Karen Sahakyan

*Corresponding Author: karen_sahakyan@ysu.am

Abstract In this article a new quadrature and cubature formulas are obtained. These formulas are formulas of Hermitian type, which exactly on some trigonometric polynomials. The question of the convergence of the quadrature process associated with these formulas is investigated. Numerical analysis of these formulas is holds. The obtained formulas may have an advantage in integrating strongly oscillating functions.

MSC2010 41A55,65D30,65D32

$$\left\{ \begin{array}{l} 1 - a_1 x_1 + a_1^2 x_2 - a_1^3 x_3 + \dots + (-1)^n a_1^n x_n = 0 \\ 1 - a_2 x_1 + a_2^2 x_2 - a_2^3 x_3 + \dots + (-1)^n a_2^n x_n = 0 \\ 1 - a_3 x_1 + a_3^2 x_2 - a_3^3 x_3 + \dots + (-1)^n a_3^n x_n = 0 \\ \vdots \\ 1 - a_n x_1 + a_n^2 x_2 - a_n^3 x_3 + \dots + (-1)^n a_n^n x_n = 0, \end{array} \right. \quad (1)$$

where $a_1, a_2, \dots, a_n$ are real or complex numbers, is

$$\begin{aligned} x_1 &= \frac{a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1}}{a_1 a_2 \dots a_n}, \\ &\dots\dots\dots \\ x_{n-2} &= \frac{a_1 a_2 + a_1 a_3 + \dots + a_1 a_n + a_2 a_3 + a_2 a_4 + \dots + a_2 a_n + \dots + a_{n-1} a_n}{a_1 a_2 \dots a_n}, \\ x_{n-1} &= \frac{a_1 + a_2 + \dots + a_n}{a_1 a_2 \dots a_n}, \\ x_n &= \frac{1}{a_1 a_2 \dots a_n}, \end{aligned}$$

Proof. Note that the system of equations (1) is equivalent to a polynomial of the

$$P(t) = 1 - x_1 t + x_2 t^2 - x_3 t^3 + \dots + (-1)^n x_n t^n,$$

whose roots are the numbers $a_1, a_2, \dots, a_n$.

Then by Vieta's theorem

$$\begin{aligned} a_1 a_2 \dots a_n &= \frac{1}{x_n} \implies x_n = \frac{1}{a_1 a_2 \dots a_n}, \\ a_1 + a_2 + \dots + a_n &= \frac{x_{n-1}}{x_n} \implies x_{n-1} = \frac{a_1 + a_2 + \dots + a_n}{a_1 a_2 \dots a_n}, \\ a_1 a_2 + a_1 a_3 + \dots + a_1 a_n + a_2 a_3 + a_2 a_4 + \dots + a_2 a_n + \dots + a_{n-1} a_n &= \frac{x_{n-2}}{x_n} \implies \\ x_{n-2} &= \frac{a_1 a_2 + a_1 a_3 + \dots + a_1 a_n + a_2 a_3 + a_2 a_4 + \dots + a_2 a_n + \dots + a_{n-1} a_n}{a_1 a_2 \dots a_n}, \\ &\dots\dots\dots \\ a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1} &= \frac{x_1}{x_n} \implies \\ x_1 &= \frac{a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1}}{a_1 a_2 \dots a_n}. \end{aligned}$$

The lemma is proved. □

Let $f(x) \in C^{2p}[-1, 1]$ and $x_k = \frac{2(k-N)}{2N+1}$ ($k = 0, \pm 1, \dots, \pm 2N$) is uniform grid on the segment $[-1, 1]$ with step $h = \frac{2}{2N+1}$.

Then the following statement holds.

Theorem 1. *The quadrature formula*

$$\int_{-1}^1 f(x) dx \approx h \sum_{k=0}^{2N} \left(A_0 f(x_k) + A_2 h^2 f''(x_k) + \dots + A_{2p} h^{2p} f^{(2p)}(x_k) \right), \quad (2)$$

where

$$\begin{aligned} A_0 &= 1, \quad A_2 = \frac{1}{\pi^2} \frac{4^2 6^2 \dots (2p)^2 + 2^2 6^2 \dots (2p)^2 + \dots + 2^2 4^2 \dots (2p-2)^2}{2^2 4^2 \dots (2p)^2}, \dots, \\ A_{2p-4} &= \frac{1}{\pi^{2p-4}} \frac{2^2 4^2 + \dots + 2^2 (2p)^2 + 4^2 6^2 + \dots + (2p-2)^2 (2p)^2}{2^2 4^2 \dots (2p)^2}, \\ A_{2p-2} &= \frac{1}{\pi^{2p-2}} \frac{2^2 + 4^2 + \dots + (2p)^2}{2^2 4^2 \dots (2p)^2}, \\ A_{2p} &= \frac{1}{\pi^{2p}} \frac{1}{2^2 4^2 \dots (2p)^2}. \end{aligned}$$

has trigonometric degree of accuracy equal to $(p+1)(2N+1)-1$.

Proof. To prove the statement should make sure that the quadrature formula (2) is exact for the functions $f_m = e^{i\pi m x}$, for $m = 0, \pm 1, \pm 2, \dots, \pm((p+1)(2N+1)-1)$.

When $f_0(x) = 1$ from (2) $\implies 2 = h \sum_{k=0}^{2N} A_0 \implies A_0 = 1$.

When $f_m(x) = e^{i\pi m x}$, $n = \pm 1, \pm 2, \dots, \pm((p+1)(2N+1)-1)$ from (2) hence

$$0 = h \sum_{k=0}^{2N} \left(e^{i\pi m x_k} + A_2 h^2 (i\pi m)^2 e^{i\pi m x_k} + \dots + A_{2p} h^{2p} (i\pi m)^{2p} e^{i\pi m x_k} \right)$$

Theorem 3.

$$\lim_{N \rightarrow \infty} S_N(f) = \int_{\Omega} f(x) dx \quad (4)$$

Proof. Let

$$R_N(f) = \int_{\Omega} f(x) dx - h^n \sum_{k=0}^{2N} (A_0 f(x_k) + A_2 h^2 \Delta f(x_k) + A_4 h^4 \Delta^2 f(x_k) + \dots + A_{2p} h^{2p} \Delta^{2p} f(x_k))$$

because of

$$\int_{\Omega} T_m(x) dx = h^n \sum_{k=0}^{2N} (A_0 T_m(x_k) + A_2 h^2 \Delta T_m(x_k) + A_4 h^4 \Delta^2 T_m(x_k) + \dots + A_{2p} h^{2p} \Delta^{2p} T_m(x_k))$$

for trigonometric polynomials $T_m(x)$ which degree does not exceed $(p+1)(2N+1) - 1$. Then will be

$$R_N(f) = \int_{\Omega} (f(x) - T_m(x)) dx - h^n \sum_{k=0}^{2N} (A_0 (f(x_k) - T_m(x_k)) + A_2 h^2 \Delta (f(x_k) - T_m(x_k)) + A_4 h^4 \Delta^2 (f(x_k) - T_m(x_k)) + \dots + A_{2p} h^{2p} \Delta^{2p} (f(x_k) - T_m(x_k)))$$

Then

$$|R_N(f)| \leq \int_{\Omega} |f(x) - T_m(x)| dx + h^n \sum_{k=0}^{2N} (A_0 |f(x_k) - T_m(x_k)| + A_2 h^2 |\Delta (f(x_k) - T_m(x_k))| + A_4 h^4 |\Delta^2 (f(x_k) - T_m(x_k))| + \dots + A_{2p} h^{2p} |\Delta^{2p} (f(x_k) - T_m(x_k))|)$$

by Jackson's inequality [2, 3] will be obtained

$$|R_N(f)| \sim o(N^{-2p})$$

which proved the theorem. □

Remark 2. If the function $f(x) \in C^r(\Omega)$ where $r \geq 2p$ then $|R_N(f)| \sim o(N^{-r})$.

4 The numerical analysis of the formulas

Let us consider a few special cases the formulas derived above.

Let $p = 1$, then from Theorem 1, $A_0 = 1$, $A_2 = \frac{1}{\pi^2} \frac{1}{2^2} = \frac{1}{\pi^2} \frac{1}{4}$ and formula (2) is

$$\int_{-1}^1 f(x) dx \approx h \sum_{k=0}^{2N} \left(f(x_k) + \frac{h^2}{4\pi^2} f''(x_k) \right), \quad (5)$$

where $h = \frac{2}{2N+1}$, $x_k = \frac{2(k-N)}{2N+1}$.

This formula can be obtained by integrating the Hermitian interpolation formula received by the author in [4].

Let $p = 2$, then from Theorem 1, $A_0 = 1$, $A_2 = \frac{1}{\pi^2} \frac{2^2 + 4^2}{2^2 4^2} = \frac{1}{\pi^2} \frac{5}{16}$, $A_4 = \frac{1}{\pi^4} \frac{1}{2^2 4^2} = \frac{1}{\pi^4} \frac{1}{64}$, and formula (2) is

$$\int_{-1}^1 f(x) dx \approx h \sum_{k=0}^{2N} \left(f(x_k) + \frac{5h^2}{16\pi^2} f''(x_k) + \frac{h^4}{64\pi^4} f^{(4)}(x_k) \right), \quad (6)$$

where $h = \frac{2}{2N+1}$, $x_k = \frac{2(k-N)}{2N+1}$.

Let $p = 1$, $n = 2$, then from Theorem 2, $A_0 = 1$, $A_2 = \frac{1}{2} \frac{1}{\pi^2} \frac{1}{2^2} = \frac{1}{8\pi^2}$ and formula (3) is

$$\int_{-1}^1 \int_{-1}^1 f(x_1, x_2) dx_1 dx_2 \approx h^2 \sum_{k_1=0}^{2N} \sum_{k_2=0}^{2N} \left(f(x_{k_1}, x_{k_2}) + \frac{h^2}{8\pi^2} \Delta f(x_{k_1}, x_{k_2}) \right), \quad (7)$$

where $h = \frac{2}{2N+1}$, $x_{k_1} = \frac{2(k_1-N)}{2N+1}$, $x_{k_2} = \frac{2(k_2-N)}{2N+1}$ and $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}$.

The resulting formulas can be more efficiency for integration a strongly oscillating functions. Consider the following example.

Table 1. Numerical experiment

Approximate value of integral finding by	Number of nodal points	Approximation error
Gauss formulas	6	-1.44803
ClenshawCurtisOscillatoryRule	142	-0.0033701060748317913
ClenshawCurtisRule	134	$4.904357265890502 \times 10^{-7}$
GaussKronrodRule	142	-0.0033701060748317913
LobattoKronrodRule	116	-0.09157826508013064
MonteCarloRule	100	-0.003370106074832069

Example 1. Let consider as a sample the next function $f(x) = xe^{-x^2} + \cos(20\pi x)$.

Then exact value of integral is $\int_{-1}^1 f(x) dx = 0$, and approximate value of integral finding by formulas (5) if $N = 1$ is $-2.3684757858670005 \times 10^{-15}$.

Below is a table of numerical results obtained by other methods of numerical integration of this function.

The advantage of the new method of integration in this case is obvious.

The numerical experiment performed using the built-in capabilities of Wolfram *Mathematica* 9.0 [5].

Example 2. Let consider as a sample the next function $f(x) = \cos(10 \sin(5\pi x))$.

An approximate value of integral finding by formulas (5) if $N = 6$ is -0.491872 .

Below is numerical results obtained by method of numerical integration of this function built-in capabilities of Wolfram *Mathematica* 9.0.

```
NIntegrate[ Cos[10Sin[5πx]], {x,-1,1}, Method Automatic, AccuracyGoal Infinity, PrecisionGoal Automatic, Working-
Precision MachinePrecision, MaxPoints Automatic, MinRecursion 0, MaxRecursion Automatic]
-0.491872
```

```
NIntegrate[ Cos[10Sin[5πx]], {x,-1,1}, Method Automatic, AccuracyGoal Infinity, PrecisionGoal Automatic, Working-
Precision MachinePrecision, MaxPoints 270, MinRecursion 0, MaxRecursion Automatic]
-0.491875
```

```
NIntegrate[ Cos[10Sin[5πx]], {x,-1,1}, Method Automatic, AccuracyGoal Infinity, PrecisionGoal Automatic, Working-
Precision MachinePrecision, MaxPoints Automatic, MinRecursion 0, MaxRecursion 3]
-0.665111
```

```
NIntegrate[ Cos[10Sin[5πx]], {x,-1,1}, Method Automatic, AccuracyGoal Infinity, PrecisionGoal Automatic, Working-
Precision MachinePrecision, MaxPoints Automatic, MinRecursion 0, MaxRecursion 4]
-0.491872
```

```
NIntegrate[ Cos[10Sin[5πx]], {x,-1,1}, Method Automatic, AccuracyGoal Infinity, PrecisionGoal Automatic, Working-
Precision MachinePrecision, MaxPoints 270, MinRecursion 0, MaxRecursion 4]
-0.491875
```

As this example suggests that in this case, the required accuracy is achieved by using more than 270 nodal points and more than 4 recursion.

While applying the formula (5) for it is required a total of 13 nodal points and 0 recursion.

5 Conclusion

Due to the fact that the construction of quadrature formulas of the highest trigonometric accuracy similar to the Gauss formula is fundamentally not possible, this approach is based on the use of derivatives may serve as an alternative to the formula of Gauss. Not difficult to foresee that these formulas can compete with well-known quadrature formulas, and in some cases even have the advantage. In the case of multiple integrals especially.

REFERENCES

- [1] B. Boyanov, Optimal quadrature formulae, *Russian Math. Surveys*, 60:6, 2005, pp.1035-1055.
- [2] S. M. Nikol'skii, Approximation of functions of several variables and imbedding theorems, *Springer* (1975) (Translated from Russian).
- [3] W. Plesniak, Multivariate Jackson Inequality, *Journal of Computational and Applied Mathematics*, Volume 233, Issue 3, 2009, pp.815-820.

- [4] K.P. Sahakyan, Hermite trigonometric interpolation, *East Journal on Approximations*, Volume 12, Number 4, 2006, pp.441-449.
- [5] Wolfram Website, <http://www.wolfram.com>.