, 19, 21, 22].

Below we have given some applications of our results.

6. Applications

A1. As an application of our Theorem 1, we consider the following generating relation [11, 12]:

\[
\sum_{n=0}^{\infty} \left( \frac{n + m}{n} \right) Y_n^{\alpha_n-nk} (x; k) t^n = (1 + t)^a \exp \left\{ x - x(1 + t)^{1/k} \right\} Y_m^{\alpha_n} \left( x(1 + t)^{1/k}; k \right).
\]

If in our theorem, we take \( a_n = \left( \frac{n + m}{n} \right) \), then

\[
G(x, w) = (1 + w)^{1+\alpha-k} \exp \left\{ x - x(1 + w)^{1/k} \right\} Y_m^{\alpha_n} (x(1 + w)^{1/k}; k).
\]

Therefore by the application of our Theorem 1, we get the following generalization of the result (36):

\[
(1 + kt + vt)^{1+\alpha-k} \exp \left\{ x - x(1 + kt + vt)^{1/k} \right\} Y_m^{\alpha_n} \left( x(1 + kt + vt)^{1/k}; k \right)
\]

\[
= \sum_{n=0}^{\infty} Y_{n+m}^{\alpha_n-nk} (x; k) \sigma_n (v) t^n,
\]

where

\[
\sigma_n (v) = \sum_{p=0}^{n} a_p \left( \frac{n + r}{p + r} \right) v^p.
\]

A2. As an application of Theorem 2, we consider the following generating relation [9,11,12]:

\[
\sum_{n=0}^{\infty} \left( \frac{n + r}{n} \right) Y_n^{\alpha_n+r} (x; k) t^n = (1 - t)^{1-a-rk} \exp \left\{ x - x(1 - t)^{1/k} \right\} \times
\]
\[ \times Y^\alpha_r \left( x(1-t)^{-\frac{1}{k}}; k \right). \]  

(38)

If in our theorem, we take \( a_n = \binom{n + r}{n} \), then

\[ G(x,w) = (1-w)^{-\frac{1}{k}} \exp \left\{ x - x(1-w)^{-\frac{1}{k}} \right\} Y^\alpha_r \left( x(1-w)^{-\frac{1}{k}}; k \right). \]

Therefore by the application of our Theorem 2 we get the following generalization of (38):

\[ (1+kt)^{\frac{1+\nu}{1}} \left( 1-vt \right)^{\frac{1+\nu}{k}} \exp \left\{ x - x(1+kt)^{-\frac{1}{k}}(1-vt)^{-\frac{1}{k}} \right\} \]
\[ \times Y^\alpha_r \left( x(1+kt)^{-\frac{1}{k}}(1-vt)^{-\frac{1}{k}}; k \right) \]
\[ = \sum_{m=0}^{\infty} t^m \sum_{p=0}^{n} a_p k^{n-p} \binom{n+r}{p+v} Y^\alpha_{n+r} \left( x; k \right) v^p. \]  

(39)

For \( k = 1 \), we get the corresponding generalization of the result involving generalized Laguerre polynomials, \( L_n^{(\alpha)}(x) \).

7. Conclusions

From the above analysis, it is clear that whenever one knows a generating relations of the form (1, 3) then the corresponding bilateral generating relations can at once be written down from (2, 4). So one can get a large number of bilateral generating relations by attributing different suitable values to \( a_n \) in (1, 3).

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