Skew - Commuting Derivations of Noncommutative Prime Rings

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Abstract The main purpose of this paper is study and investigate a skew-commuting and skew-centralizing d and g be a derivations on noncommutative prime ring and semiprime ring R, we obtain the derivation d(R)=0 (resp. g(R)=0).

Keywords Skew-commuting, Derivation, Noncommutative Prime Ring, Semiprime Ring

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1. Introduction

Derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring (see, [5]). Derivations play a significant role in determining whether a ring is commutative, see ( [1],[3],[4],[18],[19] and [20]).Derivations can also be useful in other fields. For example, derivations play a role in the calculation of the eigenvalues of matrices (see, [2]) which is important in mathematics and other sciences, business and engineering. Derivations also are used in quantum physics(see, [18]). Derivations can be added and subtracted and we still get a derivation, but when we compose a derivation with itself we do not necessarily get a derivation. The history of commuting and centralizing mappings goes back to (1955) when Divinsky [6] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner[7]has proved that the existence of a non-zero centralizing derivation on prime ring forces the ring to be commutative (Posner's second theorem). Luch [8]generalized the Divinsky result, we have just mentioned above, to arbitrary prime ring. In[9] M.N.Daif, proved that, let R be a semiprime ring and d a derivation of R with d^2≠0 . If [d,x,d,y]=0 for all x, y ∈ R, then R contains a non-zero central ideal. M.N.Daif and H.E. Bell [10] proved that, let R be a semiprime ring admitting a derivation d for which either xy+d(xy)= yx+d(yx) for all x, y ∈ R or xy-d(xy)= yx-d(yx) for all x, y ∈ R , then R is commutative. V.DeFilippis [11] proved that, when R is a prime ring let d a non-zero derivation of R , U≠(0) a two-sided ideal of R , such that d([x,y])=[x,y] for all x,y ∈ U , then R is commutative. Recently A.H. Majeed and Mehsin Jabel [12] , give some results as , let R be a 2-torsion free semiprime ring and U a non-zero ideal of R .R admitting a non-zero derivation d satisfying d([d(x),d(y)])=[x,y] for all x,y ∈ U. If d acts as a homomorphism, then R contains a non-zero central ideal. Our aim in this paper is to investigate skew-commuting d and g be derivations on noncommutative prime ring and semiprime ring R.

2. Preliminaries

Throughout R will represent an associative ring with identity, Z(R) denoted to the center of R , R is said to be n-torsion free, where n ≠ 0 is an integer, if whenever n x= 0,with x ∈ R, then x = 0. We recall that R is semiprime if xRx = (0) implies x = 0 and it is prime if xRy = (0) implies x = 0 or y = 0. A prime ring is semiprime but the converse is not true in general. An additive mapping d: R→R is called a derivation if d(xy) = d(x)y+ xd(y) holds for all x ,y ∈ R , and is said to be n-centralizing on U (resp. n-commuting on U ) , if [x^n ,d(x)] ∈ Z(R) holds for all x ∈ U (resp. [x^n ,d(x)]= 0 holds for all x ∈ U) , where n be a positive integer . Also is called skew-centralizing on subset U of R (resp. skew-commuting on subset U of R) if d(x)x+xd(x) ∈ Z(R) holds for all x ∈ U (resp.d(x)x+xd(x)=0 holds for all x ∈ U),and d acts as a homomorphism on U(resp. anti-homomorphism on U) if d(xy)= d(x)d(y) holds for all x,y ∈ U (resp. if d(xy)= d(y)d(x) holds for all x,y ∈ U). We write [x,y] for xy – yx and make extensive use of basic commutator identities [xy,z]=x[y,z]+ [x,z]y and [x,yz]=y [x,z] +[x,y]z . In some parts of the proof our theorems(3.1 and 3.2), we using same technique in [21].

First we list the lemmas which will be needed in the sequel.

Lemma1[7]
If $d$ is commuting derivation on noncommutative prime ring, then $d=0$.

**Lemma 2 [13: Theorem 1.2]**
Let $S$ be a set and $R$ a semiprime ring. If functions $d$ and $g$ of $S$ into $R$ satisfy $d(s)xg(t)=g(s)d(t)$ for all $s,t\in S$, $x \in R$, then there exists idempotents $a_i$, $\alpha_i \in C$ and an invertible element $\lambda \in C$ such that $a_i \alpha_j = \delta_{ij}$, for $i \neq j$.

**Lemma 3 [14: Theorem 2]**
Let $R$ be a semiprime ring and $U$ a non-zero ideal of $R$. If $d$ is a derivation of $R$ which is centralizing on $U$, then $d$ is commuting derivation on $U$.

**Lemma 4 [15: Lemma 4]**
Let $R$ be a semiprime ring and $U$ a non-zero ideal of $R$. If $d$ is a derivation of $R$ which is centralizing on $U$, then $d$ is commuting derivation on $R$, $d=0$.

**3. The Main Results**

**Theorem 3.1**
Let $R$ be a noncommutative prime ring, $d$ and $g$ be a derivations of $R$. If $R$ admits to satisfy $d(x)x+g(x)\in Z(R)$ for all $x \in R$, then $d(R)=0$ (resp. $g(R)=0$) or $w(d)(d)$ is skew commutating on $R$. Thus

\[ [d(x)+g(x),y]=0 \quad \text{for all } x,y \in R. \]

Now from (6) and (7), we obtain $w(d)(d)(d)(d)=0$ for all $x,y \in R$. Since $w \in Z(R)$, this relation gives

\[ w^2(d)(d)(d)(d)=0 \quad \text{for all } x,y \in R. \]

Replacing $y$ by $zy$, with using (8), we get $w^2z(d)(d)(d)=0$ for all $x,y \in R$, which implies

\[ wzw(d)(d)(d)=0 \quad \text{for all } x,y \in R. \]

Replacing $z$ by $[d(x)+g(x)]z$ and since $R$ is prime ring, which implies

\[ w[d(x)+g(x),y]=0 \quad \text{for all } x,y \in R. \]

Also from (2), we obtain

\[ [d(x)+d(w)x+g(w)x+wg(x),y]=0 \quad \text{for all } x,y \in R. \]

Thus from (6) and (7), we obtain $w(d)(d)(d)(d)=0$ for all $x,y \in R$. Since $w \in Z(R)$, this relation gives

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\[ wzw[d(x)+g(x),y]=0 \quad \text{for all } x,y \in R. \]

Since $w \in Z(R)$, this

\[ [d(x)+g(x),y]=0 \quad \text{for all } x,y \in R. \]
central for all \( w \in Z(R) \).

**Theorem 3.2**

Let \( R \) be a noncommutative prime ring, \( d \) be a skew-centralizing derivation of \( R \) (resp. \( g \) be a skew-centralizing derivation of \( R \)), if \( R \) admits to satisfy \( d(x)+xg(x) \in Z(R) \) for all \( x \in R \). Then \( d(R)=0 \) (resp. \( g(R)=0 \)).

**Proof:** Let \( x_0 \in R \) and \( c=d(x_0)x_0+x_0g(x_0) \). Thus, according to our hypothesis, we obtain \( c \in Z(R) \). Then by Theorem 3.1, we get \( cd \) and \( cg \) are commuting, then \( [cd(x),y]=0 \) for all \( x,y \in R \). Then \( c(d(x))=c \) for all \( x \in R \). Hence \( cd(x)=cy+xd(y)+d(y)x=cd(x) \) for all \( x \in R \). Then \( cd(x)c=cd(x) \) for all \( x,y \in R \). Therefore, the first identity of (15) becomes \( cd(x)c=cd(x) \) for all \( x,y \in R \). (16)

Replacing \( y \) by \(-x \) in (16), we obtain \( cd(x)c=cd(x) \) for all \( x \in R \). Thus, we get \( cd(x)c=cd(x) \) for all \( x \in R \). Hence \( cd(x)c=cd(x) \) for all \( x \in R \). Then \( cd(x)c=cd(x) \) for all \( x \in R \). Since \( cg \) is central, therefore, analogously, it follows that \( cg(x)=0 \) for all \( x \in R \). Hence \( cd(x)=cd(x) \) for all \( x \in R \). Hence \( cd(x)c=cd(x) \) for all \( x \in R \). Thus from these relations, we obtain \( cd(x)c=cd(x) \) for all \( x \in R \).

In particular, \( cd(x)c=cd(x) \) for all \( x \in R \). Since \( c \) is an arbitrary element of \( R \), therefore, \( cd(x)c=cd(x) \) for all \( x \in R \). (17)

If we taking \( d(x)=g(x) \), then \( d(x)+xg(x)=0 \) for all \( x \in R \). Then by using Lemma 3, we obtain \( d(R)=0 \) (resp. \( g(R)=0 \)).

If \( d(x)\neq g(x) \), this case lead to \( d(x)x+xg(x) \in Z(R) \) for all \( x \in R \). By Theorem 3.1, we complete our proof.

**Theorem 3.3**

Let \( R \) be a 2-torsion free semiprime ring with cancellation property. If \( R \) admits a derivation \( d \) to satisfy

(i) \( d \) acts as a skew-commuting on \( R \).

(ii) \( d \) acts as a skew-xentralizing on \( R \). Then \( d(R) \) is commuting on \( R \).

**Proof:** (i) Since \( d \) is skew-commuting, then \( d(x)+xg(x)=0 \) for all \( x \in R \). (18)

Left multiplying (18) by \( x \), we obtain \( x(x)+x^2d(x)=0 \) for all \( x \in R \). (19)

From (18), we get \( d(x)=0 \) for all \( x \in R \). (20)

In (20) replacing \( x \) by \( xy \), we obtain \( d(x)+y=x^2d(x)+y^2d(x)=0 \) for all \( x,y \in R \). According to (20), a above equation become \( d(xy)+y=0 \) for all \( x,y \in R \). Then \( d(xy)+y=0 \) for all \( x,y \in R \). Replacing \( y \) by \( x^2 \) and according to (20), we arrived to \( d(x^2)+x^2d(x)=0 \) for all \( x \in R \). (21)

Then \( x^2d(x)=x^2d(x)x^2 \) for all \( x \in R \).

By substituting (21) in (19), we get \( x^2d(x)+x^2d(x)=0 \) for all \( x \in R \). Then \( [x,d(x)]=0 \) for all \( x \in R \). Then apply the cancellation property on \( x \), we get \( [x,d(x)]=0 \) for all \( x \in R \). Then \( d(R) \) is commuting on \( R \).

(ii) We will discuss, when \( d \) acts as a skew-centralizing on \( R \).

Then we have \( d(x)x+x(x) \in Z(R) \) for all \( x \in R \). \( d(x)x \in Z(R) \) for all \( x \in R \). i.e.

\[ [d(x),r]=0 \] for all \( x,r \in R \). (22)

Also, by replacing \( r \) by \( x \) in (22), we obtain \( d(x)+x=0 \) for all \( x \in R \). Then \( d(x)x+x^2d(x)=0 \) for all \( x \in R \). Then \( d(x)x^2-x^2d(x)=0 \) for all \( x \in R \). Then \( [d(x),x]=0 \) for all \( x \in R \). (23)

In (22), replacing \( x \) by \( x+y \), we obtain \( [d(x+y)+d(y)+d(y),r]=0 \) for all \( x,y,r \in R \). According to (22), we obtain \( [d(x)+y,x]=0 \) for all \( x,y \in R \). Replacing \( y \) by \( x \), we obtain \( [d(x)+x^2d(x)+x^2d(x),r]=0 \) for all \( x,r \in R \). According to (22) and (23), we get \( [x^2d(x)+x^2d(x),r]=0 \) for all \( x,r \in R \). Then \( 2d(x)^2d(x),r]=0 \) for all \( x,r \in R \). Since \( R \) is 2-torsion free, we obtain \( [x^2d(x),r]=0 \) for all \( x,r \in R \). Then \( [x^2d(x),r]=0 \) for all \( x,r \in R \). According to (22), above equation become \( 2d(x)^2d(x),r]=0 \) for all \( x,r \in R \). Replacing \( r \) by \( x \), we obtain \( 2x^2d(x),r]=0 \) for all \( x \in R \). Applying the cancellation property on \( x^2 \), we get \( [d(x),x]=0 \) for all \( x \in R \). We complete the proof of theorem.
Theorem 3.4

Let R be a 2-torsion free noncommutative prime ring. If R admits a derivation d to satisfy one of following
(i) d acts as a homomorphism on R. Then d(R)=0.
(ii) d acts as an anti-homomorphism on R. Then d(R)=0.

Proof: (i) d acts as a homomorphism on R. We have d is a derivation, then
\[ d(xy)=d(x)y+xd(y) \] for all x, y ∈ R. Then
\[ d(xyz)=d(xyz)+d(xyz) = d(xyz)+d(xyz) \] for all x, y, z ∈ R. Since d acts as a homomorphism, then \[ d(xy)z=d(x)yz+xd(yz) \] for all x, y, z ∈ R. Therefore, d is skew-commuting on R, i.e. d(R)=0.

Theorem 3.3 and Theorem 3.4, we can’t exclude the condition char.R≠2, \( d \) is skew-centralizing on R. Then by Theorem3.2, we obtain d(R)=0. Then \[ \{d(x)2,x\}=0 \] for all x ∈ R. Thus d is skew–centralizing and skew-commuting on R, i.e. d(R)=0.

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